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A NUMERICAL STUDY ON THE INSTABILITY OF THREE-DIMENSIONAL BOUNDARY LAYERS

By Iwao YAMASHITA* and Masaki TAKEMATSU**

This paper examines the linear instability of a laminar boundary layer in the presence of a cross flow component. For wave-like disturbances propagating in various directions in the resultant three-dimensional basic flow, the disturbance equations (originally due to Stuart) are solved by computor numerical methods and the growth rate as well as the phase velocity are determined as functions of the wave number and the Reynolds number.

It is found that three-dimensional flows of the type considered can generate highly amplified disturbances with zero phase velocity for a wide range of Reynolds numbers. The particular mode of disturbances correspond well, in all essential respects, with the non-propagating vortices which have been observed experimentally in the three-dimensional boundary layer on a swept-back wing or on a rotating disk.

1. Introduction

Observing the air flow over swept-back wings by an 'evaporation' methods, Gray (unpublished) noticed that in the three-dimensional boundary layer the onset of transition is preceded by the appearance of a set of regularly spaced vortices fixed relative to the wing surface; the phenomenon is absent in the two-dimensional flow on unswept wings. Subsequently, Gregory and Walker (see Gregory, Stuart and Walker[1)]) demonstrated, by means of the china-clay visualization technique, that a similar phenomenon also occurs in the three-dimensional boundary layer on a rotating disk. A closer examination of the flow by an acoustic method indicated that some of the dominating modes of disturbances in the unstable region have indeed non-propagating property in agreement with the implication of the china-clay records. The present paper is a continuation of an effort of Stuart[1)] to account for the stationary vortex phenomenon characteristic to those three-dimensional boundary layers.

The foundations of the stability analyses for three-dimensional boundary layers were laid down by Squire, Dunn & Lin[2)] and Stuart[1)]. In particular, Stuart examined the general equations of motion in curvilinear co-ordinates by superposing a small wave-like disturbance of an arbitrary geometrical configulation and showed that in a localized region of a three-dimensional boundary layer, the velocity component in the direction of propagation of

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the disturbance can be regarded approximately as a two-dimensional flow for stability purposes; that is, the usual two-dimensional stability theory can be applied to each of a continuous set of component profiles (corresponding to different directions) of the three-dimensional basic flow.

Component profiles of particular interest in connection with the experimental observations are those in directions of the wave fronts of the observed stationary vortices. If the above conclusion due to Stuart is correct, hypothetical two-dimensional flows with such velocity profiles must admit the existence of stationary disturbances in consistence with the observations. In fact, Stuart showed that there is certainly one particular profile which generates neutral disturbances having zero phase velocity in the inviscid limit. The profile is denoted by J-profile in the literature, and the associated neutral disturbances have been supposed so far to correspond to the observed vortices mainly on the ground that the direction of the observed wave fronts is in close agreement with the direction predicted by the calculation of J-profile. Although the theoretical wave number in the inviscid limit turned out to be several times larger than the experimental value, the discrepancy was simply attributed to the effect of fluid viscosity.

Apart from the discrepancy in the wave number, however, the correlation between the theory and the experiment regarding the vortex phenomenon seems to be unsatisfactory in another essential respect. As mentioned above the observed stationary vortices are known to be of dominant mode with large amplitudes, while the theoretical non-propagating disturbances are only of neutral mode. It is quite unlikely that neutral disturbances appear to be such dominant ones in natural conditions; such dominant vortices must be associated with some highly amplified disturbances rather than neutral disturbances. Then is it possible for J-profile to generate amplified disturbances with zero phase velocity besides the neutral ones? The primary objective of the present paper is to answer this question and provide a more satisfactory explanation for the stationary vortex phenomenon.

In this paper, the disturbance equations are solved not only for J-profile but also for other neighbouring profiles by applying direct numerical methods, and the growth rate as well as the phase velocity of possible disturbances are determined as functions of the wave number and the Reynolds number. The results are then discussed in comparison with the previous experimental results concerning the vortex phenomenon. A similar numerical study on the problem was previously made by Brown[3] from the practical interest of predicting the critical Reynolds number, but his computation was restricted to the neutral case and no mention was made of the vortex phenomenon.

2. Statement of the problem

As stated in the introduction, the foundations of the stability of three-dimensional flows have been laid down by several authors and are available in the literature. For the sake of completeness, however, we begin by recapitulating the problem formulation.

Consider a steady three-dimensional boundary layer flow of velocity $\mathbf{V}$ over a (curved) surface, and superimpose on this basic flow a non-steady small disturbance $\mathbf{v}$. Then, to the first order in $\mathbf{v}$, the Navier-Stokes and continuity equations give the following disturbance equations in vector form:

$$\left.\begin{aligned}&\frac{\partial \mathbf{v}}{\partial t}-\mathbf{V}\times\mathrm{rot}\,\mathbf{v}-\mathbf{v}\times\mathrm{rot}\,\mathbf{V}=-\mathrm{grad}(p+\mathbf{V}\cdot\mathbf{v})\\&-R^{-1}\mathrm{rot}\;(\mathrm{rot}\,\mathbf{v}),\qquad \mathrm{div}\,\mathbf{v}=0,\end{aligned}\right\}\tag{1}$$

where p is the disturbance pressure, R being the Reynolds number based on a reference length δ and a reference velocity $\mathbf{V}_*$; here, and in what follows, all quantities are made dimensionless with respect to δ and $\mathbf{V}_*$.

For further manipulation of the equations, it is convenient to introduce orthogonal curvilinear coordinates (x, y, z) such that x and y are on the surface and z is normal to the surface. In view of the fact that the observed vortices are regularly spaced, we assume the disturbance velocity and pressure in the form

$$\left.\begin{aligned}\mathbf{v}\equiv (u,\ v,\ w)&=(u_1,\ v_1,\ w_1)\ \exp\,[i\alpha(x-ct)],\\ p&=p_1\ \exp\,[i\alpha(x-ct)],\end{aligned}\right\}\tag{2}$$

by choosing the coordinate x in the direction of its periodicity. If we substitute these expressions in eq. (1) and eliminate the pressure p, we obtain a set of three equations for the amplitude functions u_1, v_1, w_1. In the present problem, the effect of the curvature of the surface is not essential and may be neglected, since the type of instability considered appears for both flat and curved surfaces. Then it can be shown that, in a localized region of a thin boundary layer, the set of equations are reduced to

$$(U-c)\ (w_1''-\alpha^2w_1)-U''w_1+\frac{i}{\alpha R}(w_1''''-2\alpha w_1''+\alpha^4 w_1)=0, \qquad (3)_1$$

$$(U-c)v_1+\frac{i}{\alpha R}(v_1''-\alpha^2 v_1)-\frac{i}{\alpha}w_1V'=0, \qquad (3)_2$$

$$i\alpha u_1+w_1'=0, \qquad (3)_3$$

where the primes indicate differentiation with respect to z; U and V are the velocity components of the basic flow in the x-and y-directions, respectively. The first equation is nothing but the well-known Orr-Sommerfeld equation in the two-dimensional stability theory. It will be seen that this equation plays an essential role also in the present three-dimensional problem.

The appropriate boundary conditions are $u_1=v_1=w_1=0$ for $z=0$ and ∞, or, from eq. $(3)_3$,

$$v_1=w_1=w_1'=0 \qquad \text{for} \qquad z=0,\ \infty. \tag{4}$$

The homogeneous boundary conditions involving w_1, together with eq. $(3)_1$, lead to an eigen-relation of the form $D\ (c;\ \alpha,\ \alpha R)=0$, from which we can determine $c=c_r+ic_i$ as a function of α and R: According as c_i takes positive, zero or negative values, the type of disturbance is amplified, neutral or damped, respectively as time goes on. Once the eigenvalue problem for w_1 has been solved, the other components u_1 and v_1 are determined from eqs. $(3)_2$, $(3)_3$ and the associated boundary conditions.

Thus, so far as eigenvalues are concerned, the mathematical problem is formally the same as that of the usual two-dimensional theory. In the present context, however, the velocity function U appearing in the problem represents the component profile of the three-dimensional basic flow in the direction of propagation of the disturbance under consideration, and the function generally takes more complicated forms than those encountered in usual two-dimensional stability problems. In the following, we describe briefly possible forms of the velocity function in a typical three-dimensional boundary layer.

Velocity components of the basic flow: At any particular point on a surface, we can resolve the basic velocity into components along, and normal to, the 'external potential streamline'. The tangential component is of the ordinary boundary-layer type, whereas the normal component ('secondary flow') is of the wall-jet type. For any other direction, the velocity profile is given by a combination of these two extreme cases as

$$U_\varepsilon(z)=V_0\{F_n(z)\cos\varepsilon+F_t(z)\sin\varepsilon\},$$

where ε is an angle relative to the direction normal to the potential streamline; F_n and F_t represent the velocity distributions in the normal and tangential directions, respectively. Fig. 1 illustrates a representative class of velocity profiles calculated by the above equation. We note in passing that there is one particular profile (J) for which the point of inflexion lies just at the point of zero velcity. As mentioned in the introduction, this J-profile was of decisive significance in Stuart's explanation for the stationary vortex

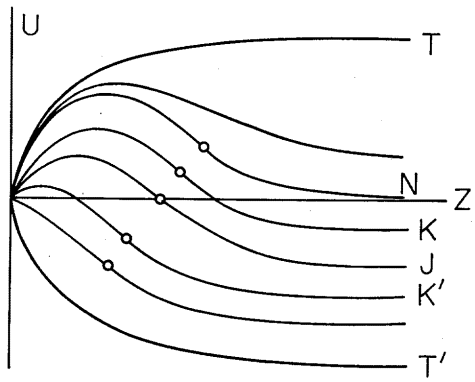

Fig. 1 Velocity components of the basic flow (○, points of inflexion).

phenomenon; there the phenomenon was interpreted as the 'inflexion-point instability' associated with this particular profile. In the present work, attention is directed also to other neighbouring profiles such as K and K′, as well as to J-profile.

As an illustrative example, let us consider a rather simple case where the velocity functions F_n and F_t in the extreme directions are defined in the form $F_n=M\ (e^{-z}-e^{-2z})$, $F_t=1-e^{-z}$, so that

$$U_\varepsilon(z)=V_0\left[(1-e^{-z})\sin\varepsilon+M\,(e^{-z}-e^{-2z})\cos\varepsilon\right]. \tag{5}$$

Practically, this represents the velocity components in the rotating-disk case with suction, M being a suction parameter. Outside the boundary layer the velocity approaches the value $V_0 \sin\varepsilon$, which is negative for J and neighbouring profiles as is illustrated in Fig. 1. Then, it may be convenient to normalize the expression (5) by $-V_0\sin\varepsilon$ and write

$$U(z)=-1+(m+1)e^{-z}-me^{-2z}, \tag{6}$$

where the subscript ε has been omitted. For $m=3$, $U=U''=0$ at $z=\log_e 3$. Thus we see that $m=3$ gives J-profile: If we take m slightly larger (smaller) than 3, we have K-(K′-) profiles. For the present purpose of accounting for the vortex phenomenon, it may suffice to deal with this simple velocity function.

3. Numerical methods of solution

3.1. Solution of the inviscid problem

The instability under consideration is obviously a 'non-viscous' type of instability that can occur even at infinite Reynolds number, and fluid viscosity may be expected to have only a slight influence on its behaviour. For this kind of instability, the solution of the simplified problem with viscosity neglected is particularly important as the first step in the treatment of the full viscous problem.

Letting formally $R=\infty$ in eq. $(3)_1$, we have the inviscid equation

$$(U-c)(w_1''-\alpha^2 w_1)-U''w_1=0. \tag{7}$$

The appropriate boundary conditions in this limiting case are

$$w_1'(z_0)+\alpha w_1(z_0)=0, \tag{8}$$

$$w_1(0)=0, \tag{9}$$

where z_0 denotes a point outside the boundary layer. Note that eq. (7) exhibits logarithmic singularity at z_c where $U(z_c)=c$ (c.f. Fig. 2). This simplified boundary-value problem has been solved for various velocity profiles in connection with the stability of two-dimensional flows, but so far there have been few attempts to solve the problem for the class of velocity profiles*

* Those profiles are similar in form to the mean velocity profile of a 'separated' flow.

concerned here.

The most effective way of solving the problem for such profiles may be the use of a computer-aided numerical method. This enables us to do the task without any restriction on the values of parameters involved. In applying the numerical method, we introduce the new dependent variable

$$G(z;\alpha, c)=-\alpha w_1(z)/w_1'(z).$$

In terms of G, eq. (7) is reduced to the first-order equation

$$G'-\alpha\,(G^2-1)-\frac{U''G^2}{\alpha(U-c)}=0, \tag{10}$$

and the boundary conditions (8) and (9) become

$$G(z_0;\ \alpha,\ c)=1, \tag{11}$$

$$G(0;\ \alpha,\ c)=0, \tag{12}$$

respectively. It may be seen that these expressions are more convenient for numerical solution than the original ones.

Our purpose here is to find particular combinations of the parameter α and c that allow the first-order equation (10) to have a solution satisfying the two conditions (11) and (12) simultaneously; the values of α and c for which this is true are referred to as eigenvalues. The computational procedure employed for this purpose is as follows: i) choose a complex value of c properly for an assigned real value of α; ii) starting from the

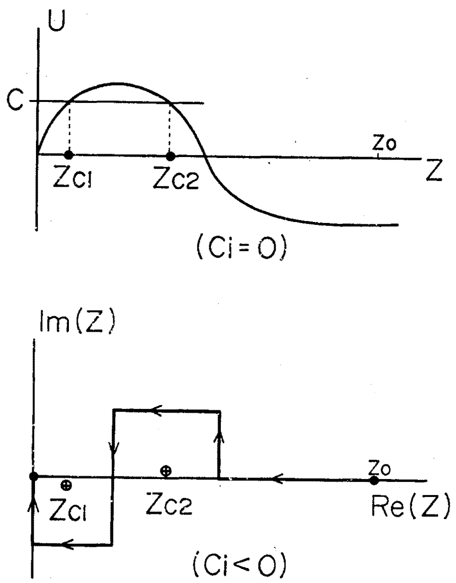

Fig. 2 Singularities and path of integration.

'initial' condition (11), integrate eq. (10) downwards to the wall ($z=0$); iii) repeat the integration, adjusting the value of c by trial and error, until the integrated value G (0; α, c) satisfies the 'eigenvalue criterion' (12) with sufficient accuracy. The integration of eq. (10) is carried out by a standard Runge-Kutta method along a path in the complex z-plane avoiding singularities in a proper way* as illustrated in Fig. 2.

3.2. Solution of the full viscous problem

We now proceed to deal with the full viscous problem defined by eq. $(3)_1$ together with the four boundary conditions in (4). In applying a numerical method to the problem, the range of integration must be finite, and hence it is necessary to replace the two conditions specified at $z=\infty$ by equivalent conditions at a finite point of z.

Outside the boundary layer where U is substantially constant ($=-1$ in the present problem), a solution of eq. (3) vanishing at infinity can be found in a closed form:

$$w_1=A_1e^{-\alpha z}+A_2e^{-\gamma z};\ z\gtrsim z_0, \tag{13}$$

where $\gamma=[\alpha^2-i\alpha R(1+c)]^{1/2}$, $R_e[\gamma]>0$, A_1 and A_2 being unknown constants. Using this expression in the usual continuation conditions at $z=z_0$ and eliminating the A's, we are led to the following conditions:

$$\left.\begin{aligned} &w_1''(z_0)+(\alpha+\gamma)w_1'(z_0)+\alpha\gamma w_1(z_0)=0,\\ &w_1'''(z_0)+\gamma w_1''(z_0)-\alpha^2 w_1'(z_0)-\alpha^2\gamma w_1(z_0)=0. \end{aligned}\right\} \tag{14}$$

The use of these two conditions instead of the vanishing conditions at $z=\infty$ allows us to remove the semi-infinite region $z\gtrsim z_0$ from the problem. The relevant conditions in (4) are now

$$w_1(0)=w_1'(0)=0. \tag{4$'$}$$

To obtain machine solutions, we subdivide the finite inner region into equally spaced small intervals and at each grid-point we express all the derivatives appering in the problem in central differences of the neighbouring five discrete variables. Then, eqs. $(3)_1$ $(4)'$ and (14) in differential form lead to a set of linear algebraic equations, which may be expressible as

$$[\boldsymbol{D}]\cdot[\boldsymbol{W}]=0, \tag{15}$$

where $\boldsymbol{D}(\alpha,\ R,\ c)$ is a square matrix and $\boldsymbol{W}$ is a colum matrix representing w_1 at discrete grid-points. The set of homogeneous equations has a non-trivial solution only if the determinant of the coefficient matrix vanishes; that is

$$|\ \boldsymbol{D}\ (\alpha,\ R,\ c)\ |=0. \tag{16}$$

Thus the required eigenvalues are determind as zeros of the determinant.

* See, e.g., Lin[4] for the rule of selecting the path of integration around the singularities.

The coefficient matrix $\boldsymbol{D}$ is much the same as the one treated in a previous paper[5] of one of the authors, except for the two rows corresponding to the surface conditions (4)′.

A trial-and-error method is used for the search of zeros. For each choice of α and R, the determinant values are evaluated over grids in the complex c-plane by the sweeping out method; such a scan of determinant values is repeated, adjusting c properly by trial and error, until a possible location of a zero is indicated; the exact location of a zero (i.e. an eigenvalue c) is then determined by interpolation. It is convenient to initiate the search for eigenvalues from high Reynolds number range, where a good 'initial guess' is available from the preliminary inviscid computation. This is particularly so in the present problem since the stability characteristics are fairly insensitive to change in the Reynolds number. (By the way, it is to be noted that without 'initial guess' numerical search for eigenvalues would be very inefficient.) The step size in the independent variable z is selected tentatively with reference to the analytical nature of eq. $(3)_1$ and the question of whether the selected value is small enough may be settled *a posteriori* by spot checks. In computing the determinant of large size, about 18 decimal digits are retained to secure a sufficient digid accuracy in the final results.

4. Results and discussion

By the foregoing methods stability computations were carried out for J-profile and neighbouring K- and K′- type profiles (see Fig. 1), which had been expected to generate non-propagating mode of disturbances in consistence with the observations. In the following examples, use was made of eq. (6) to illustrate those velocity profiles: The value of the parameter m must be taken equal to 3.0 for J-profile, while values of m of 3.5 and 2.6 were selected for K- and K′- profiles, respectively. Note that in practice eq. (6) simulates well the basic velocity field in the rotating-disk case (without suction)[1].

Inviscid eigenvalues: Fig. 3 shows the inviscid eigenvalues for J-profile. In this case instability ($c_i>0$) is predicted for wave numbers in the range $0<\alpha<\alpha_s$ $(=1.5)$. In particular at $\alpha=\alpha_s$, the phase velocity c_r as well as the growth rate αc_i become zero. This particular eigenvalue was first found by Stuart[1] and the indicated neutral stationary disturbance has been presumed so far to correspond to the observed stationary vortex phenomenon. However, we see from the figure that this may not be the case. Note that $c_r=0$ occurs also for another wave number α'_s $(<\alpha_s)$, and that the corresponding growth rate takes a large (though not the largest) positive value. Thus, the fact is that there is another amplified mode of non-propagating disturbance* besides the neutral one found by Stuart. It seems very difficult

* The existence of such an additional non-propagating disturbance does not seem to have been pointed out previously.

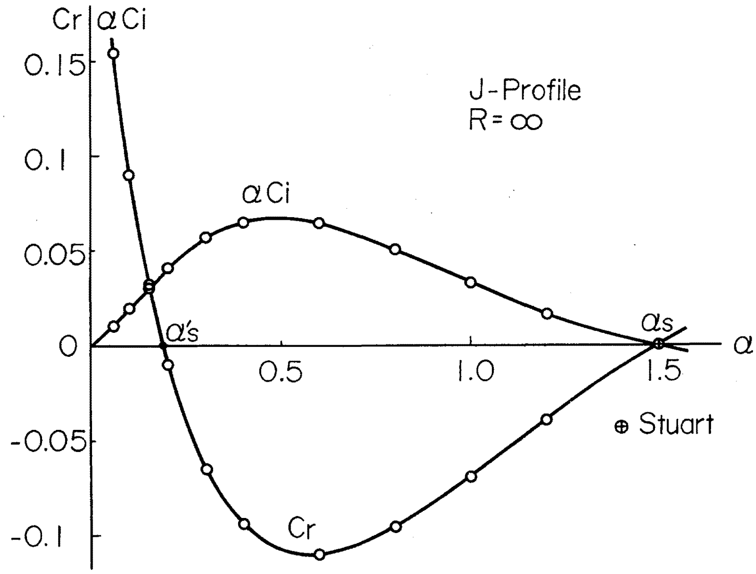

Fig. 3 Inviscid eigenvalues.

to consider a mechanism in which such a neutral disturbance exceeds eventually the amplified disturbance of the same non-propagating nature in amplitude.

The inviscid wave number α'_s of the amplified disturbance is about 0.2, whereas the dimensionless wave number of the observed stationary vortices is known to be about $\alpha_s/4$ ($\doteqdot 0.4$) in the rotating-disk case. Thus, neither of the theoretical disturbances with $c_r=0$ accounts for the spacing of the observed vortex pattern. This is not surprising since fluid viscosity is entirely neglected at present.

Fig. 4 shows the inviscid results for K- profile. As in the case of J-profile, there are *two* distinct wave numbers α_s and α'_s ($\alpha_s > \alpha'_s$) for which $c_r=0$, and the growth rate for the lower wave number α'_s is larger than the one for α_s. K'- type profiles exhibit similar characteristics except that non-propagating disturbances with α_s are all damped. Thus, the existence of amplified non-propagating disturbances with α'_s is a common feature of velocity profiles of the type considered. In this connection it is interesting to note that the neutral stationary disturbance is possible only for one particular profile J among a continuous set of component profiles.

In view of the fact that the vortex phenomenon occurs at fairly low Reynolds numbers, the above inviscid results must be taken with some reservation for the purpose of comparison with experiment. In fact, we have seen that the inviscid wave numbers of possible non-propagating disturbances differ considerably from that of observed vortices. However, it may well be said at this point that the familiar neutral stationary disturbance associated with J-profile is not worth consideration at least in the explanation of the vortex phenomenon. Any further conclusion must be made in the light of viscous eigenvalues which are presented below.

Viscous eigenvalues: Curves of neutral stability for K- and K′- profiles are partially shown in Fig. 5; the one for J-profile, which lies in between,

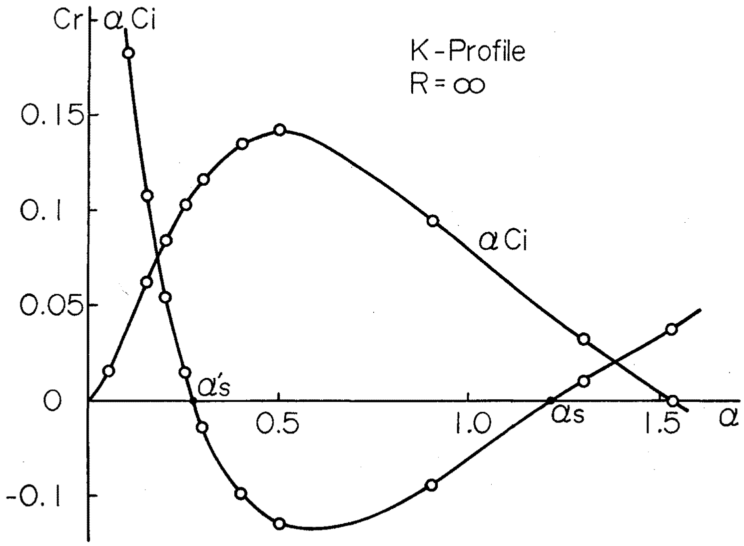

Fig. 4 Inviscid eigenvalues.

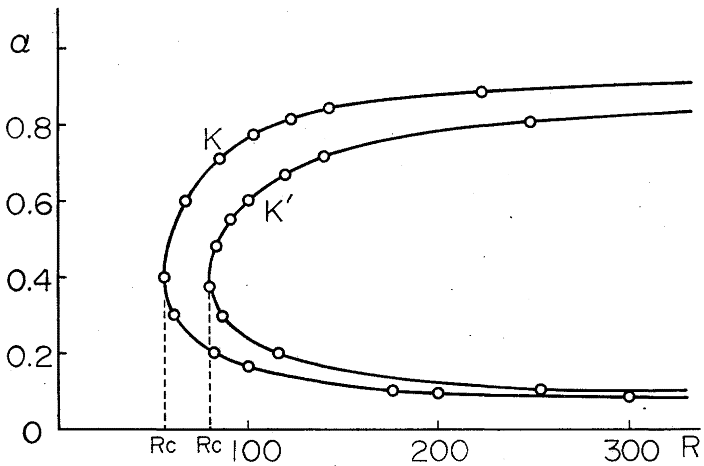

Fig. 5 Curves of neutral stability.

is omitted in the figure. Examples of cross plots at various Reynolds numbers in the unstable range are presented in Figs. 6–9. These figures together with Figs. 3 and 4 may provide a complete picture of instability characteristics of the relevant velocity profiles.

At each finite Reynolds number, as at $R=\infty$, there are *two* distinct wave numbers α_s' and α_s ($\alpha_s'<\alpha_s$) for which $c_r=0$. Note in particular that in the case of K- profile (see Figs. 7–9), the growth rate reaches a maximum nearly at $\alpha=\alpha_s'$ for each Reynolds number in the range $150<R<\infty$. This is certainly a very favourable condition for the mode of non-propagating disturbance to grow most rapidly and eventually attain the largest amplitude. Note also that the value of α_s' changes only slightly over the wide range of Reynolds numbers and is on an average in fair agreement with the wave number of the observed vortices on a rotating disk. Such a small spread of wave number may result in a slight irregularity in the spacing of the

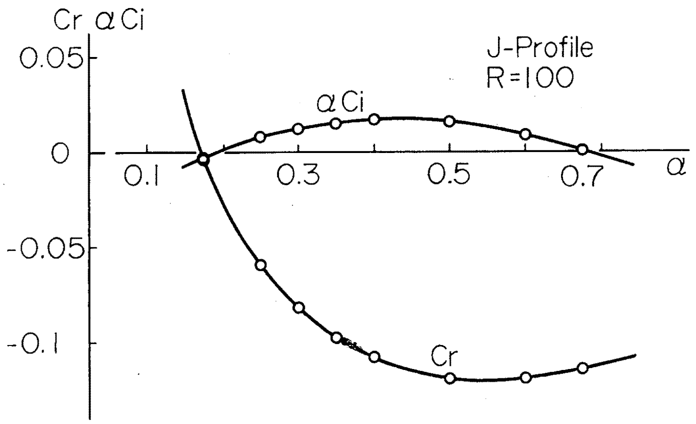

Fig. 6 Viscous eigenvalues.

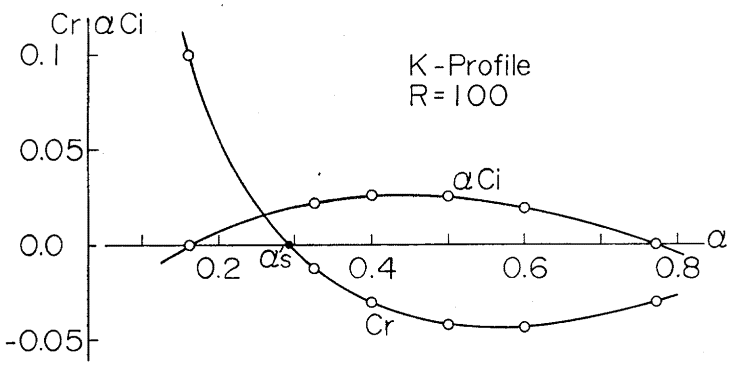

Fig. 7 Viscous eigenvalues.

possible stationary vortex pattern; this is in fact the case[1]. Thus, one mode of non-propagating disturbances generated by K- type profiles correspond well with the observed stationary vortices in all essential respects. The other possible mode of non-propagating disturbances which are less amplified and have too high wave numbers (α_s) appear to be unrelated to the vortex phenomenon of interest. In the cases of J-profile and K′- type profiles, on the other hand, none of highly amplified disturbances exhibits non-propagating nature at finite Reynolds numbers as illustrated in Fig. 6.

Fig. 10 is the c_r-R representation of the neutral stability curve for K-

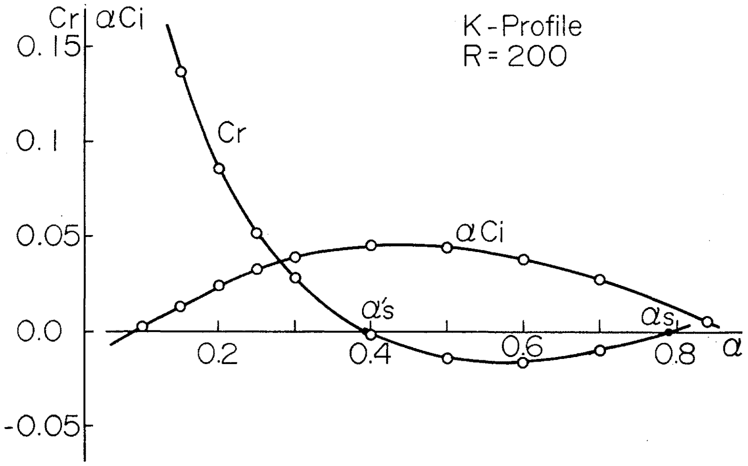

Fig. 8 Viscous eigenvalues.

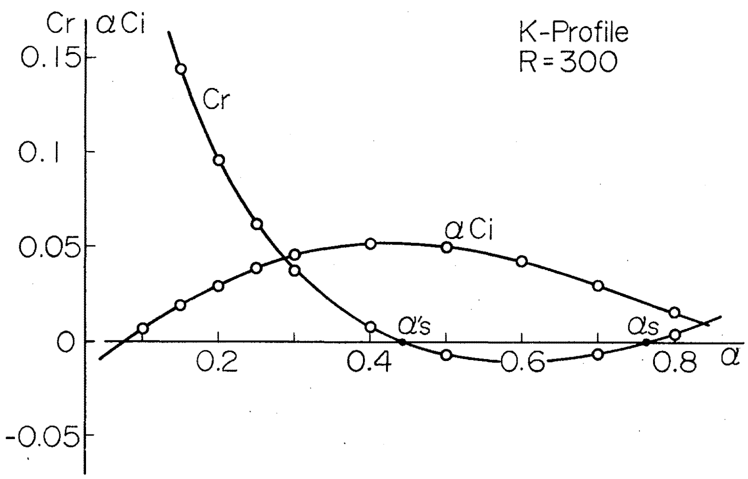

Fig. 9 Viscous eigenvalues.

profile. As is seen in Figs. 7-9, the variation of c_r with respect to α at R=const. is not monotonic; c_r takes a minimum value within the unstable wave number range. The broken line in the figure represents such minimum values of c_r at various values of R. The unstable domain in the c_r–R plane is then bounded by the c_r-minimum curve and a part of the neutral curve. The same is also true of J-profile and K′-type profiles. It is remarkable that such a non-neutral curve defines (a part of) the boundary of an unstable domain.

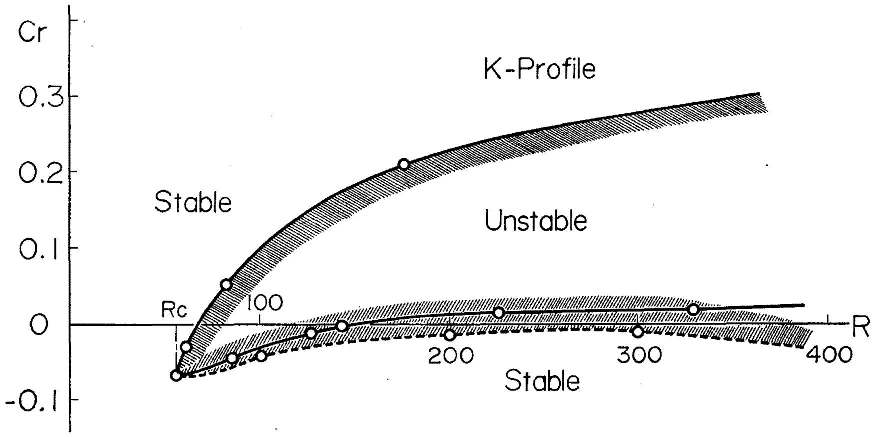

Fig. 10 c_r-R representation of the neutral stability curve for K-profile (——, neutral curves; ········, C_r-minimum curve).

5. Conclusions

Stability computations were carried out numerically for the class of velocity profiles (K-J-K′ in Fig. 1) which might be relevant to the stationary vortex phenomenon in three-dimensional boundary layers. An important new result is that each of those component profiles generates *two* distinct modes of non-propagating disturbance with different wave numbers at each finite Reynolds number as well as in the inviscid limit; the one with the lower wave number always exhibits a larger growth rate than the other with the higher wave number. Detailed properties of those non-propagating disturbances depend upon the shape of the velocity profiles (or, more precisely, upon the location of inflexion-points of the profiles) by which they are generated. In particular, the non-propagating disturbances of the lower wave number mode generated by K-profiles exhibit the largest growth rates over a wide range of Reynolds numbers, in addition their wave numbers being in fair agreement with that of the observed stationary vortices. Then we may as well conjecture that the mode of disturbances generated by K-profiles grow most rapidly and eventually form the vortex pattern as visualized by the china-clay technique. On the other hand, the non-propagating disturbances which are possible for J-profile and K′-profiles are mostly damped at finite Reynolds number of interest; moreover, their wave numbers differ considerably from that of the observed vortices. Obviously it is unreasonable to relate such disturbances to the stationary vortex phenomenon.

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