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ON THE USE OF WINDOWS FOR THE COMPUTATION OF POWER SPECTRA

by Kunio Rikiishi* and Hisashi Mitsuyasu**

The effects of data window and lag window on the estimated power spectra are investigated. It is shown that not only the theoretical basis for the use of data window are flimsy but also the use of data window, in some cases, has a demerit of giving undesirable modification to original time series. One of the best way of estimating power spectrum is to use original time series, the range of which is selected previously so as to be connected smoothly for the cyclic use of the data. It is also shown that in order to estimate reliable power spectrum autocovariance function is required to tend to zero for the maximum lag and to join smoothly to its cyclic extension. In this respect the use of the lag window, such as Hann's one, satisfies approximately these two requirements.

1. Introduction

Since the practical method for spectral analysis of random phenomena through autocovariance function (mean lagged products) was established by Blackman and Tukey[1], it has been widely used for a long time. However, the rediscovery in 1965 of the fast Fourier transform (F. F. T.) algorithm by Cooley and Tukey[2] has introduced an alternative method for spectrum estimation. The relationship between these two methods which will be referred to hereinafter as the lagged products method and the F. F. T. method respectively has been discussed recently by the present authors[3] : it has been shown that the spectral estimates by the lagged products method are identical with certain running averages of the raw spectra computed by the F. F. T. method, and that the only difference between the two methods is that of spectral window.

In the F. F. T. method, data window has been often recommended to use in addition to spectral window. Then it follows from our previous study that data window should be used also for the lagged products method. In the lagged products method, however, data window has not been used. This implies that data window is unnecessary for the F. F. T. method provided that its use is unnecessary for the lagged products method.

The main purpose of the present paper is to discuss the validity of the use

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of data window for the spectrum estimation. In this connection, the effects of spectral window, data window and lag window on the estimated power spectra are first described.

2. Spectral Window

In the lagged products method, power spectra are obtained by the Fourier transform of autocovariance function. Let $\eta(t)$, $t=0, 1, 2, \ldots\ldots, N-1$ be a finite sample from a stationary, stochastic sequence, and P_k ($k=0, 1, 2, \ldots\ldots, M_0$), P_k' ($k'=0, 1, 2, \ldots\ldots, M$) be power spectra for the cases of maximum lag number M_0 ($=N/2$) and M (an arbitrary integer; $M \leqq M_0$) respectively. Then the following equation will be derived[2].

$$P'_{k'} = \sum_{k=0}^{M_0} P_k \cdot g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right) = P_k * g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right) \tag{1}$$

where

$$g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right) = \frac{1}{2M}\left[\sin\left(\frac{k}{M_0}+\frac{k'}{M}\right)\pi M \cdot \cot\frac{1}{2}\left(\frac{k}{M_0}+\frac{k'}{M}\right)\pi \right.$$
$$\left. + \sin\left(\frac{k}{M_0}-\frac{k'}{M}\right)\pi M \cdot \cot\frac{1}{2}\left(\frac{k}{M_0}-\frac{k'}{M}\right)\pi\right]. \tag{2}$$

($*$; digital convolution)

In the ordinary way, power spectra $P'_{k'}$ are smoothed in the frequency domain with, for example, Hanning's weights (0.25, 0.50, 0.25) or Hamming's weights (0.23, 0.54, 0.23). Then the equation (1) is replaced by

$$\widetilde{P}_{k'} = \sum_{k=0}^{M} P_k \cdot g_2\left(\frac{k}{M_0}, \frac{k'}{M}\right) = P_k * g_2\left(\frac{k}{M_0}, \frac{k'}{M}\right) \tag{3}$$

or

$$\widetilde{P}_k' = \sum_{k=0}^{M_0} P_k \cdot g_3\left(\frac{k}{M_0}, \frac{k'}{M}\right) = P_k * g_3\left(\frac{k}{M_0}, \frac{k'}{M}\right) \tag{4}$$

where

$$g_2\left(\frac{k}{M_0}, \frac{k'}{M}\right) = 0.25\, g_0\left(\frac{k}{M_0}, \frac{k'-1}{M}\right)$$
$$+ 0.50\, g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right) + 0.25\, g_0\left(\frac{k}{M_0}, \frac{k'+1}{M}\right) \tag{5}$$

$$g_3\left(\frac{k}{M_0}, \frac{k'}{M}\right) = 0.23\, g_0\left(\frac{k}{M_0}, \frac{k'-1}{M}\right)$$
$$+ 0.54\, g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right) + 0.23\, g_0\left(\frac{k}{M_0}, \frac{k'+1}{M}\right) \tag{6}$$

Here, g_0, g_2 and g_3 are equivalent respectively to the spectral windows Q_0, Q_2 and Q_3 which were extensively investigated by Blackman and Tukey[1]. The spectral windows g_0, g_2 and g_3 for the case of $M_0=512$ and $M=64$ are shown in Fig. 1 together with the corresponding triangular filter which is often used in the F. F. T. method.

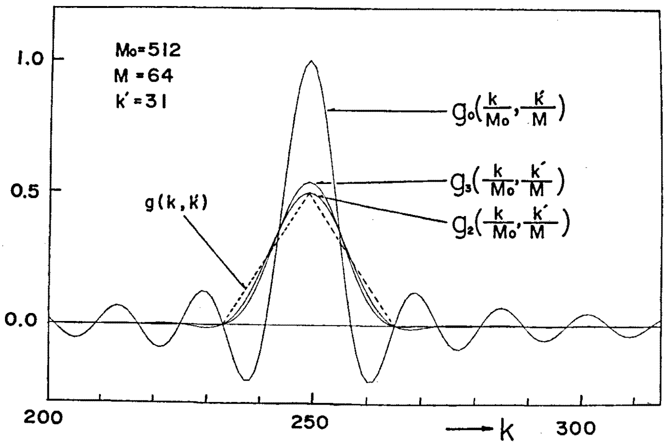
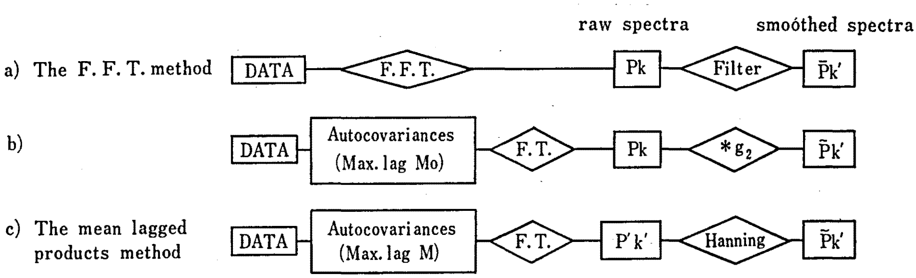


Fig. 1 Spectral windows g_0, g_2, g_2 and g.

Equations (3) and (4) mean that smoothed power spectra for the case of maximum lag number M are identical with certain running averages of raw power spectra for the case of maximum lag number M_0*. These equations are important in that they show the relationship between the lagged products method and the F. F. T. method. In order to see more clearly the relationship, the following flow charts of the practical procedure of spectrum computation are presented taking the cases of spectral window g_2 and Hanning's weights.



F. T. ; Fourier transform
F. F. T. ; fast Fourier transform

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*) In the previous study[3] by the present authors, three different ways of computing autocovariance function were examined, and it was concluded that the way in which the periodicity of time series was assumed was the the most reasonable one. In such a case, raw spectra for the case of maximum lag number M_0 are identical with periodgrams, i, e., raw spectra by the F. F. T. method.

These flow charts show that smoothed spectra by the methods (b) and (c) are identical with each other, and that raw spectra by the method (b) are equal to those by the F. F. T. method (a). Accordingly it is concluded that the only difference between the lagged products method and the F. F. T. method is that of spectral window, and that the F. F. T. method without data window is equivalent basically to the lagged products method. It follows that spectral window is not related directly to data window.

3. Data Window

Data window has been often considered to take an important role in the F. F. T. method because its use seemed to be equivalent to the operation of spectral window* (or lag window) in the lagged products method, or because its use seemed to confine leakage effect within a narrow frequency range. As discussed in the preceding section, however, data window is not related to spectral window. Hence the merit of confining leakage effect will be examined in this section by using both artificial and observed data of wind waves.

3.1. Examples of Data Window

The following four kinds of typical data window shown in Fig. 2 are considered as examples, though we can use other various data windows of arbitrary

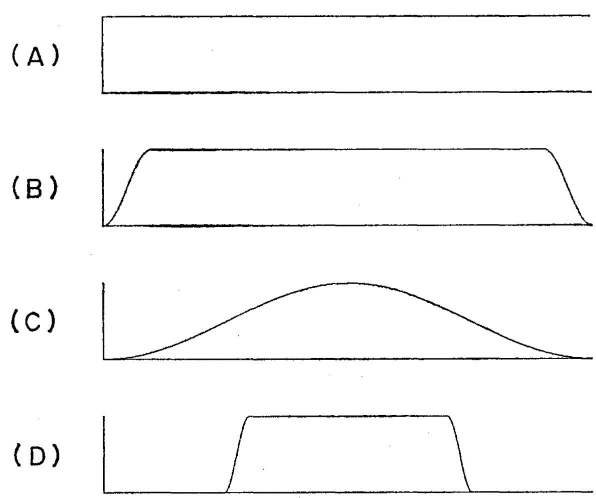

Fig. 2 Examples of data window: (A), do nothing; (B), half cosine bells; (C), cosine bell; (D), modified half cosine bells.

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*) The relationship between spectral window and data window shown by Welch holds only when spectral estimates are averaged over the ensemble of time series. In the ordinary computation of power spectra, however, such averaging is not taken.

shape in practical computation.

Data window (A) is a transparent window which is given by

$$w_t = 1 \ . \qquad (t = 0, 1, \ldots, N-1) \tag{7}$$

As this window has nothing to do with original data, it can be named "do nothing window".

Data window (B) is given by

$$w_t = \begin{cases} \frac{1}{2}\left(1 - \cos\frac{\pi t}{0.1N}\right) & (t = 0,\ 1, \ldots\ldots, 0.1N-1) \\ 1 & (t = 0.1N, \ldots\ldots, 0.9N-1) \\ \frac{1}{2}\left(1 - \cos\frac{\pi(N-t)}{0.1N}\right) & (t = 0.9N, \ldots\ldots\ldots, N-1) \end{cases} \tag{8}$$

This window is usually called as half cosine bells[8], because 10 % of data are reduced at both ends proportionally to cosine functions. Here $0.1N$ and $0.9N$, real numbers generally, are substituted by integers of their neighborhood.

Data window (C) is given by

$$w_t = \frac{1}{2}\left(1 - \cos\frac{2\pi}{N}t\right) \qquad (t = 0,\ 1, \ldots\ldots\ldots, N-1) \tag{9}$$

This data window is usually called as cosine bell[8], because a whole data is reduced by this window proprotionally to a cosine function.

Data window (D) is composed of a half cosine bells at the middle part and zeros at the both ends. In the F. F. T. method many of the available computer programs are written for $N=2^n$, where n is a positive integer. In some cases, however, the number of data points does not satisfy this condition, and it has been recommended to extend the data points to that number by adding zeros to the both ends of the data. Data window (D) corresponds to such a procedure.

3. 2. Leakage Error

Before going into the discussion of the effect of data window, general features of leakage error is first described.

As the discrete Fourier transform is based on the periodicity of time series, the exact frequency of input data can be distinguished in a case where the integral multiple of the wave period is equal to the length of the record. But in other cases, power spectra leak out in the neighborhood of the genuine frequency because the Fourier coefficients are calculated only at frequencies such that $f_k = k/N\Delta t$, where k's are positive integers and Δt is a sampling interval. This is named leakage error. The lagged products method also cannot avoid the leakage troubles.

As examples, the following five artificial time series are considered.

$$\eta(t) = \sqrt{2} \sin\left(\frac{2\pi}{N} kt\right) \qquad (t = 0,\ 1, \ldots\ldots,\ N-1) \tag{10}$$

where $k = 31.1$, 31.3, 31.5, 31.7, 31.9 and $N=256$.

The leakage errors in estimated spectra for these series are shown in Fig. 3. As is evident from Eq. (10), all of the input time series have a single line spectrum with power of unity. For the cases $k=31.1$ and $k=31.9$, the leakage error falls off rapidly because the genuine frequency is very close to f_{31} or f_{32}, whereas for the case of $k=31.5$, it spreads widely to higher and lower frequency because the input frequency is exactly in the middle between f_{31} and f_{32}. Such features of leakage error are expected not only for the values of k such that $31<k<32$ but also for arbitrary values.

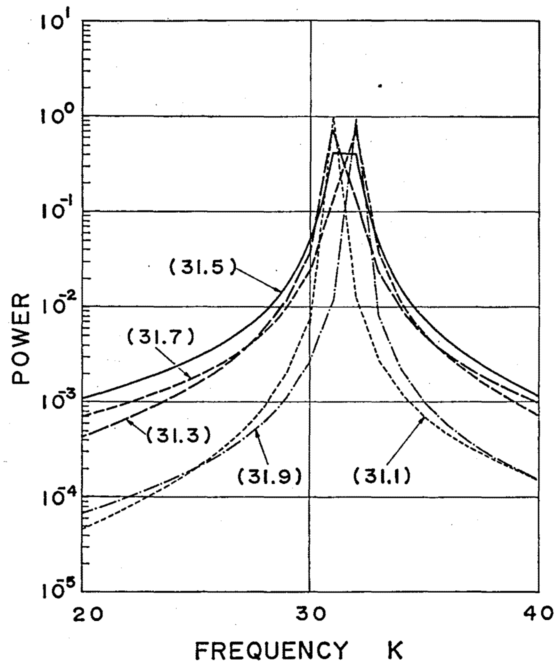

Fig. 3. Leakage error for artificial data ($N=256$). The artificial time series are given by equation (10) for $k=31.1$, 31.3, 31.5, 31.7, 31.9.

3. 3. Effect of Data Window

In order to see the effect of data window on the leakage error, we consider an artificial time series given by

$$\eta(t) = \sqrt{2} \sin\left(\frac{2\pi}{N} 31.5t\right) \qquad (t=0,\ 1, \ldots\ldots,\ N-1) \tag{11}$$

that has given the most broad leakage effect in Fig. 3. Figure 4 shows the

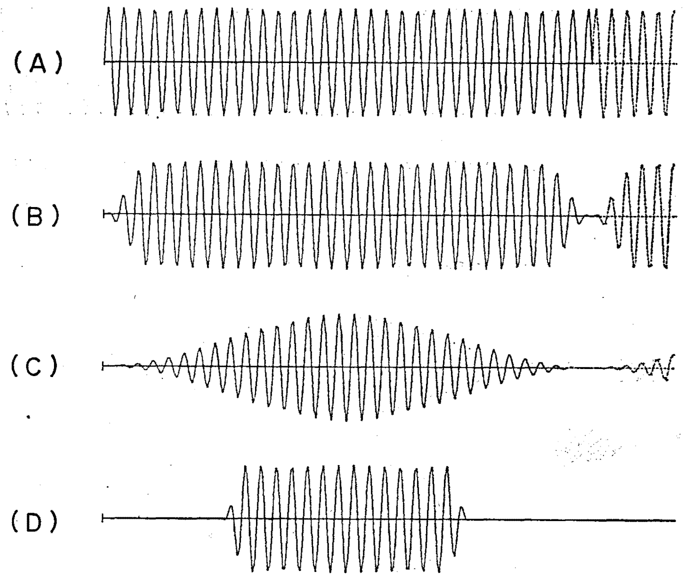

Fig. 4 Artificial wave data modified by the four data windows (A), (B), (C), and (D). The artificial time series is given by equation (11). The dotted lines represent periodically extended time series (N=256).

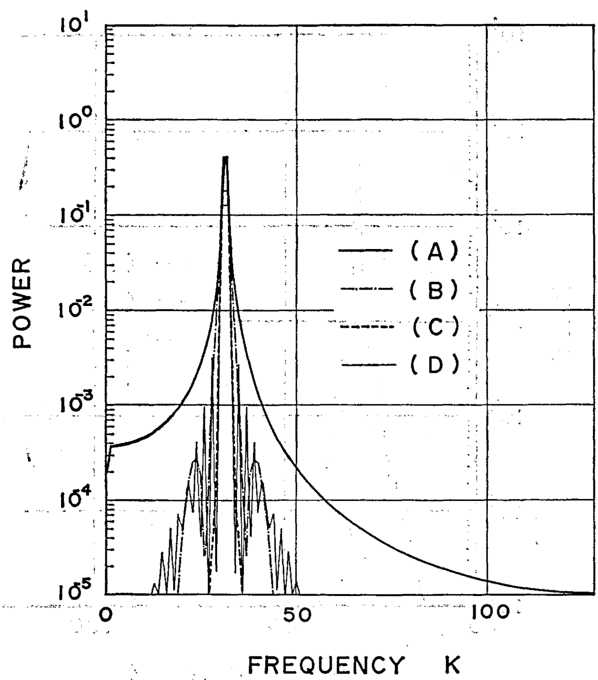

Fig. 5 Power spectra for the artificial data given by equation (11).

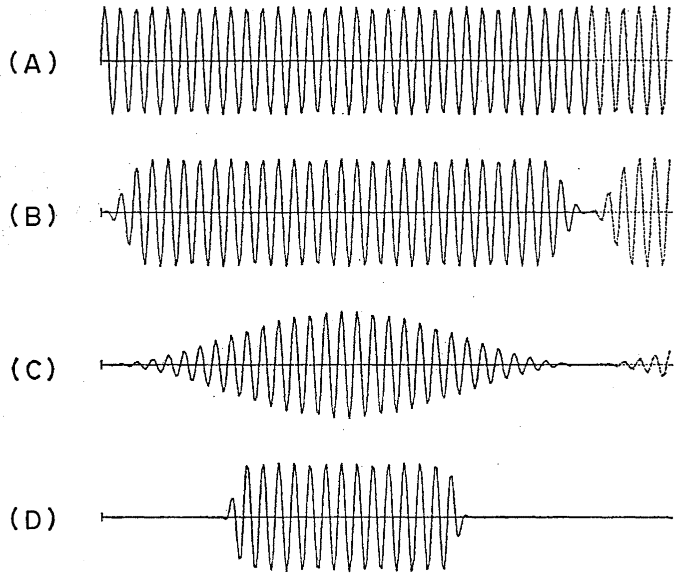

Fig. 6 Artificial wave data modified by the four data windows (A), (B), (C) and (D). The artificial time series is given by equation (12). The dotted lines represent periodically extended time series (N=256).

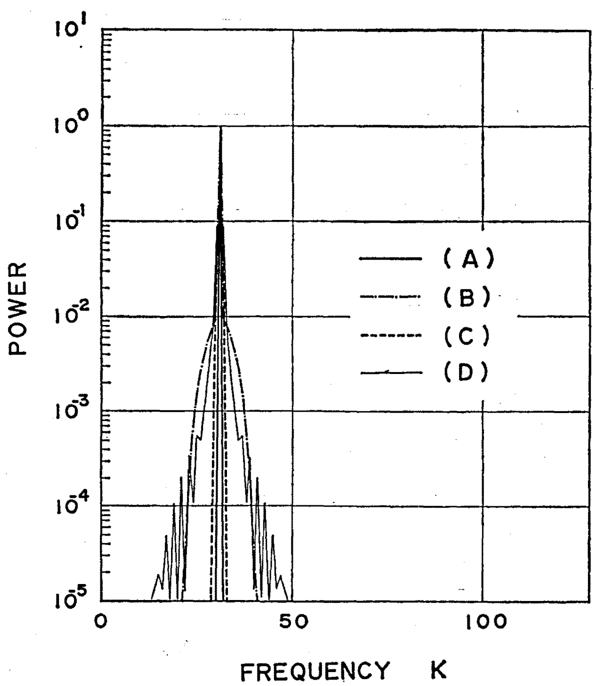

Fig. 7 Power spectra for the artificial data given by equation (12).

modified wave data which are obtained by multiplying the original wave data (Eq. 11) by the above four data windows. The broken lines represent artificial time series obtained by extending each modified wave data periodically for $t=N$, $N+1$, ... It is apparent that the original wave data is modified to a higher extent in order of (B), (C) and (D). At the same time, an unnatural joint of the wave data (A) at $t=N$ is improved for the wave data (B), (C) and (D).

On inspecting the corresponding power spectra in Fig. 5, leakage error is seen over the wide frequency range for (A), but it is confined within a fairly narrow frequency range around the genuine frequency for (C). The leakage error is confined also for (B) and (D), though the fall is not monotonous with frequency: two peaks for (B) and complicated peaks for (D) are seen.

As another example, we consider the following artificial time series ($N=256$)

$$\eta(t)=\sqrt{2}\sin\left(\frac{2\pi}{N}31.0t\right) \qquad (t=0,\ 1, \ldots\ldots, N-1) \qquad (12)$$

that gives basically no leakage error, and apply the four data windows previously described. As shown in Fig. 6, the original wave data is modified to a higher extent in order of (B), (C) and (D) as well as the former case. In this case individual time series are connected smoothly with their cyclic extensions.

Power spectra for these time series are shown in Fig. 7, and it is confirmed that there is naturally no leakage error in the case of (A). However, not a small amount of leakage error can be seen in the cases (B), (C) and (D), though it falls rapidly for high and low frequency sides in the case of (C). Evidently these leakage errors are caused by the use of data window, and then the argument that data window provides reasonable bounds on the effect of leakage proves to be not always true. For the records of measured wind wave, one cannot confirm generally whether data window confines or spreads the leakage effect, and then it is not necessarily reasonable to assume that data window is generally effective for confining the leakage effect within a narrow frequency range.

Modification of wave data by data window yields the loss of variance (or energy) of time series. The reduced energy is usually devided by

$$U=\frac{1}{N}\sum_{t=0}^{N-1} w_t^2 \qquad (13)$$

to correct the energy loss. According to the recent study by Briscoe (1972)[9], however, the ratio of the variance of the original wave record to that of the wave record modified by data window (C) has not given a constant value determined theoretically, but it has scattered around the theoretical correction factor. Here it should be noted that power spectra shown in Fig. 5 and Fig. 7 have not been corrected by such correction factors.

Finally the wave data of observed wind waves are shown in Fig. 8 together with the wave data modified by the data windows previously described ($N=1024$).

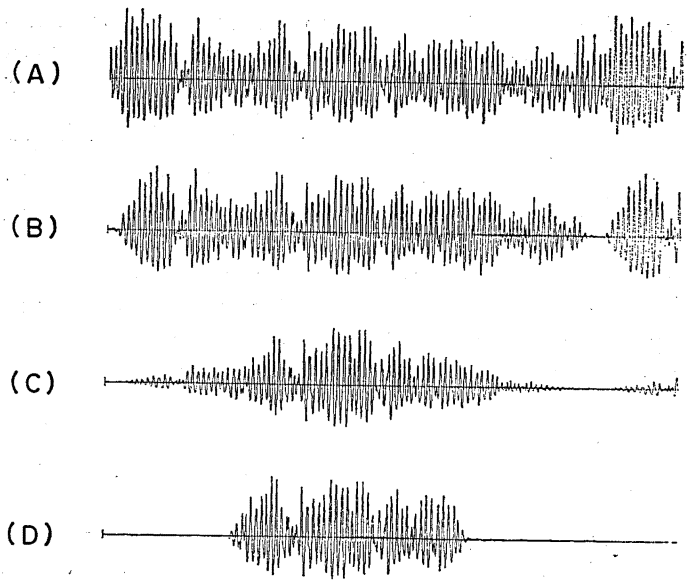

Fig. 8 Wind wave data modified by the four data windows (A), (B), (C) and (D). The dotted lines represent periodically extended time series (N=1024).

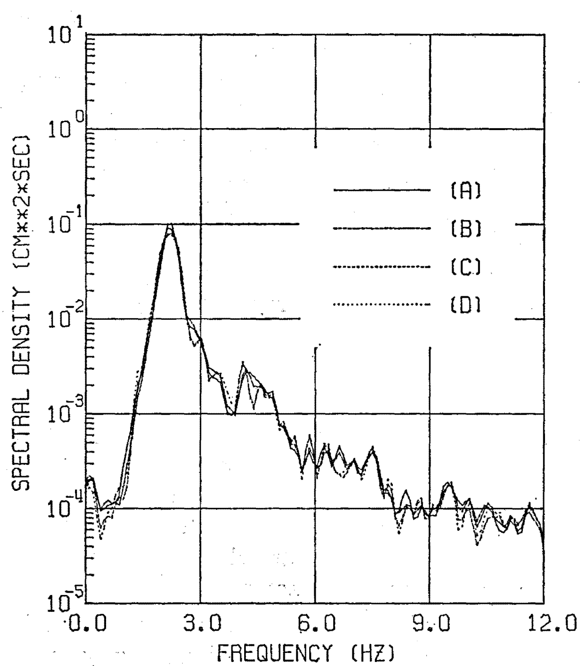

Fig. 9 Spectral density for the wind wave data.

Figure 9 shows the corresponding power spectra smoothed by a triangular filter (c) (II=4, JJ=8)*. Spectral estimates have been corrected for the present cases by the corresponding correction factors determined from the ratio of the variance of the original wave record to that of the wave record modified by the data windows. Minor differences of spectral estimates among the four cases are seen over the wide frequency range. It is noteworthy, however, that the differences are not so remarkable as might be expected from the large modification of waveform by the data windows.

4. Lag Window

If the first $(M+1)$ terms of autocovariance function $R(\tau)$ $(\tau=0, 1, \ldots\ldots, M)$ are obtained from a sample of finite and discrete time series, then power spectra P_k $(k=0, 1, \ldots\ldots, M)$ can be estimated by

$$P_k=\frac{1}{M}\sum_{\tau=-M}^{M-1} R(\tau)\cos\frac{\pi}{M}k\tau \qquad (k=1, 2, \ldots\ldots, M-1)^{**} \tag{14}$$

assuming $R(-\tau)=R(\tau)$.

Accordingly $R(\tau)$ are expressed by the inverse Fourier transform of P_k:

$$R(\tau)=\sum_{k=0}^{M} P_k \cos\frac{\pi}{M}k\tau\ . \tag{15}$$

Although $R(\tau)$ is originally defined only for $|\tau|=0, 1, \ldots\ldots, M$, it is assumed to be a periodic function with a period of $2M$ by equation (15) once we take the discrete Fourier transform of $R(\tau)$. In other words, estimated spectra P_k correspond to a periodic function $R(\tau)$. Accordingly $R(\tau)$ is expected to show the maximum value at every $2M$ (see Fig. 10). In the case of random process, however , autocovariance function obtained from measured data tends to zero with increasing τ, and we may assume $R(\tau)$ tends to zero for sufficiently large τ. These facts appear to be inconsistent with each other. As is discussed below, however, there is no essential contradiction between them.

We consider an extended autocovariance function $R'(\tau)$ obtained by adding $(n-1)M$ zeros at both bends of $R(\tau)$:

$$R'(\tau)=\begin{cases} 0 & (\tau=-nM,\ -nM+1, \ldots\ldots\ldots\ldots, -M-1) \\ R(\tau) & (\tau=-M,\ -M+1, \ldots\ldots\ldots\ldots, M-1) \\ 0 & (\tau=M,\ M+1, \ldots\ldots\ldots\ldots, nM-1) \end{cases} \tag{16}$$

Power spectra corresponding to $R'(\tau)$ are given by

$$P'_{k'}=\frac{1}{nM}\sum_{\tau=-nM}^{nM-1} R'(\tau)\cos\frac{\pi}{nM}k'\tau \tag{17}$$

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* See reference (3).

** The cases of $k=0$ and $k=M$ are omitted for simplicity.

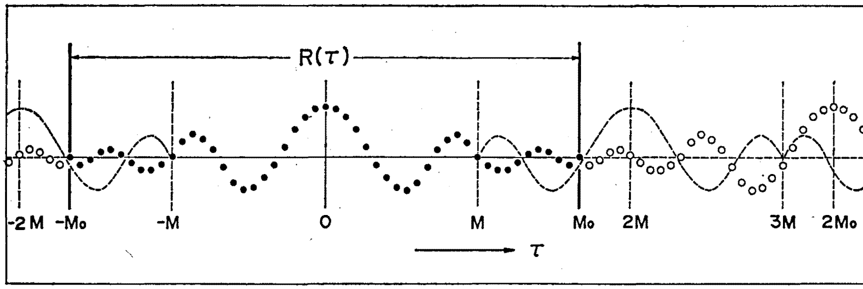

Fig. 10 Autocovariance function extended by assuming its periodicity.

$$=\frac{1}{nM}\sum_{\tau=-M}^{M-1} R(\tau)\cos\frac{\pi}{nM}k'\tau$$

$$=\sum_{k=0}^{M} P_k\cdot g'\left(\frac{k}{M},\frac{k'}{nM}\right)=P_k * g'\left(\frac{k}{M},\frac{k'}{nM}\right) \tag{18}$$

where

$$g'\left(\frac{k}{M},\frac{k'}{nM}\right)=\frac{1}{2nM}\left[\sin\left(\frac{k}{M}+\frac{k'}{nM}\right)\pi M\cdot\cot\frac{1}{2}\left(\frac{k}{M}+\frac{k'}{nM}\right)\pi\right.$$

$$\left.+\sin\left(\frac{k}{M}-\frac{k'}{nM}\right)\pi M\cdot\cot\frac{1}{2}\left(\frac{k}{M}-\frac{k'}{nM}\right)\pi\right] \tag{19}$$

Equation (18) has the same form as Eq. (1) which has shown the relationship between the lagged products method and the F. F. T. method. Substituting nk for k' in Eq. (17), the relationship between P_k and P'_{nk}* is given by

$$P'_{nk}=\frac{1}{n}P_k\ . \tag{20}$$

As the elementary frequency bandwidth for the two cases are given respectively by

$$\Delta f=\frac{1}{2M\Delta t} \qquad \text{(for maximum lag numbe } M) \tag{21}$$

$$\Delta f'=\frac{1}{2nM\Delta t}=\frac{1}{n}\Delta f \quad \text{(for maximum lag number } nM) \tag{22}$$

it follows that. ϕ'_{nk} is equal to ϕ_k (ϕ denotes spectral density, namely line spectrum devided by elementary frequency bandwidth). This means that both the original autocovariances and the extended autocovariances yield the identical spectral density for the frequency such that $f=k/2M\Delta t$.

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*) P'_{nk} corresponds to P_k because line spectra $P'_{k'}$ are defined n times as closely as P_k.

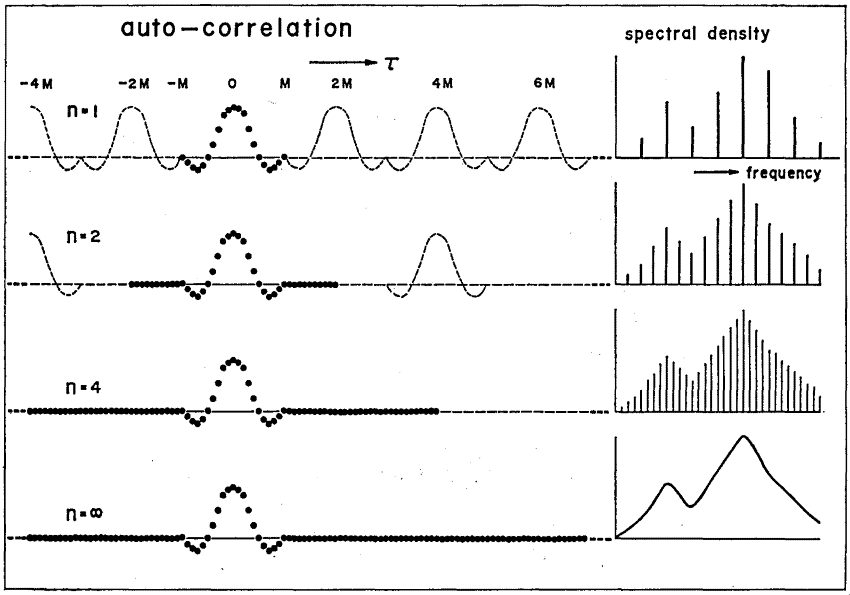

Fig. 11 Schematic diagram of the the relationship between autocorrelation function (normalized autocovariance function) and spectral density.

The above consideration is naturally valid for arbitrary number of n. Schematic diagram of the relationship between autocovariance function and corresponding spectral density is shown in Fig. 11 for the case of $n=1, 2, 4, \ldots\ldots \infty$. Considering the case of $n=\infty$, autocovariance function is extended as follows:

$$R' = \begin{cases} R(\tau) & (\tau = -M,\ -M+1, \ldots\ldots, 0, \ldots, M-1) \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Evidently the spectral density corresponding to $R'(\tau)$ is identical with that corresponding to $R(\tau)$ for the arbitrary frequency such that $f=k/2M\Delta t$. In other words, $R(\tau)$ ($|\tau| = 0, 1, \ldots\ldots, M$) is equivalent to $R'(\tau)$ ($|\tau| = 0,1, \ldots\ldots \infty$), though the former gives line spectra and the latter continuous ones. Here it should be noted that the continuous spectra are not identical with the true continuous spectra estimated from infinite record. They are seeming continuous spectra due to artificial addition of zeros, and cannot be different essentially from line spectra.

The correspondence of $R(\tau)$ to $R'(\tau)$ discussed above leads to the conclusion that two conditions are required of autocovariance function for estimating reliable power spectra. Firstly the autocovariances are required to join smoothly to additional autocovariances obtained by Eq. (15) for $\tau=M+1,\ M+2, \ldots\ldots\ldots$, because true autocovariances obtained from natural phenomena may con-

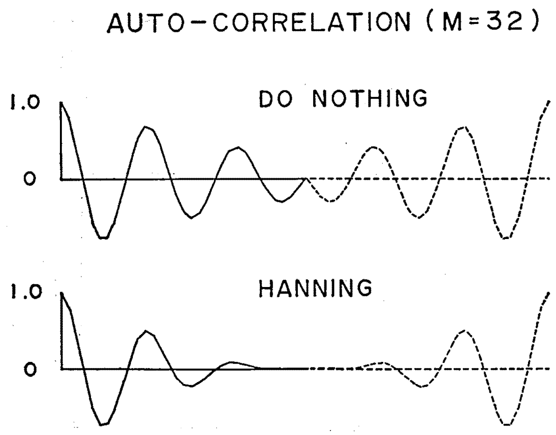

Fig. 12 Autocorrelation function for the wind wave data (N=2048, M=32). Broken lines show periodically extended autocorrelations.

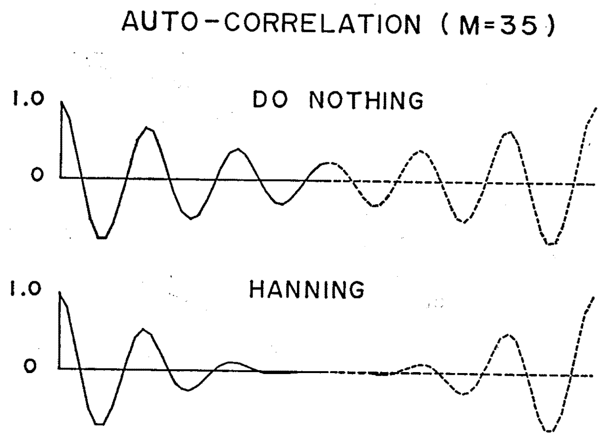

Fig. 13 Autocorrelation function for the wind wave data (N=2048, M=35). Broken lines show periodically extended autocorrelations.

tinue always naturally. Arbitrary selection of maximum lag number does not satisfy generally this condition. Secondly the autocovariance function is required to tend to zero at $\tau=M$, because true autocovariances of random process are assumed to be zero for large lag numbers such that $|\tau|>M$*. The favorable effects of the two conditions can be well demonstrated by numerical experiments using a sample from wind wave records (N=2048).

*) Note that this condition is not reasonable when the wave records are regular in time and can be described approximately by single or more line spectra.

The normalized autocovariance function (autocorrelation function) of the wind wave data for the case of maximum lag number $M=32$ is shown in the upper portion of Fig. 12. In this case, we can see unnatural continuation of the function to its additional extension. Spectral density obtained from this autocovariances, shown in Fig. 14 with a solid line, indicates extremely unnatural estimates with complicated form. On the other hand, the normalized autocovariance function for the case of maximum lag number $M=35$, shown in the upper portion of Fig. 13, indicates improved continuation, and the corresponding spectral densiy is reasonable as shown in Fig. 15 with a solid line. However, another requirement of autocovariance function, tending to zero at the maximum lag, is not yet satisfied by this autocovariances.

Here the use of lag window eliminate the difficulty. We adopt Hann's lag window, as an example, given by

$$d_i = \frac{1}{2}\left(1 + \cos\frac{\pi}{M}i\right) \tag{24}$$

for practical computation. The operation of Hann's lag window is entirely equivalent to smoothing raw spectra with Hanning's weights (0.25, 0.50, 0.25).

The original autocovariances for the case of maximum lag number $M=32$ are multiplied by Hann's lag window yielding modified autocovariances which joins naturally to its extension (see Fig. 12; lower portion). As shown in Fig. 14 with a broken line, the corresponding spectral density indicates reasonable estimates. The autocovariances modified by Hann's lag window for the case of maximum lag number $M=35$ tend to zero gradually (see Fig. 13; lower portion), and the corresponding spectral density also indicates reasonable estimates as shown in Fig. 15 with a broken line. The approximate coincidence of the power spectra obtained from the two autocovariances of different maximum lag number, each of which is modified by Hann's lag window, implies that the use of lag window satisfies the two requirements of autocovariance function at a time.

5. Discussion

For the discussion on the validity of the use of data window, the same consideration as has been shown in the preceding section for autocovariance function and lag window is applied to time series and data window.

We consider first the extended time series with $(n-1)\,N$ zeros at the end of original data. The same calculation as equations (16)$\sim$(22) shows that the Fourier coefficients are reduced by a factor of n in comparison with the results from the original data, or that the periodgrams are reduced by a factor of n^2. As the elementary frequency bandwidth is also reduced by a factor of n, the spectral density is finally reduced by a factor of n. Consequently, if we adopt n as a correction factor for correcting the energy loss due to the addition of zeros, the extended time series with $(n-1)\,N$ zeros is equivalent to the original time

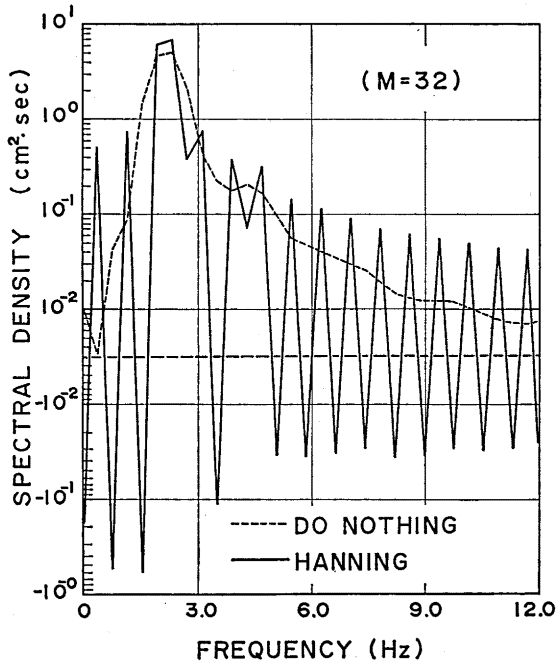

Fig. 14 Spectral density for the wind wave data estimated by the lagged products method (M=32).

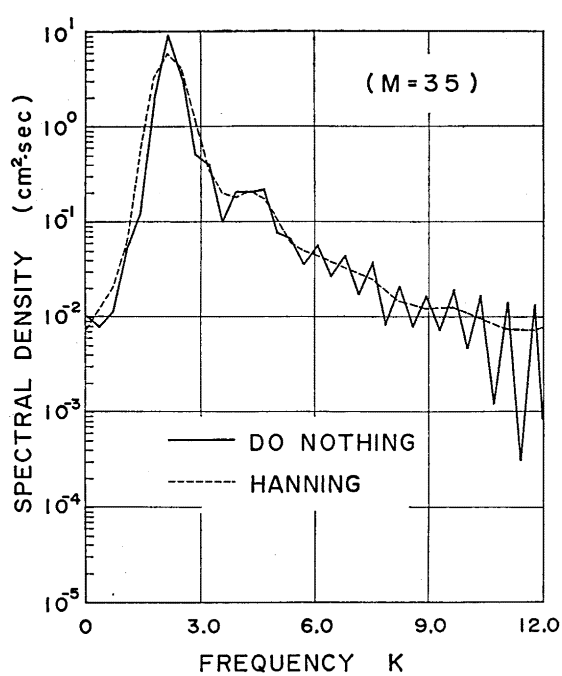

Fig 15. Spectral density for the wind wave data estimated by the lagged products method (M=35).

series. Therefore, the procedure of adding zeros to original data seems to have some grounds even if the number of additional zeros is not equal to $(n-1)\,N$, where n is arbitrary integer. Thus data window (D) does not necessarily modify the original power spectra so significantly as might be expected from modified waveform. Indeed, power spectra estimated through data window (D) differ little from those estimated through other kinds of data window (see Fig. 8 and Fig. 9).

Consider secondly the condition that original data is expected to join smoothly to its cyclic extension. As random time series, from which the finite data is sampled, will continue always smoothly, this condition is quite reasonable. In this respect data window may have some merit as well as lag window.

In considering finally the condition of "tending to zero at the both ends," however, significant difference is found between the effect of lag window and that of data window. Although the condition is quite reasonable for autocovariance function, it does not seem to have any statistical grounds for random time series. Data window changes original stationary time series to less stationary one, because the modified waveform shows artificial beats when we extend it periodically. Moreover, data window seems to distort the statistical properties of original time series.

Modification of waveform through data window yields inevitably modification of power spectra, and then we are confronted with a question whether the modified power spectra are more reasonable than the original power spectra. Considering the effect of confining leakage effect within a narrow freqnency range, data window (C) appears most adequate from Fig. 5 and Fig. 7. This interpretation, however, cannot be applied to actual data of wind waves, because the leakage errors caused by many spectral components cannot be summed linearly* and the total leakage error cannot be evaluated in reality. In other words, the discussion on the leakage effect is not conclusive for examing the validity of the use of data window.

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*) Taking the case of data window (C) as example, the relationship between the Fourier coefficients obtained from original time series $(a_k,\, b_k)$ and those obtained from modified time series by data window (C) $(A_k,\, B_k)$ can be expressed as follows:

$$A_k = -\frac{1}{4}a_{k-1} + \frac{1}{2}a_k - \frac{1}{4}a_{k+1}$$

$$B_k = -\frac{1}{4}b_{k-1} + \frac{1}{2}b_k - \frac{1}{4}b_{k+1}\ .$$

Hence power spectra for the modified time series is given by

$$\frac{1}{2}(A_k^2+B_k^2) = \frac{1}{32}\left\{(a^2_{k-1}+b^2_{k-1}) + 4(a_k^2+b_k^2) + (a^2_{k+1}+b^2_{k+1})\right\}$$

$$+\frac{1}{32}\left\{2(a_{k-1}a_{k+1}+b_{k-1}b_{k+1}) - 4(a_{k-1}a_k+b_{k-1}b_k) - 4(a_k a_{k+1}+b_k b_{k+1})\right\}$$

The second term indicates nonlinear coupling of the different frequency components induced by the data window.

Generally, the time history of wind waves are considered to be stationary in some short time and to contains no discontinuity in it. Therefore, even for a sample, the wave data of finite length, it may be desirable that the data is nearly stationary and join smoothly to its cyclic extension. Indeed data window has some merit for the latter condition but at the same time it has a demerit too for the former. In order to satisfy the above two requirements, the very original time series, whose space of time is selected previously so as to join smoothly and naturally to its cyclic extension, should be used. Such a sample may be most desirable for estimating reliable power spectra.

6. Conclnding Remarks

The present investigation on the use of windows may be summarized briefly as follows:

(1) Theoretical basis for the use of data window seems missing firstly because data window is not generally related to spectral window, secondly because the operation of data window does not always confine leakage effect within a narrow frequency range, thirdly because data window changes artificially original stationary time series to less stationary one. The use of original time series, a space of time of which is selected previously so that the time series joins smoothly and naturally to its cyclic extension, may be the most desirable for analyzing random process.

(2) For the spectral analysis of random process, the following two conditions are required of autocovariance function : autocovariance function is expected to join smoothly to its cyclic extension, and tend to zero for the maximum lag. The operation of Hann's lag window, for example, modifies the autocovariances to satisfy these two requirement.

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References

1) Blackman, R. B. and Tukey, J. W., *The Measurement of Power Spectra*, Dover, New York, 1958.
2) Cooley, J. W. and Tukey, J. W., An algorithm for the machine calculation of complex Fourier Series, Mathematics of Computation, Vol. 19, 1965.
3) Rikiishi, K. and Mitsuyasu, H., On the methods of Computing Power Spectrum (In Japanese), Bull. Res. Inst. Appl. Mech., No. 39, 1973.

4) Tukey, J. W., An introduction to the calculations of Numerical Spectral Analysis, In, *Spectral analysis of Time Series,* Harris B. (Editor), Wiley, 1967.
5) Welch, P. D., The use of the fast Fourier Transform for estimation of spectra: A method based on time averaging over short, modefied periodograms, IEEE Transaction on Audio and Electroacoustics, AU-15 (2), 1967.
6) Cooley, J. W., P. Lewis, A. W. and Welch P. P., The Application of the fast Fourier Transform algorithm to the estimation of spectra and cross-spectra, J. Sound and Vibration, Vol. 12, No. 3, 1970.
7) Maling, G. C., Morrey, W. T. and Lang W. W., Digital Determination of Third-Octave and Full-Octave Spectra of Acoustical Noise, IEEE Transaction on Audio and Electroacoustics, AU-15 (2), 1970.
8) Bingham, C., Godfrey, M. D. and Tukey J. W., Modern techniques in power spectrum estimation, IEEE Transaction on Audio and Electroacoustics, AU-15 (2), 1967.
9) Briscoe, M. G., Energy loss in surface wave spectra due to data windowing, Deep-Sea Research, Vol. 19, 1972.

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