Correction to: Conley index theory without index pairs. I: The point-set level theory

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CORRECTION TO: CONLEY INDEX THEORY WITHOUT INDEX PAIRS. I: THE POINT-SET LEVEL THEORY

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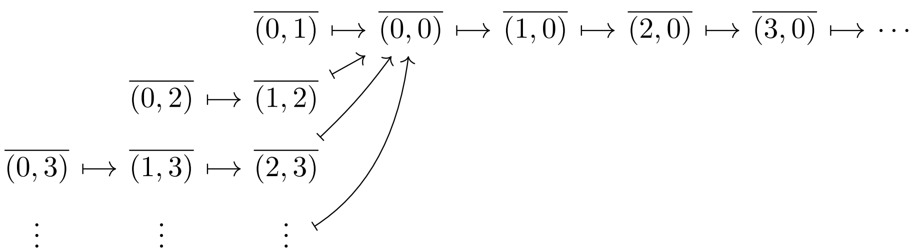
Lemmas 5.4 and 10.4 in the original article are wrong. The first and the third paragraphs of the proof of Lemma 5.4 are correct, hence the largest f-invariant subset of E exists and is contained in $I_f(E)$. However, the second paragraph is incorrect. In fact, the following counterexample exhibits that the subset $I_f(E)$ is not necessarily f-invariant:

Example 1. Let $X = (\mathbb{N} \times \mathbb{N})/{\sim}$, where the equivalence relation $\sim$ is defined by

$$(a, b) \sim (a', b') \iff ((a = a' \text{ and } b = b') \text{ or } (a \geqslant b \text{ and } a - a' = b - b')).$$

Define a self-map $f : X \to X$ by $f(\overline{(a, b)}) = \overline{(a + 1, b)}$. Thus, f can be described by the following diagram:



Put $E = X$. We see that

$$I_f(E) = \bigcap_{a,b\in\mathbb{N}} f^a \left(\bigcap_{i=0}^{a+b} f^{-i}(E) \right) = \bigcap_{a\in\mathbb{N}} f^a(X) = \left\{ \overline{(n, 0)} \;\middle|\; n \in \mathbb{N} \right\}$$

and hence

$$f(I_f(E)) = \left\{ \overline{(n, 0)} \;\middle|\; n \in \mathbb{N}_{\geqslant 1} \right\}.$$

Thus, we have $I_f(E) \neq f(I_f(E))$, i.e. $I_f(E)$ is not f-invariant.

Remark 2. It is obvious that $I_f(E)$ is a subset of E satisfying $I_f(E) \subset \operatorname{Dom} f$ and $f(I_f(E)) \subset I_f(E)$. The issue is that $I_f(E) \subset f(I_f(E))$ may not hold.

There is an analogous counterexample to Lemma 10.4:

Example 3. Put $X = (\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0})/{\sim}$, where $\sim$ is the equivalence relation defined by

$$(a, b) \sim (a', b') \iff ((a = a' \text{ and } b = b') \text{ or } (a \geqslant b \text{ and } a - a' = b - b')).$$

Define a semiflow $F : \mathbb{R}_{\geqslant 0} \times X \to X$ by $F(t, \overline{(a, b)}) = \overline{(a + t, b)}$. Put $E = X$. Then, $I_F(E)$ is not F-invariant.

Fortunately, the failures of Lemmas 5.4 and 10.4 do not affect the other parts of the original article. This is because the following two weaker lemmas, which are enough for our purpose, hold true:

Lemma 4. *Let $f\colon X \rightharpoonup X$ be a continuous partial self-map on a locally compact Hausdorff space X. Let K be a compact subset of $\operatorname{Dom} f$. Then, $I_f(K)$ is the largest f-invariant subset of K.*

Proof. It suffices to verify that $I_f(K) \subset f(I_f(K))$. In other words, it is enough to see that the intersection $f^{-1}(x) \cap I_f(K)$ is nonempty for any $x \in I_f(K)$. Observe that, for each $a, b \in \mathbb{N}$,

$$f^{-1}(x) \cap f^a\left(\bigcap_{i=0}^{a+b} f^{-i}(K)\right)$$

is a closed subset of K. Furthermore, it is nonempty since

$$x \in I_f(K) \subset f^{a+1}\left(\bigcap_{i=0}^{a+b} f^{-i}(K)\right) = f\left(f^a\left(\bigcap_{i=0}^{a+b} f^{-i}(K)\right)\right).$$

From Remark 5.3 in the original article and the compactness of K, we conclude that

$$f^{-1}(x) \cap I_f(K) = \bigcap_{a,b\in\mathbb{N}} f^{-1}(x) \cap f^a\left(\bigcap_{i=0}^{a+b} f^{-i}(K)\right) \neq \varnothing. \qquad \square$$

Lemma 5. *Let $F\colon \mathbb{R}_{\geqslant 0} \times X \rightharpoonup X$ be a continuous partial semiflow on a locally compact Hausdorff space X. Let K be a compact subset of X such that $[0, \varepsilon] \times K \subset \operatorname{Dom} F$ for some $\varepsilon \in \mathbb{R}_{>0}$. Then, $I_F(K)$ is the largest F-invariant subset of K.*

Proof. Using Lemma 9.2 in the original article and arguing as in the proof of Lemma 4, we see that $I_f(K) \subset f^t(I_F(K))$ (and hence, $I_f(K) = f^t(I_F(K))$) holds for $t \in [0, \varepsilon]$. This implies $I_f(K) = f^t(I_F(K))$ for any $t \in \mathbb{R}_{\geqslant 0}$. □

We finally remark that there is another important situation in which Lemmas 5.4 and 10.4 are true:

Definition 6. We say that a partial map $f\colon X \rightharpoonup Y$ is *injective* if it is injective as a map from $\operatorname{Dom} X$ to Y.

Lemma 7. *Let $f\colon X \rightharpoonup X$ be an injective partial self-map on a set X. Let E be a subset of X. Then, $I_f(E)$ is the largest f-invariant subset of E.*

Proof. If f is injective, the second paragraph of the proof of Lemma 5.4 in the original article is valid as written; direct images *under injective maps* commute with intersections. □

Lemma 8. *Let $F\colon \mathbb{R}_{\geqslant 0} \times X \rightharpoonup X$ be a partial semiflow on a set X such that $f^t\colon X \rightharpoonup X$ is injective for any $t \in \mathbb{R}_{\geqslant 0}$. Let E be a subset of X. Then, $I_F(E)$ is the largest F-invariant subset of E.*

Proof. The same as the proof of Lemma 7. □

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