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YAMADA, Hikoji
Research Institute for Applied Mechanics, Kyushu University : Formerly Professor

OKABE, Jun-ichi
Research Institute for Applied Mechanics, Kyushu University : Professor

TORII, Hirakazu
Chubu Institute of Technology : Lecturer

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SOURCE DISTRIBUTION AND CONFORMAL TRANSFORMATION

A contribution to the slender body theory in ship hydrodynamics

By Hikoji YAMADA,* Jun-ichi OKABE,** and Hirakazu TORII***

It is well-known that a solution of Neumann's problem with respect to flowfield prevailing outside an arbitrary closed curve may be yielded by distributing sources along that contour. In this paper, a method is presented of constructing in practice a solution with the aid of a transformation through which the domain exterior to the closed curve given on the physical plane is mapped conformally onto the domain exterior to the unit circle on the image plane. During this procedure, special regard must be paid to the many-valuedness of the transformation function. Image sources would necessarily give rise to image flowfield, and the essential point of our method lies in expressing explicitly the fact that the summation of contributions from all these image sources should satisfy the boundary conditions imposed on the image plane. The technique and the formulas thus acquired are applied to the slender body theory in ship hydrodynamics, and it is attained not only to improve the accuracy of the approximation method but also to understand a thin ship and a flat ship as particular idealizations in the slender body theory.

1. Introduction

Legitimacy of a mathematical procedure in which a ship is to be replaced by sources (and sinks) distributed on its surface, in order to produce the flowfield of an inviscid fluid around the ship form, can be proved by making use of Green's formula in the theory of potential functions. Numerical works involved in dealing with the source distributions are quite laborious, however, and for the procedure of computations a number of variations were devised according to the degree of accuracy required. As a means powerful but still

* Formerly Professor, Research Institute for Applied Mechanics, Kyushu University, Fukuoka, Japan, and now Professor, Chubu Institute of Technology, Nagoya, Japan.
** Professor, Research Institute for Applied Mechanics.
*** Lecturer, Chubu Institute of Technology.

comparatively simple among them, we may mention the slender body theory, invented originally for theoretical treatments in aerodynamics of aircrafts[1] and applied later for ship hydrodynamics.[2-5] The vital point of this method lies in solving a *plane* potential problem of determining a linear density of source distribution along a contour in such a way that a given distribution of the normal velocity may be realized eventually at every point on an arbitrary sectional curve (a plane closed curve) of a ship (or speaking strictly, a double ship).

Since this source distribution is one of the forms of the solution of Neumann's problem concerning a plane potential, its properties are well-known. However, carrying out numerical computations in practice for a given contour is sometimes quite laborious, and for the purpose of determining the source distribution a technique is used of mapping the original contour onto the unit circle by means of a conformal transformation. This is underlain by the fact that the appropriate source distribution on the circumference of a circle may be worked out without difficulty.[2,4] The present paper is concerned with this computation procedure, and its purpose is to improve the accuracy of the approximate solution adopted up to the present by removing its shortcomings in particular which ceases to be negligible when a sectional contour of a double ship deviates largely from a circle.

2. Theoretical Considerations

Let a given closed circuit C on the physical plane $z(=x+iy)$ be described as $z=z(s)$, where s denotes the arclength measured along C from an arbitrary origin situated on that curve, and is nothing more than the coordinate for a point lying on it. Let $n(s)$ give the distribution along C of the normal velocity of an incompressible fluid flowing outward across C. The flowfield prevailing exterior to C is determined by the distribution of the normal velocity, and let us write the complex potential of the field as $W=W(z)$. Letting further $m(s)$ represent the linear density of sources, which should be distributed on C in such a way that the same flowfield may be established outside, then obviously we have

$$W(z) = \frac{1}{2\pi}\oint m(s)\ln(z-z(s))\mathrm{d}s, \tag{2.1}$$

where $m(s)$, the strength of the source distribution, is defined in terms of the efflux of the fluid per unit time. How to evaluate $m(s)$ out of a given $n(s)$; this is the central part of the problem we are now going to deal with.

As stated before, with a view to employing a conformal transformation in order to solve the present problem, let us introduce so-called Bieberbach's transformation function, F say, such that

$$z = F(\zeta) = p\left(\zeta+\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\cdots\right), \tag{2.2}$$

i.e. a function which maps with one-to-one correspondence the domain exterior to C on the z plane onto that exterior to the unit circle on the ζ plane and the point at infinity of one plane onto that of the other. The correspondence through the medium of this function between the contour C and the unit circle $\Gamma(\zeta=\exp(i\sigma))$ may be expressed in terms of a relationship between s and σ, i.e. $s=s(\sigma)$, or inversely $\sigma=\sigma(s)$, where σ denotes the argument of a point lying upon Γ. One or the other of the last two equations makes it possible for us to put

$$z(s) = x(s)+\mathrm{i}y(s) = F(\exp(\mathrm{i}\sigma)). \tag{2.3}$$

On transformation of quantities appearing in equation (2.1) into those depending upon ζ, $W(z)$ may be defined also on this plane. Writing it as $W(\zeta)$, we can evaluate it in the form

$$W(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} m(s)\ln\{F(\zeta)-F(\exp(\mathrm{i}\sigma))\}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\,\mathrm{d}\sigma. \tag{2.4}$$

Let us take up first the function $F(\zeta)-F(\exp(\mathrm{i}\sigma))$ contained in the integrand on the right-hand side. Since on the z plane this reduces to $z-z(s)$, this is found not to vanish at a point z lying outside C, namely at a point exterior to the unit circle Γ. It is clear, therefore, that this function does vanish only at points lying on or interior to Γ. Writing down its form in full, we should have

$$F(\zeta)-F(\exp(\mathrm{i}\sigma)) = p(\zeta-\zeta_0)\left(1-\frac{\zeta_1}{\zeta}\right)\left(1-\frac{\zeta_2}{\zeta}\right)\cdots. \tag{2.5}$$

The zeros of the function $F(\zeta)-F(\exp(\mathrm{i}\sigma))$ are $\zeta_0(=\exp(\mathrm{i}\sigma))$, $\zeta_1(\sigma)$, $\zeta_2(\sigma),\cdots$, and they satisfy the conditions $|\zeta_i|<1$ $(i=1, 2, \cdots)$. Introducing equation (2.5) into (2.4), we have

$$\begin{aligned} W(\zeta) &= \text{const.}+\frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\ln(\zeta-\zeta_0(\sigma))\mathrm{d}\sigma \\ &\quad+\sum_{i=1}\frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\ln\left(1-\frac{\zeta_i(\sigma)}{\zeta}\right)\mathrm{d}\sigma, \end{aligned} \tag{2.6}$$

where
$$\mu(\sigma) = m(s(\sigma))\frac{\mathrm{d}s}{\mathrm{d}\sigma} = m(s(\sigma))\left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\zeta=\exp(\mathrm{i}\sigma)}; \tag{2.7}$$

thus out of an unknown $m(s)$, another one defined as $\mu(\sigma)$ has been derived.

The expression (2.6) clarifies the many-valuedness of the transformation function (2.2). Namely, in contrast with a single contour C $(z=z(s))$ lying on the z plane, a number of contours $\zeta=\zeta_i(\sigma)$ $(i=0, 1, 2, \cdots)$ are given rise to on the ζ plane, along each of which sources are distributed with the density $\mu(\sigma)$ aside from an isolated point source situated at the origin. The flowfield exterior to Γ results out of all these sources. Although, in some computations already published, approximation was made use of that the outer field was created simply out of the source distribution along the unit circle, accuracy of this approximation becomes deteriorated as the form of the contour C deviates from a circle, as shown in the next section.

If the logarithmic functions are expanded, equation (2.6) is found to become

$$W(\zeta) = \text{const.} + A_0 \ln \zeta - \frac{A_1}{\zeta} - \frac{A_2}{2\zeta^2} - \frac{A_3}{3\zeta^3} - \cdots, \tag{2.8}$$

where $A_0 = \overline{\mu(\sigma)}$,

and $A_j = \sum_{i=0} \overline{(\zeta_i(\sigma))^j \mu(\sigma)} \qquad (j = 1, 2, \cdots);$ (2.9)

an overbar on the right-hand side means that the average value with respect to σ over the range $(0, 2\pi)$ should be taken. Although the value of A_0 (or $\bar{\mu}$) is always real, those of A_j $(j \geqq 1)$ are generally complex.

Let us examine properties of the function $W(\zeta)$ defined by equation (2.4). This function gives the flowfield outside the unit circle, and provided the distribution of the normal component of velocity on the unit circle is given, $\nu(\sigma)$ say, this Neumann's problem can be solved without difficulty. In the present case, as is well-known, with the aid of the functional relationship in the conformal transformation, $\nu(\sigma)$ may be computed out of $n(s)$ as a known function of σ in the following way:

$$\nu(\sigma) = n(s(\sigma)) \frac{ds}{d\sigma} = n(s(\sigma)) \left| \frac{dz}{d\zeta} \right|_{\zeta = \exp(i\sigma)} . \tag{2.10}$$

Let us write down its Fourier expansion in a form

$$\nu(\sigma) = \sum_{k=0} \nu_k^{(c)} \cos(k\sigma) + \sum_{k=1} \nu_k^{(s)} \sin(k\sigma). \tag{2.11}$$

Incidentally, it should be noticed that the relationship between $\nu(\sigma)$ and $n(s)$ is the same as that between $\mu(\sigma)$ and $m(s)$, and that accordingly any linear relationship between $\mu(\sigma)$ and $\nu(\sigma)$ is to be found identical with that existing between $m(s)$ and $n(s)$ without regard to a particular transformation function $F(\zeta)$. This fact will be made use of later.

On the other hand, $\nu(\sigma)$ can be evaluated also out of equation (2.8). Namely, substituting $r\exp(i\sigma)$ for ζ in this expression and taking the real part, we can readily show that $\phi(r, \sigma)$, the velocity potential of the flowfield, may be expressed in the form

$$\phi = \text{const.} + A_0 \ln r - \frac{1}{r} (A_1^{(r)} \cos \sigma + A_1^{(i)} \sin \sigma)$$
$$- \frac{1}{2r^2} (A_2^{(r)} \cos(2\sigma) + A_2^{(i)} \sin(2\sigma)) - \cdots, \tag{2.12}$$

where the superscripts (r) and (i) denote the real and the imaginary part, respectively. From this, we have

$$\nu(\sigma) = (\partial\phi/\partial r)_{r=1} = A_0 + A_1^{(r)} \cos \sigma + A_1^{(i)} \sin \sigma$$
$$+ A_2^{(r)} \cos(2\sigma) + A_2^{(i)} \sin(2\sigma) + \cdots. \tag{2.13}$$

Since equations (2. 11) and (2. 13) should be identical to each other, by comparing coefficients of corresponding terms in both of these expressions, we obtain

$$\nu_0^{(c)} = A_0, \quad \nu_k^{(c)} = A_k^{(r)}, \quad \text{and} \quad \nu_k^{(s)} = A_k^{(i)} \quad (k = 1, 2, 3, \cdots), \tag{2. 14}$$

or we may put them in alternative forms which follow:

$$\nu_0^{(c)} = A_0 = \frac{1}{2\pi}\int_0^{2\pi} \mu(\sigma)\mathrm{d}\sigma,$$

and
$$\nu_k^{(c)} + \mathrm{i}\nu_k^{(s)} = A_k = \sum_{j=0} \frac{1}{2\pi}\int_0^{2\pi} (\zeta_j(\sigma))^k \mu(\sigma)\mathrm{d}\sigma \quad (k = 1, 2, 3, \cdots). \tag{2. 15}$$

By solving equation (2. 14) or (2. 15) with regard to $\mu(\sigma)$, the unknown, we might evaluate it in terms of $\nu(\sigma)$.

However, in order to carry out this calculation, it would be necessary to re-examine here the computation scheme for A_k's. In their definitions shown as (2. 9), $\zeta_i(\sigma)$'s are employed, which are rather difficult to evaluate. Returning to equation (2. 5),

$$\begin{aligned} F(\zeta) - F(\zeta_0) &= p(\zeta - \zeta_0)\left\{1 - \frac{a_1}{\zeta\zeta_0} - \frac{a_2}{\zeta^2\zeta_0^2}(\zeta + \zeta_0) - \frac{a_3}{\zeta^3\zeta_0^3}(\zeta^2 + \zeta\zeta_0 + \zeta_0^2) - \cdots\right\} \\ &= p\zeta\left(1 - \frac{\zeta_0}{\zeta}\right)\left\{1 - \frac{1}{\zeta}\left(\frac{a_1}{\zeta_0} + \frac{a_2}{\zeta_0^2} + \frac{a_3}{\zeta_0^3} + \frac{a_4}{\zeta_0^4} + \cdots\right)\right. \\ &\qquad \left. - \frac{1}{\zeta^2}\left(\frac{a_2}{\zeta_0} + \frac{a_3}{\zeta_0^2} + \frac{a_4}{\zeta_0^3} + \cdots\right) - \frac{1}{\zeta^3}\left(\frac{a_3}{\zeta_0} + \frac{a_4}{\zeta_0^2} + \cdots\right) - \cdots\right\}. \end{aligned}$$

By virtue of this relation, we are led to

$$\begin{aligned} \ln(F(\zeta) - F(\zeta_0)) &= \text{const.} + \ln\zeta + \ln\left(1 - \frac{\zeta_0}{\zeta}\right) \\ &\quad + \ln\left\{1 - \frac{1}{\zeta}\left(\frac{a_1}{\zeta_0} + \frac{a_2}{\zeta_0^2} + \frac{a_3}{\zeta_0^3} + \frac{a_4}{\zeta_0^4} + \cdots\right)\right. \\ &\qquad \left. - \frac{1}{\zeta^2}\left(\frac{a_2}{\zeta_0} + \frac{a_3}{\zeta_0^2} + \frac{a_4}{\zeta_0^3} + \cdots\right) - \cdots\right\} \\ &= \text{const.} + \ln\zeta - \sum_{k=1}^{\infty} \frac{1}{k}\left[\left(\frac{\zeta_0}{\zeta}\right)^k + \left\{\frac{1}{\zeta}\left(\frac{a_1}{\zeta_0} + \frac{a_2}{\zeta_0^2} + \frac{a_3}{\zeta_0^3} + \frac{a_4}{\zeta_0^4} + \cdots\right)\right.\right. \\ &\qquad \left.\left. + \frac{1}{\zeta^2}\left(\frac{a_2}{\zeta_0} + \frac{a_3}{\zeta_0^2} + \frac{a_4}{\zeta_0^3} + \cdots\right) + \cdots\right\}^k\right] \\ &= \text{const.} + \ln\zeta - \frac{B_1(\sigma)}{\zeta} - \frac{B_2(\sigma)}{2\zeta^2} - \cdots, \end{aligned} \tag{2. 16}$$

where all of $B_j(\sigma)$'s may be expressed as linear combinations of $\exp(\mathrm{i}k\sigma)$, k being an integer, positive or negative. Suffice it to notice this fact for the time being; forms $B_j(\sigma)$ will be exemplified in the next section. Provided equation (2.16) is introduced into the right-hand side of (2.4), $W(\zeta)$ may be given explicitly, and since it should be identical with (2.8), we can formulate a computation procedure for A_j:

$$A_j = \overline{B_j(\sigma)\mu(\sigma)} \qquad (j = 1, 2, 3, \cdots). \tag{2.17}$$

Finally, the Fourier series representation of $\mu(\sigma)$, i.e.

$$\mu(\sigma) = \sum_{k=0} \mu_k^{(c)} \cos(k\sigma) + \sum_{k=1} \mu_k^{(s)} \sin(k\sigma), \tag{2.18}$$

may be introduced into equation (2.17). The right-hand side of this equation yields an expression linear and homogeneous with regard to $\mu_i^{(c)}$ and $\mu_j^{(s)}$ ($i = 0, 1, 2, \cdots$, and $j = 1, 2, \cdots$), and provided A_k ($k = 1, 2, \cdots$) in equation (2.15) is expressed in a similar manner, a system of simultaneous linear equations of $\mu_k^{(c)}$ and $\mu_k^{(s)}$ can be derived. When they are solved, $\mu_k^{(c)}$, $\mu_k^{(s)}$ and accordingly, $\mu(\sigma)$ can be determined in terms of $\nu_0^{(c)}$, $\nu_k^{(c)}$, and $\nu_k^{(s)}$ ($k = 1, 2, \cdots$).

In order to arrive at the final result by inverting $\mu(\sigma)$ into $m(s)$, we have only to use equation (2.7) inversely, namely

$$m(s) = \mu(\sigma(s)) \frac{\mathrm{d}\sigma}{\mathrm{d}s} = \mu(\sigma(s)) \left| \frac{\mathrm{d}\zeta}{\mathrm{d}z} \right|_{z=z(s)}. \tag{2.19}$$

3. Illustrative Examples

The contour of a cross section of a ship might be regarded as symmetrical right and left. Furthermore, since for a ship advancing with a uniform speed, the surface of water might be replaced by its mean position to the first approximation,[5] the flow around a ship body can be approximated, as well-known, by the lower half of that taking place around a so-called double ship. (In what follows, the term 'a double ship' will be abbreviated simply as 'a ship' unless confusion is possible.) Thus C, the contour of a cross section of a ship, is found to be symmetrical right and left as well as up and down, and accordingly also the flow around it has the same property. This symmetry simplifies the computations considerably as shown below.

Namely, in the first place, correspondence between the z plane and the ζ plane may be written as

$$z = F(\zeta) = p\left(\zeta + \frac{a_1}{\zeta} + \frac{a_3}{\zeta^3} + \frac{a_5}{\zeta^5} + \cdots\right), \tag{3.1}$$

where $p > 0$ and a_1, a_2, a_3, $\cdots$ are all real numbers. Next, at this time, $n(s)$, the distribution of normal velocity on C, is found to have the same property of symmetry for the behavior of a ship presupposed in the slender body theory, and eventually $m(s)$ also has the same symmetry. It results from this fact that the following special type of Fourier cosine expansions is valid for both of $\mu(\sigma)$ and $\nu(\sigma)$, the distributions of the source and the normal velocity, respectively, on the circumference of the unit circle on the ζ plane:

$$\mu(\sigma) = \sum_{i=0} \mu_{2i} \cos(2i\sigma), \tag{3.2}$$

and
$$\nu(\sigma) = \sum_{j=0} \nu_{2j}\cos(2j\sigma). \tag{3.3}$$

With a view to investigating in this section qualitative characteristics, above all, appearing in practical problems, we take in the first place a Joukowsky transformation

$$z = F(\zeta) = p\left(\zeta + \frac{a_1}{\zeta}\right), \tag{3.4}$$

where $p > 0$ and $a_1 \leqq 0$. As easily verified, we have

$$\begin{aligned}\frac{ds}{d\sigma} = \left|\frac{dz}{d\zeta}\right|_{\zeta=\exp(i\sigma)} &= p|1 - a_1\exp(-2i\sigma)| \\ &= p(1+a_1^2)^{1/2}\left(1 - \frac{2a_1}{1+a_1^2}\cos(2\sigma)\right)^{1/2},\end{aligned} \tag{3.5}$$

and
$$\ln(F(\zeta) - F(\zeta_0)) = \text{const.} - \ln\zeta + \ln(\zeta - \zeta_0) + \ln\left(\zeta - \frac{a_1}{\zeta_0}\right), \tag{3.6}$$

or expanding the right-hand side,

$$\ln(F(\zeta) - F(\zeta_0)) = \text{const.} + \ln\zeta - \sum_{k=1}^{\infty}\frac{1}{k}\left\{\zeta_0^k + \left(\frac{a_1}{\zeta_0}\right)^k\right\}\frac{1}{\zeta^k}. \tag{3.7}$$

With the aid of (3.6), therefore, (2.6) is found to become

$$\begin{aligned}W(\zeta) = \text{const.} - \bar{\mu}\ln\zeta &+ \frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\ln(\zeta - \exp(i\sigma))d\sigma \\ &+ \frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\ln(\zeta - a_1\exp(-i\sigma))d\sigma,\end{aligned} \tag{3.8}$$

and likewise with the use of (3.7), equation (2.8) reduces to

$$\begin{aligned}W(\zeta) &= \text{const.} + \bar{\mu}\ln\zeta \\ &- \sum_{k=1}^{\infty}\frac{1}{2\pi k\zeta^k}\int_0^{2\pi}\mu(\sigma)\{\exp(ik\sigma) + a_1^k\exp(-ik\sigma)\}d\sigma.\end{aligned} \tag{3.9}$$

If we take notice that the last term on the right-hand side of (3.8) can be written

$$\frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\ln(\zeta - a_1\exp(i\sigma))d\sigma$$

by virtue of the symmetry relation $\mu(\sigma) = \mu(-\sigma)$, it is readily understood that equation (3.8) shows clearly the construction of the flowfield around the unit circle Γ: the source distributed on a single contour C on the z plane turns out to be decomposed, when we pass to the ζ plane, into the source distributed on two circles $\Gamma(\zeta = \exp(i\sigma))$ and $\Gamma'(\zeta = a_1\exp(i\sigma))$ together with an isolated point source of the strength $-\bar{\mu}$ situated at the origin. The flowfield generated outside Γ out of all these sources corresponds in a manner of conformal transformation to that prevailing outside C on the z plane.

In computing $\mu(\sigma)$ in practice, however, it is rather simple to stand at the viewpoint represented by equation (3.9), or in other words the interpretation that the flowfield is made up by the assembly of an infinite number of multipoles superposed at the origin. Namely, upon substituting the expression shown as (3.2) for $\mu(\sigma)$ in equation (3.9) and making use of the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(k\sigma)\exp(\pm il\sigma)\mathrm{d}\sigma = \frac{1}{2}\delta_{kl}, \tag{3.10}$$

where k and l denote integers and δ_{kl} is Kronecker's delta, we find

$$W(\zeta) = \text{const.}+\mu_0\ln\zeta - \sum_{k=1}\frac{1+a_1^{2k}}{2}\frac{\mu_{2k}}{2k}\frac{1}{\zeta^{2k}}. \tag{3.11}$$

On the other hand, the flowfield having the normal velocity distribution as indicated by equation (3.3) on the unit circle may be expressed without difficulty:

$$W(\zeta) = \text{const.}+\nu_0\ln\zeta - \sum_{k=1}\frac{\nu_{2k}}{2k}\frac{1}{\zeta^{2k}}. \tag{3.12}$$

Both of the flowfields shown as equations (3.11) and (3.12) become identical provided the following condition be satisfied:

$$\mu_0 = \nu_0 \quad \text{and} \quad \mu_{2k} = \frac{2}{1+a_1^{2k}}\nu_{2k} \qquad (k = 1, 2, 3, \cdots). \tag{3.13}$$

The source distribution on the circumference of Γ has been thus determined.

Combining equations (3.13) and (3.2), we have

$$\mu(\sigma) = \nu_0 + \sum_{k=1}\frac{2}{1+a_1^{2k}}\nu_{2k}\cos(2k\sigma), \tag{3.14}$$

and through transforming it back onto the contour C, we are led finally to the source distribution which has been aimed at:

$$m(s) = \left\{\nu_0 + \sum_{k=1}\frac{2}{1+a_1^{2k}}\nu_{2k}\cos(2k\sigma)\right\}\Big/ p(1+a_1^2)^{1/2}\left(1-\frac{2a_1}{1+a_1^2}\cos(2\sigma)\right)^{1/2}. \tag{3.15}$$

Let us examine the characteristics of the distributions (3.14) and (3.15) in more detail. Since the contour C which we are now dealing with is an ellipse and may be expressed in terms of the rectangular coordinates as

$$x = p(1+a_1)\cos\sigma \quad \text{and} \quad y = p(1-a_1)\sin\sigma,$$

$p(1+a_1)$ and $p(1-a_1)$ are the principal radii of that ellipse. Confining ourselves for the present to an oblate one, we put $-1 < a_1 < 0$. Writing the ratio of the two radii as ε, we have

$$\varepsilon = (1+a_1)/(1-a_1); \tag{3.16}$$

our next step will be to scrutinize the mode of change of the flowfield due to variation of ε.

(i) In the case when $\varepsilon = 1$: —we have $a_1 = 0$, and consequently C is found to be a circle. Equation (3.14) becomes

$$\mu(\sigma) = \nu_0 + 2\sum_{k=1} \nu_{2k}\cos(2k\sigma) = 2\nu(\sigma) - \nu_0 = 2\nu(\sigma) - \bar{\nu}. \qquad (3.17)$$

This is in fact the expression of $\mu(\sigma)$ used frequently as an approximation in a general case. Namely, after it is multiplied by $d\sigma/ds$ for a particular problem in question according to equation (2.19), the result is taken as $m(s)$. This is exact only when C is a circle, of course, and then from $ds/d\sigma = p$, we have $\mu = pm$, $\nu = pn$ according to equations (2.7) and (2.10), and therefore from (3.17) we obtain

$$m(s) = 2n(s) - \tilde{n}, \qquad (3.18)$$

where $\tilde{n}$ is the value of n averaged with respect to s over the range 0 through $2\pi p$. Equations (3.17) and (3.18) are found to satisfy the general rule concerning the relationship between (m, n) and (μ, ν). However, since C itself is a circle, (3.18), the resulting equation, is quite natural.

(ii) In the case when $\varepsilon = 0$: —we have $a_1 = -1$ and accordingly C becomes a segment of a vertical line (a flat plate). Equation (3.14) reduces to

$$\mu(\sigma) = \nu_0 + \sum_{k=1} \nu_{2k}\cos(2k\sigma) = \nu(\sigma), \qquad (3.19)$$

and we have readily

$$m(s) = n(s); \qquad (3.20)$$

this also is a well-known property, and at the same time is very significant for a thin ship, as will be shown in the next section. Difference between (3.18) and (3.20) is quite sensible.

(iii) In the case when $\varepsilon = 0.2$: —we get $a_1 = -2/3$ and C becomes an ellipse elongated in the direction of the y axis. Roughly speaking, it might be regarded as a representative example of a cross sectional form of a ship near the bow. Then, since we have

$$\frac{2}{1+a_1^2} \fallingdotseq 1.38, \quad \frac{2}{1+a_1^4} \fallingdotseq 1.67, \quad \frac{2}{1+a_1^6} \fallingdotseq 1.84, \cdots, \qquad (3.21)$$

the ratio μ_{2k} vs. ν_{2k} $(k = 1, 2, \cdots)$ approaches a value a little larger than 1, which is pertinent for a flat plate, as we go to small values of k, and tends to 2, that for a circle, as we go to large values of k, on the contrary. This fact might have a considerable effect in accuracy of estimating the intensity of the ship wave generated at the shoulder. Approximate estimation adopted up to the present, in which the source distribution corresponding to a circle is assumed, would yield an exaggerated value, from a theoretical standpoint at least. In transforming necessary quantities back to the z plane, we have to make use of equation (3.15), since no simple relation such as (3.18) or (3.20) is valid any longer.

Modifying computations for an ellipse a little, we can arrive at four-term approximation put to practical use sometimes: namely, instead of (3.4) we write

$$z = F(\zeta) = p\left(\zeta + \frac{a_1}{\zeta} + \frac{a_3}{\zeta^3} + \frac{a_5}{\zeta^5}\right); \qquad (3.22)$$

our next step will be to examine the problem for this case in more detail. Expansion formula corresponding to equation (2.8) will be obtained through the same technique as employed in (2.16): after somewhat laborious computations, we are led to

$$W(\zeta) = \text{const.} + \bar{\mu}\ln\zeta - \frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\left[\sum_{k=1}\frac{1}{k\zeta^k}\{\zeta_0^k + kS_k(\sigma)\}\right]\mathrm{d}\sigma, \qquad (3.23)$$

where $S_{2k}(\sigma)$ and $S_{2k+1}(\sigma)$ denote, respectively, linear homogeneous expressions of even and odd powers of $\exp(-i\sigma)$. However, as seen in equation (3.2), $\mu(\sigma)$ is a linear expression of even powers (positive and negative) of $\exp(i\sigma)$, and in the integrals with respect to σ shown above, only $\zeta_0^{2k} + 2kS_{2k}(\sigma)$ can have a contribution toward integration. So that writing only $S_{2k}(\sigma)$ in full, we have

$$S_{2k}(\sigma) = \sum_{l=1} a_{2k,2l}\zeta_0^{-2l} \quad (a_{2k,2l} = a_{2l,2k}); \qquad (3.24)$$

forms of the first several members of the numerical coefficients $a_{2k,2l}$ are given in Table 1.

Table 1 Forms of $a_{2k,2l}$

$2l \backslash 2k$	2	4	6	…
2	$\frac{1}{2}a_1^2+a_3$	$a_1a_3+a_5$	$a_1a_5+\frac{1}{2}a_3^2$	…
4	$a_1a_3+a_5$	$\frac{1}{4}a_1^4+a_1^2a_3+a_1a_5+\frac{3}{2}a_3^2$	$a_1^3a_3+a_1^2a_5+2a_1a_3^2+3a_3a_5$	…
6	$a_1a_5+\frac{1}{2}a_3^2$	$a_1^3a_3+a_1^2a_5+2a_1a_3^2+3a_3a_5$	$\frac{1}{6}a_1^6+a_1^4a_3+a_1^3a_5+\frac{9}{2}a_1^2a_3^2$ $+6a_1a_3a_5+\frac{7}{3}a_3^3+\frac{5}{2}a_5^2$	…
⋮	⋮	⋮	⋮	

Upon substitution of equation (3.2) for $\mu(\sigma)$, the integral shown as (3.23) will be evaluated in the following way. Employing the integral formulas given by equation (3.10), we have

$$W(\zeta) = \text{const.} + \mu_0\ln\zeta - \sum_{k=1}\left(\frac{1}{2}\mu_{2k} + k\sum_{l=1}a_{2k,2l}\mu_{2l}\right)\Big/2k\zeta^{2k}. \qquad (3.25)$$

Comparing this with equation (3. 12), we have a system of linear equations

$$\mu_0 = \nu_0 ,$$

and
$$\mu_{2k}+2\,k \sum_{l=1} a_{2k.2l}\mu_{2l} = 2\,\nu_{2k} \qquad (k = 1, 2, 3, \cdots), \tag{3.26}$$

which is nothing but equation (2. 14). If these equations are solved, μ_{2k} will be evaluated accordingly. When $a_1 = a_3 = a_5 = 0$, we are led to the case of a circle, and if $a_1 = -1$, $a_3 = a_5 = 0$, the problem reduces to the case of a flat plate.

As an example of a ship form, let us take the numerical values

$$p = 0.576; \quad a_1 = -0.776, \quad a_3 = -0.025, \quad \text{and} \quad a_5 = 0.015. \tag{3.27}$$

These values were read roughly out of a graphic representation of these parameters in Fig. 2 of reference 4), and may correspond more or less to the contour of a cross section of a ship at her shoulder or thereabout. In terms of them, $a_{2k,2l}$ may be calculated as shown in Table 2. Accordingly, by solving equation (3. 26), we have

Table 2 Values of $a_{2k,2l}$

$2l$ \ $2k$	2	4	6	$\cdots$
2	0. 276	0. 034	−0. 011	$\cdots$
4	0. 034	0. 065	0. 019	$\cdots$
6	−0. 011	0. 019	0. 024	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$$\begin{aligned}
\mu_0 &= \nu_0 , \\
\mu_2 &= \quad 1.296\,\nu_2-0.073\,\nu_4+0.030\,\nu_6+\cdots, \\
\mu_4 &= -0.145\,\nu_2+1.632\,\nu_4-0.109\,\nu_6+\cdots, \\
\mu_6 &= \quad 0.089\,\nu_2-0.164\,\nu_4+1.761\,\nu_6+\cdots, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{3.28}$$

By comparing these expressions with equation (3. 13), we find that the parallelism is obvious apart from minor terms contained in the former. In this case, therefore, the flow has a property quite similar to that prevailing around a slender ellipse laying its major axis parallel to the y axis. Namely, the coefficient of ν_2 in the second equation is found to be less than 2, the value pertinent for a circle, and in fact a little larger than 1, that for a flat plate. As pointed out before, this fact may have a sensible effect in the accuracy of estimating the bow waves and the stern waves.

4. Thin Ship as a Slender Body

Now, let us look back at the slender body theory. Writing L for the length of the body, M for the maximum of radii of all cross sections perpendicular to the length, then we may conclude that the slender body theory is valid as long as M/L would be sufficiently small. Take the x axis in the direction of the length, and the y and the z axes perpendicularly to it, for there can be hardly any danger of confusing the above notations with those employed in the preceding sections in the forms x, y, and $z\,(=x+iy)$. Now, if non-dimensional quantities defined as

$$(\xi, \eta, \zeta) = (x/L,\ y/M,\ z/M) \tag{4.1}$$

is introduced, ϕ, the velocity potential of the flowfield, satisfies the equation $\Delta\phi = 0$, or

$$\left(\frac{M}{L}\right)^2 \frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\zeta^2} = 0, \tag{4.2}$$

and, by virtue of the condition $M/L \ll 1$, we have

$$\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\zeta^2} = 0; \tag{4.3}$$

thus a plane potential equation is found to be valid approximately. In this section, it is assumed that the ship is at rest and the water flows in the direction of x increasing with a uniform velocity U. The y axis is taken horizontally in the transverse direction, and z vertically downward. It is next supposed that the ship form can be expressed by a pair of equations

$$y = \pm f(z;\ x), \tag{4.4}$$

and finally that n^* denotes the length measured along an outward-drawn normal, lying on the y-z plane, of the contour C of a cross section.

Rewriting equation (4.3) in terms of the old variables, we have

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{4.5}$$

and the boundary conditions to be imposed upon it should be: —at a great distance, i.e. as $|y|$ and/or $|z| \to \infty$,

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0, \tag{4.6}$$

and on C, the contour of a cross section of the ship,

$$\frac{\partial\phi}{\partial n^*} = U\frac{\partial f}{\partial x}\left\{1 + \left(\frac{\partial f}{\partial z}\right)^2\right\}^{-1/2}. \tag{4.7}$$

Although there exists one condition $M \ll L$ for M, the maximum value for the transverse dimension of the body, no other restrictions are given to the form of a cross section. Accordingly, provided M is sufficiently small compared with L, a thin ship and a flat ship are both in the region of legitimacy of the slender body theory. As a representative example, we take a thin ship, and making use of a crude approximation that the contour of its cross section can be regarded as a flat plate, we put $\partial f/\partial z \doteqdot 0$ in equation (4.7) corresponding to the thin-ship-assumption. Then, on both sides of that plate, the following condition for $n(s)$ should be valid as the boundary condition:

$$\frac{\partial\phi}{\partial n^*} = U\frac{\partial f}{\partial x}, \tag{4.8}$$

which is nothing but the assumption made by Mitchell.[6] As we saw in the preceding section, since the source density corresponding to the lateral velocity $n(s)$ is $m(s) = n(s)$ on a flat plate, the source defined as

$$m(s) = U\frac{\partial f}{\partial x} \tag{4.9}$$

is to be distributed on the central plane of the ship. It is self-evident that by substituting equation (4.9) in the general expression for wave-making resistance of a source distribution, for example equation (4.1) of reference 4), we are led to Michell's formula for the wave-making resistance.

To sum up, as long as the condition $M \ll L$ is valid, a thin ship and a flat ship are distinctly idealized cases of a slender ship. So that the slender ship theory in which the curvature of the contour of a cross section is taken into consideration is expected to yield a more accurate result, from a theoretical standpoint at least, provided we should be enabled to formulate the source distribution correctly, than the rather extreme approximation in which the ship is regarded simply as a flat plate.

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