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MODULATED LANGMUIR WAVES AND NONLINEAR LANDAU DAMPING

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The nonlinear Schrödinger equation with an integral term,

$$iu_t + \frac{P}{2}u_{xx} + Q|u|^2u + RP\int_{-\infty}^{\infty}\frac{|u(x',t)|^2}{x-x'}dx'u = 0,$$

which describes modulated Langmuir waves with the nonlinear Landau damping effect, is solved by numerical calculations. Especially, the effects of nonlinear Landau damping on solitary wave solutions are studied. For both cases, $PQ>0$ and $PQ<0$, the results show that the solitary waves deform in an asymmetric way changing its velocity.

1. Introduction

It has been shown by Taniuti and one of the authors (N. Y.) that one-dimensional nonlinear modulation of plane waves in dispersive systems can be described by the nonlinear Schrödinger equation [1]:

$$i\frac{\partial u}{\partial t} + \frac{P}{2}\frac{\partial^2 u}{\partial x^2} + Q|u|^2u = 0, \tag{1}$$

where u is the complex amplitude of a plane wave varying slowly due to modulation and P and Q are parameters which represent the strength of dispersion and nonlinearity. This equation describes a wide class of physical phenomena which involve modulational instability of water waves [2], propagation of heat pulses in anharmonic crystals [3], helical motion of a very thin vortex filament [4], nonlinear modulation of collisionless plasma waves [5],[6] and self-trapping of a light beam in colour-dispersive systems [7]. In the modulationally unstable case ($PQ>0$), the initial value problem of the nonlinear Schrödinger equation was investigated numerically by Karpman and Krushkal' [8], Yajima and Outi [9], and Satsuma and Yajima [10]. According to them a given initial disturbance breaks up to a train of solitons due to the balance between nonlinearity and dispersion. These solitons preserve their identities through the collision between them.

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The initial value problem of eq. (1) was solved analytically by Zakharov and Shabat [11]. They applied to this problem the analytical method of solving nonlinear evolution equations formulated by Lax [12]. Through the time evolution of solution, the solitons work as stable entities.

It is in general possible that there coexist waves more than one. By applying the reductive perturbation method to such a system, Oikawa and Yajima [13] showed that the system is described by the superposition of the nonlinear modulated waves each of which evolves according to eq. (1). The velocities of these modulated waves are different from their group velocities owing to the presence of the other waves.

It is well known that in real plasma systems a wave interacts strongly with the resonant particles, for example, a nonlinear modulated wave is scattered by the particles moving with the velocity equal to its group velocity. If the velocity distribution of the particle is Maxwellian, this scattering leads to the nonlinear Landau damping. Ichikawa and Taniuti [14] studied this phenomenon and modified eq. (1) to the form involving a nonlocal-nonlinear integral term:

$$i\frac{\partial u}{\partial t}+\frac{P}{2}\frac{\partial^2 u}{\partial x^2}+Q|u|^2u+R\mathrm{P}\int_{-\infty}^{\infty}\frac{|u(x',t)|^2}{x-x'}\mathrm{d}x'u=0, \tag{2}$$

where P denotes the Cauchy principal value. The effect of nonlinear Landau damping is contained in the integral term with coefficient R. The coefficient Q of the nonlinear term of eq. (1) is also modified owing to the wave-particle interactions. It is noted that for the modulation of ion waves Q changes the sign depending on the ratio of the ion temperature to the electron temperature.

We now consider the effect of nonlinear Landau damping on nonlinear wave modulations. We substitute

$$u=[\psi_0+\phi_1 e^{i(qx-\omega t)}+\phi_2 e^{-i(qx-\omega^* t)}]e^{-i\omega_0 t} \tag{3}$$

into eq. (2) and linearize with respect to the perturbed amplitude, ϕ_1 and ϕ_2, where asterisk denotes the complex conjugation. It can be readily shown that the dispersion equation is

$$\omega_0=-Q|\psi_0|^2, \tag{4}$$

$$\omega^2=\frac{1}{4}P^2q^4\left\{1-\frac{4Q}{Pq^2}|\psi_0|^2+i\frac{4R}{Pq|q|}|\psi_0|^2\right\}. \tag{5}$$

We put $\omega=\Omega+i\Gamma$, where Ω and Γ are real, to obtain

$$\Omega^2-\Gamma^2=\frac{1}{4}P^2q^4\left(1-\frac{4Q}{Pq^2}|\psi_0|^2\right), \tag{6.a}$$

$$\Omega\Gamma=\frac{1}{2}PRq|q||\psi_0|^2. \tag{6.b}$$

Without the nonlinear Landau damping, $R = 0$, eqs. (6) give the regular stability condition; the system is unstable ($\Omega = 0$, $\Gamma^2 > 0$) if $4Q|\phi_0|^2/(Pq^2) > 1$ and otherwise stable. Whilst, with $R \neq 0$, the growth rate Γ never vanishes and plane waves become unstable. If $PR > 0$, we find that $\Gamma > 0$ for the disturbance with $\Omega q > 0$ and $\Gamma < 0$ for $\Omega q < 0$. In this case the amplitude of modulated wave, thereby, grows if it propagates in the positive x-direction and damps in the opposite direction.

In this paper, eq. (2) is solved by numerical computations in order to explore how the nonlinear modulated waves, particularly the solitary waves evolve under the influence of the nonlocal-nonlinear integral term. In § 2, the modulationally unstable case ($PQ > 0$) is studied. On the other hand, the modulationally stable case ($PQ < 0$) is dealt with in § 3. In both cases R is taken to be positive.

The reductive perturbation method applies to investigate the behaviour of a slightly modulated plane wave with small nonlinear Landau damping for the case $PQ < 0$. The result essentially agrees with the equation introduced by Ott and Sudan [15], who studied the nonlinear ion acoustic wave with long wave length taking account of linear Landau damping.

The difference scheme used to solve integro-differential equation (2) is presented in Appendix.

2. Numerical Solutions to Equation (2) with $PQ > 0$ —Effect of Nonlinear Landau Damping on Envelope Solitons—

In the present case ($PQ > 0$), eq. (1) has a envelope-soliton solution, which satisfies the boundary condition that $u(x, t)$ and its derivatives vanish at $x = \pm\infty$,

$$S(x, t) = \exp[i\{(V/P)x - (V^2/2P)t + (QA^2/2)t\}]A\operatorname{sech}[(Q/P)^{1/2}A(x - Vt)], \tag{7}$$

and keeps its shape unchanged. Putting $V = 0$ we have soliton at rest,

$$S_0(x, t) = \exp[i(QA^2/2)t]A\operatorname{sech}[(Q/P)^{1/2}Ax]. \tag{7'}$$

In phrase of the Schrödinger equation, the nonlinear term of eq. (1) works as an attractive potential if $Q > 0$ and prevent a diffusion of wave packet due to the dispersion term, so that the stationary soliton solution (7) can exist.

2.1 Effect of the Nonlinear Landau Damping on Envelope Solitons.

The soliton solution (7) does not satisfy eq. (2) and deforms under the effect of nonlocal-nonlinear integral term. We now deal with the following initial value problem for eq. (2);

$$u(x, t=0) = A\operatorname{sech}[(Q/P)^{1/2}Ax]. \tag{8}$$

If $R = 0$, the solution with this initial condition is just the solition at rest, (7′). For this u, the integral term in eq. (2) is positive for $x>0$ and negative for $x<0$. This implies that the nonlinear attractive force is enhanced for $x>0$ and weakened for $x<0$ and, therefore, the wave form rises more steeply from the right than from the left. The nonlinear Landau eamping thus leads to an asymmetric deformation of wave form. The integral term produces another effect: the amplitude A slowly increases with x and then the phase $(QA^2/2)t$ of soliton solution (see eq. (7′)) advances more rapidly in larger x. As can be seen from eq. (7), this makes an effect of $V \neq 0$.

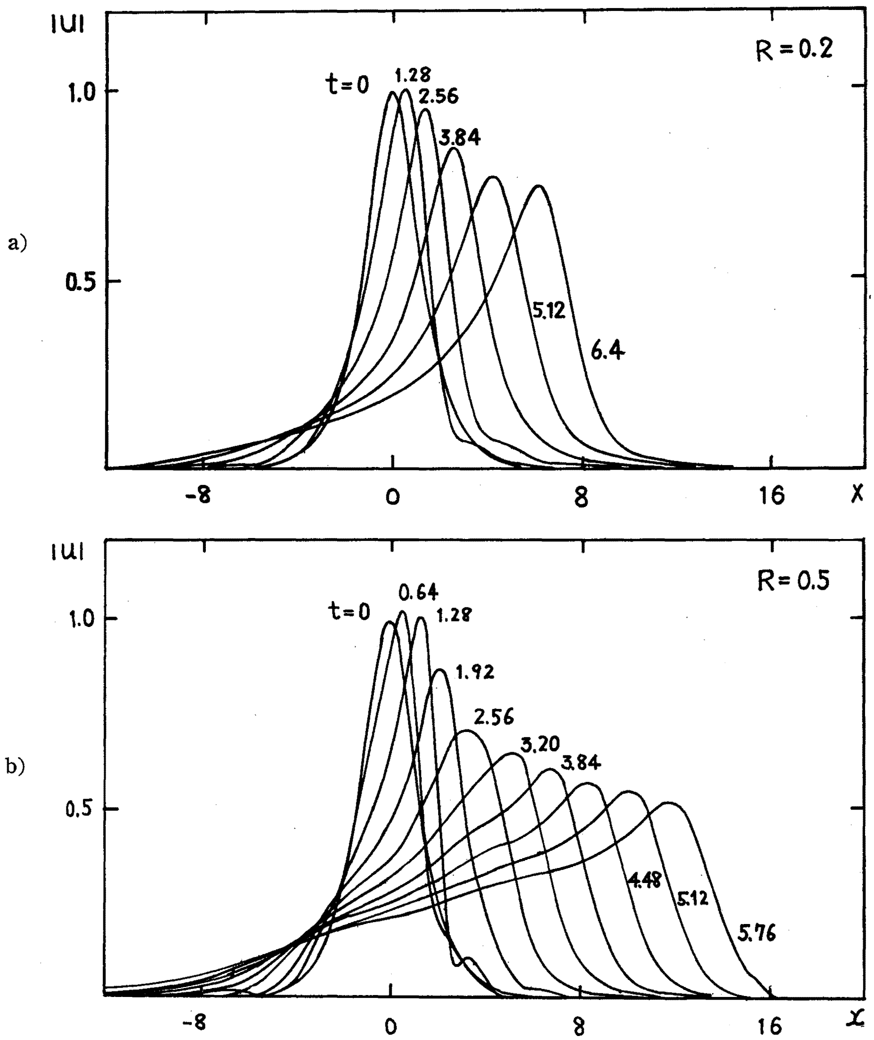

Fig. 1. Decay of Envelope Soliton
Time development of solution for the initial condition (8) is calculated for both cases, a) $R=0.2$ and b) $R=0.5$.

Therefore, the soliton which is initially at rest comes to move.

The numerical solutions to eq. (2) with initial value (8) are illustrated in Fig. (1). There we take $A=Q=P=1$. It is observed that the soliton deforms in an asymmetric way and comes to travel. This agrees with the feature shown in § 1 that the waves propagating in the positive x-direction grow and that in the negative x-direction damp.

2.2 Effect of Nonlinear Landau damping on Bound State of Envelope Solitons.

Equation (1) has a solution

$$u(x,t)=4Ae^{iQA^2t/2}\frac{\mathrm{ch}(3Bx)+3\mathrm{ch}(Bx)e^{4iQA^2t}}{\mathrm{ch}(4Bx)+4\mathrm{ch}(2Bx)+3\cos(4QA^2t)}, \tag{9}$$

which satisfies the initial condition

$$u(x,t=0)=2A\,\mathrm{sech}(Bx), \tag{10}$$

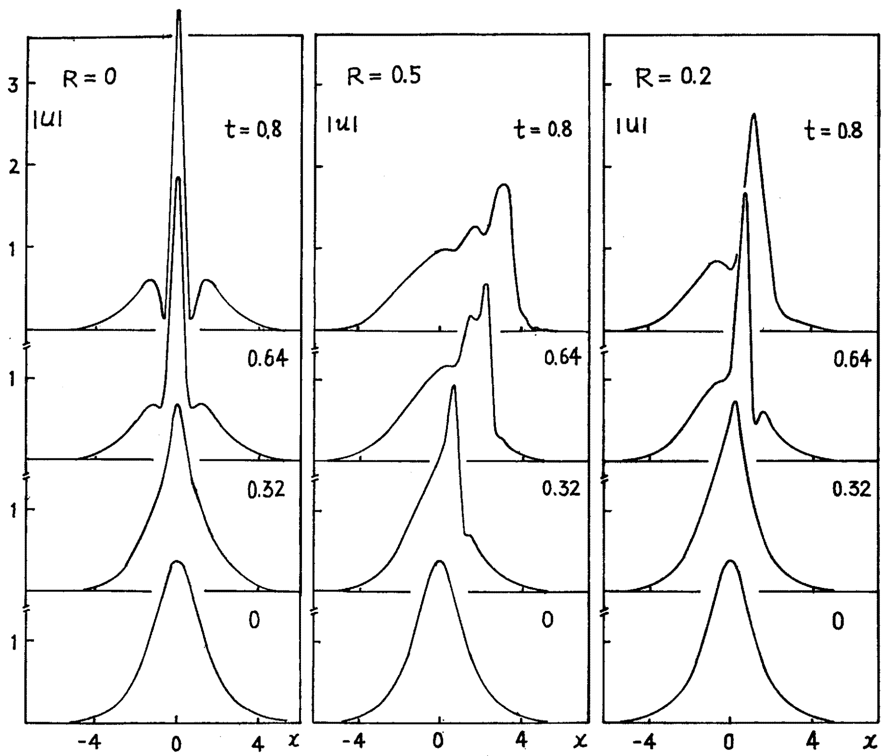

Fig. 2 Time development of solution for the initial condition (10). For comparison, the solution with $R=0$, i.e., (9), is also illustrated.

$$B=(Q/P)^{1/2}A. \tag{11}$$

This solution does not decay into a train of solitons and pulsates with a period $\pi/(2QA^2)$. It has been already shown that if a disturbance with an asymmetric imaginary part is inflicted on the bound state, it decays into constituent solitons [10].

We now solve eq. (2) with initial condition (10). The symmetry of u breaks due to the nonlinear Landau damping. Owing to such an asymmetry in u, solitons which are bound in its initial state are made to be free. Each of solitons travels with changing shape and velocity. Examples are illustrated in Fig. 2, for $A=2$, $P=Q=1$.

3. Numerical Solutions to Equation (2) with $PQ<0$.

We introduce the amplitude and the phase of u as

$$u=\sqrt{n(x,t)}\exp[i\theta(x,t)/P], \tag{12}$$

and substitute into eq. (2), to get

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0, \tag{13.a}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}-PQ\frac{\partial n}{\partial x}-\frac{P^2}{4}\frac{\partial}{\partial x}\left[n^{-1/2}\frac{\partial}{\partial x}\left(n^{-1/2}\frac{\partial n}{\partial x}\right)\right] -PR\mathrm{P}\int_{-\infty}^{\infty}\frac{1}{x-x'}\frac{\partial n(x',t)}{\partial x'}\mathrm{d}x'=0 \tag{13.b}$$

where $v=\partial\theta/\partial x$. If the higher derivative term and the nonlocal integral term are neglected, eqs. (13.a) and (13.b) reduce to the hyperbolic system of equations (note that $PQ<0$), in which the nonlinear steepening occurs. The presence of the higher derivative term, which represents the dispersive effect, prevents the nonlinear steepening and leads to an emission of solitons.

Equations (13) without the integral term have a following soliton-solution;

$$n(x,t)=n_0\left[1-A\mathrm{sech}^2\left\{\frac{c}{P}A^{1/2}(x-\lambda_\pm t)\right\}\right], \tag{14.a}$$

$$v(x,t)=\lambda_\pm\mp c(1-A)^{1/2}/\left[1-A\mathrm{sech}^2\left\{\frac{c}{P}A^{1/2}(x-\lambda_\pm t)\right\}\right], \tag{14.b}$$

$$\lambda_\pm=v_0\pm c(1-A)^{1/2},\ c=(-PQn_0)^{1/2}, \tag{14.c}$$

where n_0 and v_0 are the boundary values of n and v, respectively, at $x=\pm\infty$. It is noted that the above soliton with λ_+(or λ_-) represents the defect in the amplitude, which propagates in the positive (or negative) x-direction.

The effect of nonlinear Landau damping on such soliton solutions is studied by numerical integration of eq. (2) with the initial value

$$u=n_0^{1/2}\left[1-A\mathrm{sech}^2\left(\frac{c}{P}A^{1/2}x\right)\right]^{1/2}\exp\left[\mp i\left\{\frac{c}{P}(1-A)^{1/2}x\right.\right.$$
$$\left.\left.+\tan^{-1}\left(\left(\frac{A}{1-A}\right)^{1/2}\tanh\left[\frac{c}{P}A^{1/2}x\right]\right)\right\}\right], \tag{15}$$

which corresponds to the soliton with $\lambda_{\pm}=0$. In this case, we study the behaviour of soliton in the frame moving with it. The results are shown in Fig. 3 and 4. It is observed that the soliton damps or grows due to the

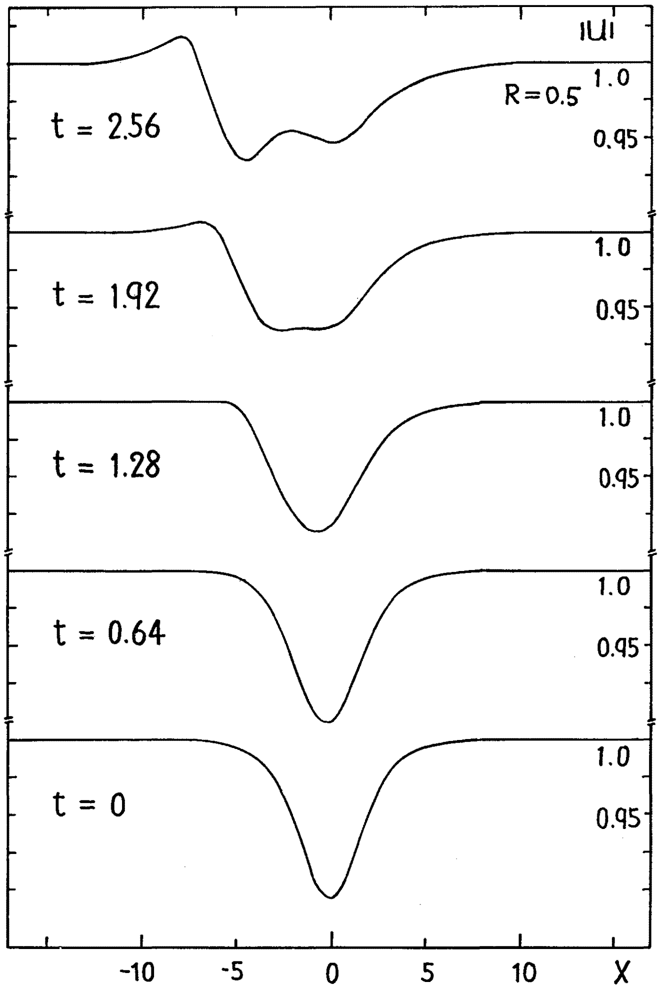

Fig. 3. Time development of solution for the initial condition (15) with λ_+. $A=0.1$, $P=-1$, $Q=1$ and $R=0.5$.

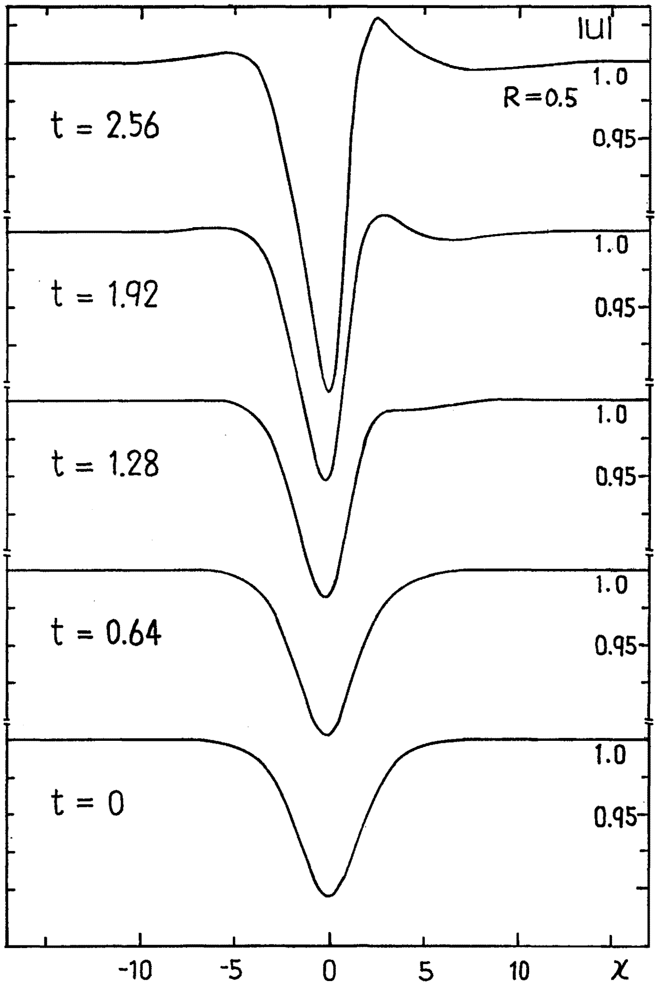

Fig. 4 Time development of solution for the initial condition (15) with λ_-. $A=0.1$, $P=-1$, $Q=1$ and $R=0.5$.

nonlinear Landau damping according as that it moves in the positive or negative x-directions.

It is interested to study the case that the nonlinear Landau damping is sufficiently small and is of the same order of magnitude as that of modulation. In this case, we can apply the reductive perturbation method[16]. Assuming the modulation to be small, we now expand n and v in powers of a small parameter ε,

$$n=n_0+\varepsilon n_1+\varepsilon^2 n_2+\cdots\cdots, \tag{16.a}$$

$$v=\qquad \varepsilon v_1+\varepsilon^2 v_2+\cdots\cdots. \tag{16.b}$$

In order to take into account appropriately the competition between two effects, nonlinear steepening and dispersion, we introduce the stretched coordinates:

$$\xi_1=\varepsilon^{1/2}(x-ct-\varepsilon^{1/2}\psi_1), \tag{17.a}$$

$$\xi_2=\varepsilon^{1/2}(x+ct-\varepsilon^{1/2}\psi_2), \tag{17.b}$$

$$\tau=\varepsilon^{3/2}t, \tag{17.c}$$

where c is given in eq. (14.c). The stretched variable ξ_1 (or ξ_2) represents the phase of soliton which travels in the positive (or negative) x-direction. The phase shifts ψ_1 and ψ_2 are due to the mutual interaction between waves moving right and left and considered as functions of ξ_1, ξ_2 and τ. Suppose that

$$R=\varepsilon r, \tag{18}$$

where r is at most of the order of unity.

Substituting eqs. (16)$\sim$(18) into eqs. (13), we obtain a set of equations to be solved for the successive powers of ε. In the lowest order, we get

$$v_1+(c/n_0)n_1=f(\xi_1,\tau), \tag{19.a}$$

$$v_1-(c/n_0)n_1=g(\xi_2,\tau). \tag{19.b}$$

The functions f and g can be obtained from the non-secularity condition of the next order equations.

In the order $\varepsilon^{5/2}$, we have

$$\frac{\partial}{\partial\xi_2}\Big[2cF+\Big\{\frac{fg}{4}-\frac{1}{8}g^2+\frac{P^2}{8c}\frac{\partial^2 g}{\partial\xi_2{}^2}-\frac{rc}{2Q}\mathrm{P}\int_{-\infty}^{\infty}\frac{g(\xi')}{\xi_2-\xi'}\mathrm{d}\xi'\Big\}\Big]$$
$$+\Big[\frac{\partial f}{\partial\tau}+\frac{3}{4}f\frac{\partial f}{\partial\xi_1}-\frac{P^2}{8c}\frac{\partial^3 f_1}{\partial\xi_1{}^3}+\frac{rc}{2Q}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi_1-\xi'}\frac{\partial f}{\partial\xi'}\Big]$$
$$+\Big\{\frac{1}{4}g-2c\frac{\partial\psi_1}{\partial\xi_2}\Big\}\frac{\partial f}{\partial\xi_1}=0, \tag{20.a}$$

$$\frac{\partial}{\partial\xi_1}\Big[2cG+\Big\{-\frac{fg}{4}+\frac{1}{8}f^2+\frac{P^2}{8c}\frac{\partial^2 f}{\partial\xi_1{}^2}-\frac{rc}{2Q}\mathrm{P}\int_{-\infty}^{\infty}\frac{f(\xi')}{\xi_1-\xi'}\mathrm{d}\xi'\Big\}\Big]$$
$$+\Big[\frac{\partial g}{\partial\tau}+\frac{3}{4}g\frac{\partial g}{\partial\xi_2}+\frac{P^2}{8c}\frac{\partial^3 g}{\partial\xi_2{}^3}-\frac{rc}{2Q}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi_2-\xi'}\frac{\partial g}{\partial\xi'}\Big]$$
$$-\Big\{\frac{1}{4}f+2c\frac{\partial\psi_2}{\partial\xi_1}\Big\}\frac{\partial g}{\partial\xi_2}=0, \tag{20.b}$$

where

$$F=u_2+(c/n_0)n_2, \tag{21.a}$$

$$G=u_2-(c/n_0)n_2. \tag{21.b}$$

The functions ψ_1 and ψ_2 are chosen such that

$$\frac{\partial\psi_1}{\partial\xi_2}=\frac{g}{8c},\quad \frac{\partial\psi_2}{\partial\xi_1}=-\frac{f}{8c}. \tag{22}$$

Substituting eq. (22) into eqs. (20. a) and (20. b) and imposing the condition that F and G are bounded (non-secularity condition) yields

$$\frac{\partial f}{\partial\tau}+\frac{3}{4}f\frac{\partial f}{\partial\xi_1}-\frac{P^2}{8c}\frac{\partial^3 f}{\partial\xi_1^3}+\frac{rc}{2Q}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi_1-\xi'}\frac{\partial f}{\partial\xi'}=0, \tag{23. a}$$

$$\frac{\partial g}{\partial\tau}+\frac{3}{4}g\frac{\partial g}{\partial\xi_2}+\frac{P^2}{8c}\frac{\partial^3 g}{\partial\xi_2^3}-\frac{rc}{2Q}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi_2-\xi'}\frac{\partial g}{\partial\xi'}=0. \tag{23. b}$$

These equations agree with that obtained by Ott and Sudan [15] in studying ion acoustic waves of finite amplitude with the linear Landau damping by electrons.

We note here that eq. (23. b) reduces to eq. (23. a) by the transformation $g\to-f$, $\tau\to-\tau$. If $r=0$, eqs. (23. a) and (23. b) have soliton solutions with negative and positive amplitudes, respectively. For the case $r\neq 0$, the negative-amplitude soliton, f, damps as time increases and the positive-amplitude soliton, g, grows. This tendency agrees with the numerical solutions with the initial condition (15) (see Fig. 3 and 4).

4. Conclusion

For $PQ>0$, the soliton damps deforming asymmetrically and changing the velocity, due to the nonlinear Landau damping. The bound state of solitons decays into a series of solitons, which behave themselves in a way similar to the above. On the other hand, the solitons for $PQ<0$ display a different character from the case $PQ>0$; the soliton propagating in the positive x-direction damps and that in the negative x-direction grows.

Further details, together with the periodic solutions to eq. (2), will be presented in the forthcoming paper.

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Appendix

Method for Numerical Solution

To solve the nonlinear Schrödinger equation (1), we replace the partial derivatives by the central difference quotients,

$$\partial u/\partial t \to (u(x, t+\Delta t) - u(x, t-\Delta t))/(2\Delta t), \qquad \text{(A-1)}$$

$$\partial^2 u/\partial x^2 \to (u(x+\Delta x, t) - 2u(x, t) + u(x-\Delta x, t))/\Delta x^2, \qquad \text{(A-2)}$$

where Δt and Δx are mesh size in the x-t space. The function $u(x, t)$ is divided into the real and imaginary parts,

$$u = X + iY, \qquad \text{(A-3)}$$

and substituted into eq. (1). We then obtain the set of difference equations:

$$X(x, t+\Delta t) = X(x, t-\Delta t) - P(\Delta t/\Delta x^2)\{Y(x+\Delta x, t) - 2Y(x, t) + Y(x-\Delta x, t)\} - 2Q\Delta t\{X(x, t)^2 + Y(x, t)^2\}Y(x, t), \qquad \text{(A-4)}$$

$$Y(x, t+\Delta t) = Y(x, t-\Delta t) + P(\Delta t/\Delta x^2)\{X(x+\Delta x, t) - 2X(x, t) + X(x-\Delta x, t)\} + 2Q\Delta t\{X(x, t)^2 + Y(x, t)^2\}X(x, t). \qquad \text{(A-5)}$$

The suitable mesh size must be chosen so that eqs. (A-4) and (A-5) may be stable. This can be estimated as follows: Linearizing eqs. (A-4) and (A-5) and taking the Fourier transforms of them yields

$$U(k, t+\Delta t) = A(k) U(k, t), \tag{A-6}$$

$$U(k, t) = \begin{bmatrix} \tilde{X}(k, t) \\ \tilde{X}(k, t-\Delta t) \\ \tilde{Y}(k, t) \\ \tilde{Y}(k, t-\Delta t) \end{bmatrix} \text{ and } A(k) = \begin{bmatrix} 0 & 1 & -B & 0 \\ 1 & 0 & 0 & 0 \\ B & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{A-7}$$

$$\begin{bmatrix} \tilde{X}(k, t) \\ \tilde{Y}(k, t) \end{bmatrix} = \int_{-\infty}^{\infty} \mathrm{d}x \exp(-ikx) \begin{bmatrix} X(x, t) \\ Y(x, t) \end{bmatrix}, \tag{A-8}$$

$$B = 4P(\Delta t/\Delta x^2)\sin^2(k\Delta x/2) + 2QN\Delta t, \tag{A-9}$$

where N is introduced as a measure of the nonlinearity. The eigenvalue, λ, of the amplification matrix A is given by

$$\lambda^2 + \lambda^{-2} = 2(1 - B^2/2). \tag{A-10}$$

The difference scheme (A-4) and (A-5) is stable if and only if $|1-B^2/2| \leqq 1$, i.e.,

$$|4P(\Delta t/\Delta x^2)\sin^2(k\Delta x/2) + 2QN\Delta t| \leqq 1,$$

or

$$\Delta t \leqq \Delta x^2 / |4P + 2QN\Delta x^2|. \tag{A-11}$$

The integral term in eq. (2) is approximated as

$$\mathrm{P}\int_{-\infty}^{\infty} \mathrm{d}x' \frac{f(x')}{x-x'} = -6\Delta x\left(\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{2}\Delta x^2 \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} + \frac{27}{200}\Delta x^4 \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}\right) + \left(\int_{-\infty}^{x-3\Delta x} + \int_{x+3\Delta x}^{\infty}\right)\mathrm{d}x' \frac{f(x')}{x-x'}.$$

In deriving the above expression $f(x')$ is expanded in powers of $(x'-x)$ for the region $x-3\Delta x \leqq x' \leqq x+3\Delta x$. The differential coefficients are approximated by the seven-points difference quotients and the integrals with respect to x' over the residual intervals are calculated by using Weddle's formula,

$$\begin{aligned} \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x')}{x-x'} dx' &= -[0.13\{f(x+3\Delta x) - f(x-3\Delta x)\} \\ &-0.48\{f(x+2\Delta x) - f(x-2\Delta x)\} + 1.65\{f(x+\Delta x) - f(x-\Delta x)\}] \\ &-0.3\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty}\Big[\Big\{\frac{f(x+(6j+3)\Delta x)}{6j+3} + \frac{f(x+(6j-3)\Delta x)}{6j-3}\Big\} \\ &+5\Big\{\frac{f(x+(6j+2)\Delta x)}{6j+2} + \frac{f(x+(6j-2)\Delta x)}{6j-2}\Big\} + \frac{f(x+6j\Delta x)}{j} \\ &+\Big\{\frac{f(x+(6j+1)\Delta x)}{6j+1} + \frac{f(x+(6j-1)\Delta x)}{6j-1}\Big\}\Big]. \end{aligned} \tag{A-12}$$

The difference scheme (A-4) and (A-5) with (A-11) and (A-12) applies to the numerical computation for initial value problems of eq. (2).