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FINITE ELEMENT ANALYSIS WITH IMPROVED ACCURACY FOR RECTANGULAR PLATE BENDING ELEMENT

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1. Introduction

A number of methods of analyzing a structure consisted of plates with stiffners have been proposed up to present.

One of the most universal and powerful method is finite element method. However, this method has a few inconvenient points as follows:

(1) Extensive calculation is usually needed to solve the simultaneous equations with a large number of unknowns.

(2) It is difficult to get continous values of stresses at the interjacent boundaries except by a few particular cases together with increasing the accuracy of the results as a whole.

To lighten the volume of calculation, the finite strip method was proposed by Y. K. Cheung[1] to analyze a kind of plate structure using a limited number of approximate polynomials as displacement functions. This paper presents a method of increasing the accuracy of the finite strip method by using general solutions which satisfy the equilibrium equation of plate bending as displacement functions. Some examples are shown for the cases of analyzing bending of plates without and with stiffners of which all edges are clamped corresponding a severe ones to analyze analytically.

2. General

This study is concerned with structure consisted of continuous plates with stiffners.

The plate is treated by separating into a number of strips, the thickness of each strip is assumed to be constant and may vary for each strip.

The local coordinates (x, y) in each strip are taken as shown in Fig. 1.*** Considering any strip, let sides ⓘ and ⓚ corresponding respectively for x=constant with arbitary geometrical and mechanical boundary conditions and side ⓤ and ⓛ corresponding y=constant supported by specified condition, i. e. clamped, free or simply supported.

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And for the case of plate structure with stiffners in y-direction, the location of plate sides is assumed to be coincided with the sides corresponding x=constant.

3. Displacement Function

3. 1. Plate bending analysis

It is assumed that the direction of load q acting on a plate is normal to its surface and that the deflection is small compared with thickness of the plate.

The equation of equilibrium in terms of deflection w at the middle plane of the plate during bending is given by

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D} \tag{1}$$

where D is the bending rigidity of plate.

The load q is assumed to be constant in x-direction or to be represented by power series with respect to x, provided that it may vary arbitrarily in y-direction.

And it can be expressed as

$$q=q_s+q_a$$

in which suffix s means symmetric and suffix a means anti-symmetric with respect to x-axis.

Accordingly, the deflection of plate can be expressed as follows :

$$w=w_s+w_a \ .$$

We use following series as the displacement functions, which satisfy exactly the homogeneous equation of (1) :

$$w_s=\sum_{n=1}^{\infty} k_n(x)\cos\alpha_n y+\sum_{m=1}^{\infty} E_m \phi_m(y)\cos\beta_m x \tag{2. a}$$

$$w_a=\sum_{n=1}^{\infty} k_n(x)\sin\bar{\alpha}_n y+\sum_{m=1}^{\infty} F_m \psi_m(y)\cos\beta_m x \tag{2. b}$$

where $\quad \alpha_n=\dfrac{(2n-1)\pi}{2b}, \quad \bar{\alpha}_n=\dfrac{n\pi}{b}, \quad \beta_m=\dfrac{(2m-1)\pi}{2a}$

$$k_n(x)=A_n\frac{\cosh\alpha_n x}{\sinh\alpha_n a}+B_n\frac{\sinh\alpha_n x}{\cosh\alpha_n a}$$

$$+C_n\frac{\alpha_n x\sinh\alpha_n x}{\sinh\alpha_n a}+D_n\frac{\alpha_n x\cosh\alpha_n x}{\cosh\alpha_n a}$$

$$\phi_m(y)=\frac{\beta_m b\cosh\beta_m y}{\cosh\beta_m b}-\frac{\beta_m y\sinh\beta_m y}{\sinh\beta_m b}$$

$$\psi_m(y) = \frac{\beta_m b \sinh \beta_m y}{\sinh \beta_m b} - \frac{\beta_m y \cosh \beta_m y}{\cosh \beta_m b} \quad .$$

Then, A_n, B_n, C_n, D_n, E_m and F_m are unknown constants to be determined by boundary conditions of the plate. Provided that both second terms in the right hand sides of equation (2. a) and (2. b) should be dropped for special case of both ⓤ and ⓛ sides being simply supported.

Hereafter, the case of symmetrical bending with respect to x-axis is shown as examples.

Anti-symmetrical case can be treated similarly as the procedure of symmetrical case and general case can be treated by superposing the results of the both case.

(i) Boundary conditions for sides ⓤ and ⓛ

For the case that the both edges of the plate in x-direction are built in, we have following conditions for both edges:

$$w_s = 0 \ , \quad \frac{\partial w_s}{\partial y} = 0 \tag{3}$$

Substituting expression (2. a) into (3), we obtain

$$\mp \sum_{n=1}^{\infty} (-1)^{n+1} \alpha_n k_n(x) \pm \sum_{m=1}^{\infty} E_m \frac{\partial \phi_m}{\partial y}\bigg|_{y=b} \cos \beta_m x = 0 \quad . \tag{4}$$

We expand $k_n(x)$ in the form of following Fourier series :

$$k_n(x) = \sum_{m=1}^{\infty} X'_{mn} \cos \beta_m x \tag{5}$$

then, we have

$$X'_{mn} = \frac{(-1)^{m+1}}{a} \left\{ A_n \frac{2\beta_m}{\alpha_n^2 + \beta_m^2} \coth \alpha_n a + C_n \frac{2\alpha_n \beta_m}{\alpha_n^2 + \beta_m^2} \left(a - \frac{2\alpha_n}{\alpha_n^2 + \beta_m^2} \coth \alpha_n a \right) \right\}$$

$$= \frac{(-1)^{m+1}}{a} X_{mn} \quad .$$

Substituting (5) in (4), we obtain

$$E_m \beta_m \Delta(\beta_m b) = -\sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{a} \alpha_n X_{mn} \tag{6}$$

where

$$\Delta(\beta_m b) = 1 + \beta_m b (\coth \beta_m b - \tanh \beta_m b).$$

The cases for other boundary conditions can be treated similarly.

(ii) Boundary conditions for sides ⓘ and ⓚ

We expand the deflections at the sides ⓘ and ⓚ and the slopes perpendicular to them in following series.

$$\text{For ⓘ side,}\begin{cases}\text{deflection:} & \sum_{n=1}^{\infty} \delta_{in}\cos\alpha_n y\\ \text{slope:} & \sum_{n=1}^{\infty} \theta_{in}\cos\alpha_n y\end{cases}$$

$$\text{For ⓚ side,}\begin{cases}\text{deflection:} & \sum_{n=1}^{\infty} \delta_{kn}\cos\alpha_n y\\ \text{slope:} & \sum_{n=1}^{\infty} \theta_{kn}\cos\alpha_n y\end{cases}$$

From the deflections for sides ⓘ and ⓚ, we obtain the following equations by using (2. a) and the above expressions

$$A_n\coth\alpha_n a+C_n\alpha_n a=\frac{\delta_{kn}+\delta_{in}}{2} \tag{7}$$

$$B_n\tanh\alpha_n a+D_n\alpha_n a=\frac{\delta_{kn}-\delta_{in}}{2}\quad . \tag{8}$$

Similarly, from the slopes for sides ⓘ and ⓚ, we obtain

$$\sum_{n=1}^{\infty}\theta_{kn}\cos\alpha_n y=\sum_{n=1}^{\infty}\left.\frac{\partial k_n}{\partial x}\right|_{x=a}\cos\alpha_n y-\sum_{m=1}^{\infty}(-1)^{m+1}E_m\phi_m(y)\beta_m \tag{9}$$

$$\sum_{n=1}^{\infty}\theta_{in}\cos\alpha_n y=\sum_{n=1}^{\infty}\left.\frac{\partial k_n}{\partial x}\right|_{x=-a}\cos\alpha_n y+\sum_{m=1}^{\infty}(-1)^{m+1}E_m\phi_m(y)\beta_m\ . \tag{10}$$

Then, we expand $\phi_m(y)$ in following Fourier series :

$$\phi_m(y)=\sum_{n=1}^{\infty}Y'_{mn}\cos\alpha_n y \tag{11}$$

from which

$$Y'_{mn}=\frac{(-1)^{n+1}}{b}\cdot\frac{4\alpha_n\beta_m^2\coth\beta_m b}{(\alpha_n^2+\beta_m^2)^2}=\frac{(-1)^{n+1}}{b}Y_{mn}\ .$$

Substituting (11) in (9) and (10), we obtain

$$B_n+D_n(1+\alpha_n a\tanh\alpha_n a)=\frac{\theta_{kn}+\theta_{in}}{2\alpha_n} \tag{12}$$

$$A_n+C_n\ (1+\alpha_n a\coth\alpha_n a)$$
$$-\sum_{m=1}^{\infty}\frac{(-1)^{m+n}}{b}E_m\frac{\beta_m}{\alpha_n}Y_{mn}=\frac{\theta_{kn}-\theta_{in}}{2\alpha_n}\quad . \tag{13}$$

We can determine the unknown constants A_n, B_n, C_n, D_n and E_m from the equations (6), (7), (8), (12) and (13).

3. 2. Approximate matrix representation of displacement functions

To calculating the deflection of the plate strip by finite terms (n, m) in the series in (2. a), we write the solution of equation (1) as follows

$$w = F \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ E_m \end{Bmatrix} \tag{2)'$$

where F is $4n+m$ row matrix.

$$F = \left[\frac{\cosh \alpha_1 x}{\sinh \alpha_1 a} \cos \alpha_1 y,\ \frac{\cosh \alpha_2 x}{\sinh \alpha_2 a} \cos \alpha_2 y, \ldots, \frac{\sinh \alpha_1 x}{\cosh \alpha_1 a} \cos \alpha_1 y, \ldots, \right.$$

$$\left. \ldots, \frac{\alpha_1 x \sinh \alpha_1 x}{\sinh \alpha_1 a} \cos \alpha_1 y, \ldots, \frac{\alpha_1 x \cosh \alpha_1 x}{\cosh \alpha_1 a} \cos \alpha_1 y,\ \ldots,\ \phi_1(y) \cos \beta_1 x, \ldots \right] .$$

From **3.1** the relation between unknown constants $\{A_1 A_2 A_3 \ldots\}^T$ and the boundary displacements at sides ⓘ and ⓚ can be represented in following form:

$$\begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ E_m \end{Bmatrix} = S \begin{Bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{Bmatrix} = SW \tag{14}$$

where S: $(4n+m,\ 4n)$ matrix (refer to Appendix 1.)
W: $4n$ column matrix representing boundary displacements
$W_j = \{\delta_{ij},\ \theta_{ij},\ \delta_{kj},\ \theta_{kj}\}^T$. $(j=1,\ 2, \ldots n)$.

Combining (2)′ with (14), we obtain

$$w = FSW . \tag{15}$$

4. Potential Energy and Equilibrium Equation for Each Strip

The curvatures of the strip are given by

$$\chi = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = TSW \tag{16}$$

where T is $(3,\ 4n+m)$ matrix. (see Appendix 2.)
And the bending and twisting moments for isotropic material are represented in following form:

$$M = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_x & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$=D\chi=DTSW \tag{17}$$

in which

$$D_x=Et^3/12(1-\nu^2),\ D_1=\nu D_x$$

$$D_{xy}=\frac{1-\nu}{2}D_x$$

t=the thickness of plate strip, ν=Poisson's ratio ,

E=Young's modulus .

The potential energy of each strip element is given by

$$U_p=\frac{1}{2}\int_{-a}^{a}\int_{-b}^{b}M^T\chi\ dxdy-\int_{-a}^{a}\int_{-b}^{b}w^Tqdxdy. \tag{18}$$

The normal load q can be expressed by following cosine series

$$q=\sum_n q_n\cos\alpha_n y=\boldsymbol{q}C_y \tag{19}$$

where $\boldsymbol{q}=[q_1q_2\ldots q_n]$: n row matrix

$C_y=\{\cos\alpha_1 y,\ \ldots,\ \cos\alpha_n y\}^T$: n column matrix.

Substituting (15), (16), (17) and (19) in (18), we obtain

$$U_p=\frac{1}{2}W^TS^T\left(\int_{-a}^{a}\int_{-b}^{b}T^TDT\,dxdy\right)SW$$

$$-W^TS^T\left(\int_{-a}^{a}\int_{-b}^{b}F^T\boldsymbol{q}C_y\,dxdy\right)$$

$$=\frac{1}{2}W^TS^TESW-W^TS^TP \tag{20}$$

where

$$E=\begin{bmatrix} e_{11} & e_{12} & \cdots\cdots & e_{1,4n+m} \\ e_{21} & e_{22} & \cdots\cdots & \\ \vdots & & & \\ e_{4n+m,\,1} & \cdots\cdots & & e_{4n+m,\,4n+m}\end{bmatrix} : 4n+m \text{ square matrix}$$

$$e_{ij}=\int_{-a}^{a}\int_{-b}^{b}(T^TDT)_{ij}dx\,dy$$

$P=\{p_1p_2\ \ldots\ldots\ p_{4n+m}\}^T$: $4n+m$ column matrix

$$p_i=\int_{-a}^{a}\int_{-b}^{b}(F^T\boldsymbol{q}C_y)_i dx\,dy\ ..$$

Double integrations in each strip of E and P are able to be obtained analytically as shown in Appendix 3 and 4.

According to the principle of minimum potential energy,

$$\frac{\partial U_p}{\partial W}=0 \quad . \tag{21}$$

Thus, we can obtain the simultaneous equation with $4n$ unknown generalized displacements,

$$(S^T E S)\ W - S^T P = 0 \tag{22}$$

in which, square matrix $k=S^T ES$ corresponds to stiffness matrix and column matrix $f=S^T P$ corresponds to load matrix in usual structural analysis like finite element method. We can obtain total equations for the whole strip by repeating and superposing above mentioned procedure for every finite strips.

5. Stiffness Matrix on Stiffners

Let us consider the stiffners in y-direction as shown in Fig. 1, fitted at the sides of y-direction for each finite strip.

Curvature and torsion of the stiffner are given by

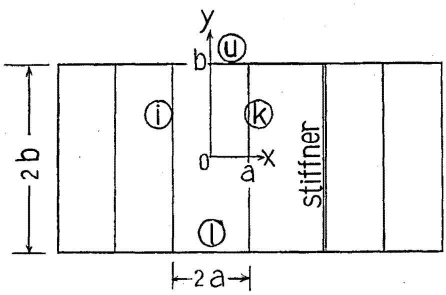

Fig. 1 Plate structure and local coodinates.

$$\chi_s=\left\{\begin{array}{c} -\left.\dfrac{\partial^2 w}{\partial y^2}\right|_{x=a} \\ \left.\dfrac{\partial^2 w}{\partial x \partial y}\right|_{x=a} \end{array}\right\}=RSW \ . \tag{32}$$

in which R is $(2, 4n+m)$ matrix. (refer to Appendix 5.)

And the bending moment and twisting moment on stiffner are represented in the form:

$$M_s=\left\{\begin{array}{c} M_x \\ M_y \end{array}\right\}=D_s\chi_s=D_sRSW \tag{24}$$

where

$$D_s=\begin{bmatrix} EI_x & 0 \\ 0 & GJ \end{bmatrix}$$

R: (2, $4n+m$) matrix (see Appendix 5)

then, EI_x is the bending rigidity of stiffner and GJ is the twisting rigidity of stiffner.

The strain energy of each stiffner can be given by

$$U_s=\frac{1}{2}\int_{-b}^{b} M_S^T \chi_S \, dy$$

$$=W^T S^T \left(\int_{-b}^{b} R^T D_S R \, dy\right) SW \, . \tag{25}$$

Then, the stiffness matrix of stiffner is represented in following form:

$$k_S=S^T G_S S \tag{26}$$

where

$$G_S=\begin{bmatrix} g_{11} g_{12} \cdots\cdots g_{1,\,4n+m} \\ g_{21} g_{22} \cdots\cdots\cdots\cdots\cdots \\ g_{4n+m,\,1} \cdots g_{4n+m,\,4n+m} \end{bmatrix} \quad : (4n+m) \text{ square matrix}$$

and

$$g_{ij}=\int_{-b}^{b} (R^T D_S R)_{ij} dy \, .$$

Each element in G_S is able to be integrated analytically as shown in Appendix 6.

For the case of plate attached with stiffners in x-direction instead of y-direction, the stiffness matrix can be obtained by similar manner as the case of plate with stiffners in y-direction, but it is not necessarily advantageous to use exact displacement functions satisfying homogeneous equation of (1) for the aim of increasing accuracy and it is recommendable to use approximate functions such as polynomials for simplicity, sacrificing the high accuracy of the result.

6. Examples

(a) Square plate

For investigating the accuracy of solutions by using expressin (2. a) as displacement function in finite strip method, we have calculated uniformly loaded square plate on the boundary condition of all edges clamped.

Numerical results on bending moment M_x and twisting moment M_{xy} are given in Fig. 2 (a) and Fig. 2 (b).

We cannot almost find the discontinuity of moment M_x and M_{xy} on the inter-

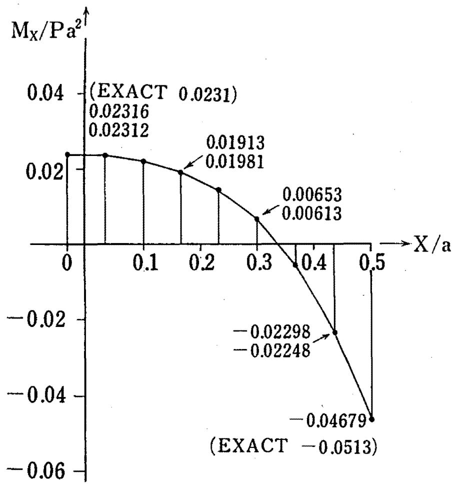

Fig. 2 (a) Bending moment M_x at y=0 (15 strips, 4 terms)

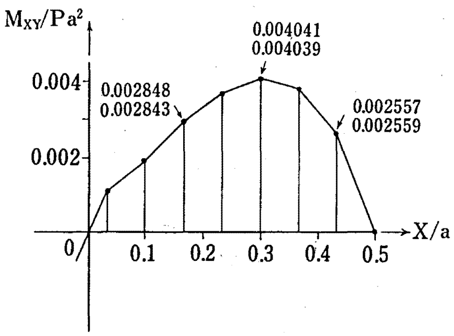

Fig. 2 (b) Twisting moment M_{xy} at y=0 (15 strips, 4 terms)

strip boundary.

For different number of strips keeping to take 4 terms of the series for every cases, the change of maximum deflection of plate is shown in Fig. 3.

(b) Rectangular plate with two stiffners

The deflection of rectangular plate with two stiffners loaded uniformly and with all clamped edges is shown in Fig. 4.
It is difficult to obtain highly accurate solution for the case of thin plate with very rigid stiffner. For such case, it is recommendable to use exact solutions of governing equations for the deflections of stiffners instead of using exact solutions for plate bendings.

7. Consideration

It has been made clear by many examples[2)] that we cannot necessarily

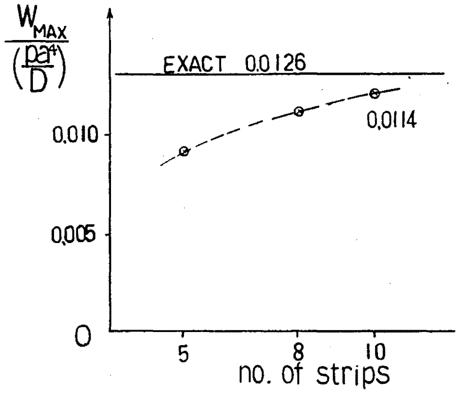

Fig. 3 The relation between maximum deflection and number of strips.

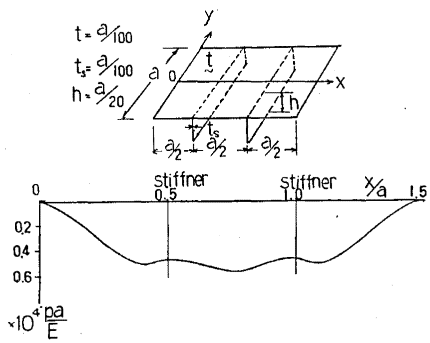

Fig. 4 Deflection of rectangular plate with two stiffners at y=0 (6 strips, 6 terms)

obtain the numerical solution with satisfactory accuracy even assuming by trial functions considered to be approximate in usual displacement method of finite element method, especially it is often ture for the values of stresses. And this stiuation is supposed to be same in the finite strip method.

For the sake of arising the discontnuity of the obtained stresses at inter-element/strip boundaries, we have generally no guarantee of accuracy of results for practically admissible size of finite element/strip mesh as far as using assumed approximate functions as displacement functions.

If minimum mesh size is limited by practical reasons, we have to find any mean of increasing accuracy of obtained values (especially stresses) without changing the mesh size.

This paper suggests a method met with such situation.

Authors used conforming functions satisfying exactly the governing equation of plate bending in the form of expanding in the finite terms of series.

Therefore we can generally expect to get the solution tending to exact value by increasing the number of terms in finite strip/element method according to the principle of minimum potential energy for plate bending problem. And thus we get the numerical solution with practically allowable continuity of stresses at the interjacent boundaries, this solution is considered to be guaranteed the accuracy, because the solution satisfies the governing equation in each finite strip/element.

It has become clear that the increasing number of terms in the series of such solutions is more effective to increase the accuracy than to increase the number of strips in the case of finite strip method. This situation is considered to be similar for the case of finite element method.

Practical examples for square plate being all edges clamped without and with stiffners are given. For the former case we got satisfactory results by using over 10 strips and 4 terms.

8. Conclusion

A method of increasing accuracy of results by means of finite strip method by using the exact general solutions as displacement functions is presented.

To save the computing time and necessary memory size of computer and to get the solution with improved accuracy than that obtained by the former numerical methods, this approach is powerful tool for the numerical stress analysis of plate structure.

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Appendix

A 1. Matrix-S

The relation between unknown constants and boundary displacement can be represented in the following form:

$$E_1 \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ E_m \end{Bmatrix} = E_2 \begin{Bmatrix} \delta_{i1} \\ \theta_{i1} \\ \vdots \end{Bmatrix} \tag{A 1}$$

From (A 1), we obtain

$$S=E_1^{-1}E_2$$

where E_1 is $(4n+m)$ square matrix and consists of following submatrices

$$(E_1)_{ij}=\begin{bmatrix} 1 & 0 & \mu_i\tanh\mu_i & 0 & 0 \\ 0 & \Omega(\mu_i)\tanh\mu_i & 0 & 0 & 0 \\ 0 & 0 & \Delta(\mu_i) & 0 & -x_{ij} \\ 0 & 0 & 0 & \Omega(\mu_i)\tanh\mu_i & 0 \\ 0 & 0 & -y_{ji} & 0 & \Delta(\lambda_i) \end{bmatrix}$$

in which

$$\mu_i=\alpha_i a\ ,\quad \lambda_i=\beta_i b$$

$$\Delta(\lambda)=1+\lambda(\coth\lambda-\tanh\lambda)$$

$$\Omega(\mu)=1-\mu(\coth\mu-\tanh\mu)$$

$$x_{ij}=\frac{4(-1)^{i+j}\beta_j^3\coth\lambda_j}{b(\alpha_i^2+\beta_j^2)^2}$$

$$y_{ij}=\frac{4(-1)^{i+j}\alpha_i^3\coth\mu_i}{a(\alpha_i^2+\beta_j^2)^2}$$

and E_2 is $(4n+m,\ 4n)$ matrix and consists of following submatrices

$$(E_2)_{ij}=\begin{bmatrix} \dfrac{\tanh\mu_i}{2} & 0 & \dfrac{\tanh\mu_i}{2} & 0 \\ -\dfrac{1+\mu_i\tanh\mu_i}{2} & -\dfrac{a}{2} & \dfrac{1+\mu_i\tanh\mu_i}{2} & -\dfrac{a}{2} \\ -\dfrac{\tanh\mu_i}{2} & -\dfrac{1}{2\alpha_i} & -\dfrac{\tanh\mu_i}{2} & \dfrac{1}{2\alpha_i} \\ \dfrac{1}{2} & \dfrac{\tanh\mu_i}{2\alpha_i} & -\dfrac{1}{2} & \dfrac{\tanh\mu_i}{2\alpha_i} \\ -\dfrac{(-1)^{i+j}\alpha_j}{(\alpha_j^2+\beta_i^2)a} & 0 & -\dfrac{(-1)^{i+j}\alpha_j}{(\alpha_j^2+\beta_i^2)a} & 0 \end{bmatrix}$$

A 2. Matrix-T

$$T=\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} \end{bmatrix}$$

where t_{ij} ($i=1, 2, 3$; $j=1, 2, \ldots, 5$) is row matrix of n or m order in which each element is shown as follows:

$$(t_{11})_n = -\frac{\alpha_n^2 \cosh\alpha_n x}{\sinh\alpha_n a}\cos\alpha_n y\ , \quad (t_{12})_n = -\frac{\alpha_n^2 \sinh\alpha_n x}{\cosh\alpha_n a}\cos\alpha_n y$$

$$(t_{13})_n = -\frac{2\alpha_n^2\cosh\alpha_n x + \alpha_n^3 x \sinh \alpha_n x}{\sinh \alpha_n a}\cos \alpha_n y$$

$$(t_{14})_n = -\frac{2\alpha_n^2 \sinh \alpha_n x + \alpha_n^3 x \cosh \alpha_n x}{\cosh \alpha_n a}\cos \alpha_n y$$

$$(t_{15})_m = \beta_m^2 \cos \beta_m x \left(\frac{\beta_m b \cosh \beta_m y}{\cosh \beta_m b} - \frac{\beta_m y \sinh \beta_m y}{\sinh \beta_m b}\right)$$

$$(t_{21})_n = \frac{\alpha_n^2 \cosh \alpha_n x}{\sinh \alpha_n a}\cos \alpha_n y\ , \quad (t_{23})_n = \frac{\alpha_n^3 x \sinh \alpha_n x}{\sinh \alpha_n a}\cos \alpha_n y$$

$$(t_{22})_n = \frac{\alpha_n^2 \sinh \alpha_n x}{\cosh \alpha_n a}\cos \alpha_n y\ , \quad (t_{24})_n = \frac{\alpha_n^3 x \cosh \alpha_n x}{\cosh \alpha_n a}\cos \alpha_n y$$

$$(t_{25})_m = -\beta_m^2 \cos \beta_m x \left(\frac{\beta_m b \cosh \beta_m y}{\cosh \beta_m b} - \frac{2\cosh \beta_m y + \beta_m y \sinh \beta_m y}{\sinh \beta_m b}\right)$$

$$(t_{31})_n = -2\frac{\alpha_n^2 \sinh \alpha_n x}{\sinh \alpha_n a}\sin \alpha_n y\ , \quad (t_{32})_n = -2\frac{\alpha_n^2 \cosh \alpha_n x}{\cosh \alpha_n a}\sin \alpha_n y$$

$$(t_{33})_n = -2\alpha_n^2 \frac{\sinh \alpha_n x + \alpha_n x \cosh \alpha_n x}{\sinh \alpha_n a}\sin \alpha_n y$$

$$(t_{34})_n = -2\alpha_n^2 \frac{\cosh \alpha_n x + \alpha_n x \sinh \alpha_n x}{\cosh \alpha_n a}\sin \alpha_n y$$

$$(t_{35})_m = -2\beta_m^2 \sin \beta_m x \left(\frac{\beta_m b \sinh \beta_m y}{\cosh \beta_m b} - \frac{\sinh \beta_m y + \beta_m y \cosh \beta_m y}{\sinh \beta_m b}\right)$$

A 3. Matrix-P

$$P = \{p_1\ p_2\ p_3\ p_4\ p_5\}^T$$

where p_i ($i=1, 2, \ldots, 5$) is column submatrix of order n or m and each element is shown as follows:

$$(p_1)_n = \frac{2bq_n}{\alpha_n}$$

$$(p_2)_n = 0$$

$$(p_3)_n = \frac{2bq_n}{\alpha_n}(\alpha_n a \coth \alpha_n a - 1)$$

$$(p_4)_n = 0$$

$$(p_5)_m = \sum_{k=1}^{n} q_k \frac{8(-1)^{k+m}\alpha_k \beta_m}{(\alpha_k^2 + \beta_m^2)^2}\coth \beta_m b$$

A 4. Matrix-E

$$E=\begin{bmatrix} e_{11} & o & e_{13} & o & e_{15} \\ & e_{22} & o & e_{24} & o \\ & & e_{33} & o & e_{35} \\ SYM. & & & e_{44} & o \\ & & & & e_{55} \end{bmatrix}$$

where e_{ij} $(i=1, 2, \ldots, 5; j=1, 2, \ldots, 5)$ is square matrix of order n or m, then

$$e_{ij}=\begin{bmatrix} e_{ij(1,1)} & e_{ij(1,2)} & \cdots\cdots & e_{ij(1,n)} \\ e_{ij(2,1)} & e_{ij(2,2)} & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ e_{ij(n,1)} & e_{ij(n,2)} & \cdots\cdots & e_{ij(n,n)} \end{bmatrix}$$

and each element of submatrix is shown as follows

$$e_{11(k,l)}=\begin{cases} o & (k\neq l) \\ \dfrac{\alpha_k^3 b\{(2\overline{D_x-D_1}+4D_{xy})\sinh 2\alpha_k a+2(2\overline{D_x-D_1}-4D_{xy})\alpha_k a\}}{\cosh 2\alpha_k a-1} & (k=l) \end{cases}$$

$$e_{22(k,l)}=\begin{cases} o & (k\neq l) \\ \dfrac{\alpha_k^3 b\{(2\overline{D_x-D_1}+4D_{xy})\sinh 2\alpha_k a-2(2\overline{D_x-D_1}-4D_{xy})\alpha_k a\}}{\cosh 2\alpha_k a+1} & (k=l) \end{cases}$$

$$e_{13(k,l)}=\begin{cases} o & (k\neq l) \\ \dfrac{\alpha_k^3 b\{(\overline{D_x-D_1}+2D_{xy})(2\alpha_k a\cosh 2\alpha_k a+\sinh 2\alpha_k a)+4(\overline{D_x-D_1}-2D_{xy})\alpha_k a\}}{\cosh 2\alpha_k a-1} & (k=l) \end{cases}$$

$$e_{24(k,l)}=\begin{cases} o & (k\neq l) \\ \dfrac{\alpha_k^3 b\{(\overline{D_x-D_1}+2D_{xy})(2\alpha_k a\cosh 2\alpha_k a+\sinh 2\alpha_k a)-4(\overline{D_x-D_1}-2D_{xy})\alpha_k a\}}{\cosh 2\alpha_k a+1} & (k=l) \end{cases}$$

$$e_{33(k,l)}=\begin{cases} o & (k\neq l) \\ \dfrac{\alpha_k^3 b}{\sinh^2\alpha_k a}\Big[D_x(4\alpha_k a+2\sinh 2\alpha_k a)+(D_x-D_1)(2\alpha_k a\cosh 2\alpha_k a-\sinh 2\alpha_k a) & \end{cases}$$

$$+(D_x-D_1)\left\{\left(\alpha_k^2 a^2+\frac{1}{2}\right)\sinh 2\alpha_k a-\alpha_k a\cosh 2\alpha_k a-\frac{2}{3}\alpha_k^3 a^3\right\}$$

$$+2D_{xy}\left\{\left(\alpha_k^2 a^2+\frac{1}{2}\right)\sinh 2\alpha_k a+\alpha_k a\cosh 2\alpha_k a+\frac{2}{3}\alpha_k^3 a^3-2\alpha_k a\right\}\Big] \qquad (k=l)$$

$$e_{44(k,l)}=\begin{cases} o & (k\neq l) \\ \dfrac{\alpha_k^3 b}{\cosh^2\alpha_k a}\Big[D_x(2\sinh 2\alpha_k a-4\alpha_k a)+(D_x-D_1)(2\alpha_k a\cosh 2\alpha_k a-\sinh 2\alpha_k a) & \end{cases}$$

$$+(D_x-D_1)\left\{\left(\alpha_k^2a^2+\frac{1}{2}\right)\sinh 2\alpha_ka-\alpha_ka\cosh 2\alpha_ka+\frac{2}{3}\alpha_k^3a^3\right\}$$

$$+2D_{xy}\left\{2\alpha_ka+\left(\alpha_k^2a^2+\frac{1}{2}\right)\sinh 2\alpha_ka+\alpha_ka\cosh 2\alpha_ka-\frac{2}{3}\alpha_k^3a^3\right\}\Bigg] \qquad (k=l)$$

$$e_{55(k,l)}=\begin{cases} o & (k\neq l)\\ (D_x-D_1)\,\beta_l^3a\Big[\left(\dfrac{2\beta_l^2b^2}{\cosh^2\beta_lb}-\dfrac{8\beta_lb}{\sinh 2\beta_lb}\right)\left(\beta_lb+\dfrac{\sinh 2\beta_lb}{2}\right) \end{cases}$$

$$-\left(\frac{2\beta_lb}{\sinh 2\beta_lb}-\frac{1}{\sinh^2\beta_lb}\right)(2\beta_lb\cosh 2\beta_lb-\sinh 2\beta_lb)\Big]$$

$$+D_x\frac{4\beta_l^3a}{\sinh^2\beta_lb}\left(\beta_lb+\frac{\sinh 2\beta_lb}{2}\right)$$

$$+4D_{xy}\beta_l^3a\left[\left(\frac{\beta_lb}{\cosh\beta_lb}-\frac{1}{\sinh\beta_lb}\right)^2\left(\frac{\sinh 2\beta_lb}{2}-\beta_lb\right)\right.$$

$$-\left(\frac{\beta_lb}{\sinh 2\beta_lb}-\frac{1}{2\sinh^2\beta_lb}\right)(2\beta_lb\cosh 2\beta_lb-\sinh 2\beta_lb)$$

$$\left.+\frac{1}{\sinh^2\beta_lb}\left\{\left(\frac{\beta_l^2b^2}{2}+\frac{1}{4}\right)\sinh 2\beta_lb-\frac{\beta_lb}{2}\cosh 2\beta_lb+\frac{1}{3}\beta_l^3b^3\right\}\right] \qquad (k=l)$$

$$e_{15(k,l)}=(D_x-D_1)\frac{8(-1)^{k+l}\alpha_k^3\beta_l^3(\alpha_k^2-\beta_l^2)\coth\beta_lb\coth\alpha_ka}{(\alpha_k^2+\beta_l^2)^3}$$

$$-D_{xy}\frac{32(-1)^{k+l}\alpha_k^5\beta_l^3\coth\beta_lb\coth\alpha_ka}{(\alpha_k^2+\beta_l^2)^3}$$

$$e_{35(k,l)}=\frac{8(-1)^{k+l}\alpha_k^3\beta_l^3\coth\beta_lb}{(\alpha_k^2+\beta_l^2)^3}\left\{(\alpha_k^2-\beta_l^2)(D_x-D_1)\alpha_ka\right.$$

$$\left.-2\frac{(\alpha_k^4+\beta_l^4)D_x+2\alpha_k^2\beta_l^2D_1}{\alpha_k^2+\beta_l^2}\coth\alpha_ka\right\}$$

A 5. Matrix-R

$$R=\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15}\\ r_{21} & r_{22} & r_{33} & r_{24} & r_{25}\end{bmatrix}$$

where r_{ij} ($i=1, 2;\ j=1, 2, \dots, 5$) is row matrix of order n or m and element is shown as follows:

$(r_{11})_n=-\alpha_n^2\coth\alpha_na\cos\alpha_ny,\ (r_{21})_n=-\alpha_n^2\sin\alpha_ny$

$(r_{12})_n=-\alpha_n^2\tanh\alpha_na\cos\alpha_ny,\ (r_{22})_n=-\alpha_n^2\sin\alpha_ny$

$(r_{13})_n=-\alpha_n^2(2\coth\alpha_na+\alpha_na)\cos\alpha_ny,\ (r_{23})_n=-\alpha_n^2(1+\alpha_na\coth\alpha_na)\sin\alpha_ny$

$(r_{14})_n=-\alpha_n^2(2\tanh\alpha_na+\alpha_na)\cos\alpha_ny,\ (r_{24})_n=-\alpha_n^2(1+\alpha_na\tanh\alpha_na)\sin\alpha_ny$

$$(r_{15})_m = 0 \qquad ,(r_{25})_m = -(-1)^{m+1}\beta_m{}^2\left(\frac{\beta_m b \sinh \beta_m y}{\cosh \beta_m b} - \frac{\sinh \beta_m y + \beta_m y \cosh \beta_m y}{\sinh \beta_m b}\right)$$

A 6. Matrix-G_s

$$G_s = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} \\ & g_{22} & g_{23} & g_{24} & g_{25} \\ & & g_{33} & g_{34} & g_{35} \\ & SYM. & & g_{44} & g_{45} \\ & & & & g_{55} \end{bmatrix}$$

where g_{ij} ($i=1, 2, \ldots, 5$; $j=1, 2, \ldots, 5$) is square matrix of order n or m, then

$$g_{ij} = \begin{bmatrix} g_{ij(1,1)} & g_{ij(1,2)} & \cdots\cdots & g_{ij(1,n)} \\ g_{ij(2,1)} & g_{ij(2,2)} & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ g_{ij(n,2)} & g_{ij(n,2)} & \cdots\cdots & g_{ij(n,n)} \end{bmatrix}$$

and each element of submatrix is shown as follows :

$$g_{11(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b \coth \alpha_k a + GJ\,\alpha_k{}^4 b & (k=l) \end{cases}$$

$$g_{12(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b + GJ\,\alpha_k{}^4 b & (k=l) \end{cases}$$

$$g_{13(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b(2\coth \alpha_k a + \alpha_k a)\coth \alpha_k a + GJ\alpha_k{}^4 b(1 + \alpha_k a \coth \alpha_k a) & (k=l) \end{cases}$$

$$g_{14(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b(2 + \alpha_k a \coth \alpha_k a) + GJ\alpha_k{}^4 b(1 + \alpha_k a \tanh \alpha_k a) & (k=l) \end{cases}$$

$$g_{15(k,l)} = g_{25(k,l)}$$

$$= 4(-1)^{k+l} GJ \frac{\alpha_k{}^2 \beta_l{}^5 \coth \beta_l b}{(\alpha_k{}^2 + \beta_l{}^2)^2}$$

$$g_{22(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b \tanh^2 \alpha_k a + GJ\,\alpha_k{}^4 b & (k=l) \end{cases}$$

$$g_{23(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b(2 + \alpha_k a \tanh \alpha_k a) + GJ\alpha_k{}^4 b(1 + \alpha_k a \coth \alpha_k a) & (k=l) \end{cases}$$

$$g_{24(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b(2 + \alpha_k a \tanh \alpha_k a)\tanh \alpha_k a + GJ\alpha_k{}^4 b(1 + \alpha_k a \tanh \alpha_k a) & (k=l) \end{cases}$$

$$g_{33(k,l)} = \begin{cases} 0 & (k \neq l) \\ EI_x \alpha_k{}^4 b(2\coth \alpha_k a + \alpha_k a)^2 + GJ\alpha_k{}^4 b(1 + \alpha_k a \coth \alpha_k a)^2 & (k=l) \end{cases}$$

$$g_{34(k,\,l)}=\begin{cases}0 & (k\neq l)\\ EI_x\,\alpha_k^4 b(2\coth\alpha_k a+\alpha_k a)(2\tanh\alpha_k a+\alpha_k a)\\ \quad +GJ\,\alpha_k^4 b(1+\alpha_k a\coth\alpha_k a)(1+\alpha_k a\tanh\alpha_k a) & (k=l)\end{cases}$$

$$g_{35(k,\,l)}=4(-1)^{k+l}GJ\frac{\alpha_k^2\beta_l^5(1+\alpha_k a\coth\alpha_k a)\coth\beta_l b}{(\alpha_k^2+\beta_l^2)^2}$$

$$g_{44(k,\,l)}=\begin{cases}0 & (k\neq l)\\ EI_x\,\alpha_k^4 b(2\tanh\alpha_k a+\alpha_k a)^2+GJ\alpha_k^4 b(1+\alpha_k a\tanh\alpha_k a)^2 & (k=l)\end{cases}$$

$$g_{45(k,\,l)}=4(-1)^{k+l}GJ\frac{\alpha_k^2\beta_l^5(1+\alpha_k a\tanh\alpha_k a)\coth\beta_l b}{(\alpha_k^2+\beta_l^2)^2}$$

$$g_{55(k,\,l)}=\begin{cases}0 & (k\neq l)\\ GJ\,\beta_l^3\Bigg[\left(\dfrac{\beta_l b}{\cosh\beta_l b}-\dfrac{1}{\sinh\beta_l b}\right)^2\left(\dfrac{\sinh 2\beta_l b}{2}-\beta_l b\right)\\ \quad -\left(\dfrac{\beta_l b}{\sinh 2\beta_l b}-\dfrac{1}{2\sinh^2\beta_l b}\right)(2\beta_l b\cosh 2\beta_l b-\sinh 2\beta_l b)\\ \quad +\dfrac{1}{\sinh^2\beta_l b}\left\{\left(\dfrac{\beta_l^2 b^2}{2}+\dfrac{1}{4}\right)\sinh 2\beta_l b-\dfrac{\beta_l b}{2}\cosh 2\beta_l b+\dfrac{\beta_l^3 b^3}{3}\right\}\Bigg] & (k=l)\end{cases}$$