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KUMAI, Toyoji
Research Institute for Applied Mechanics, Kyushu University : Professor

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NOTE

NOTES ON THE BENDING OF RECTANGULAR PLATE WITH VARIABLE STIFFNESS

By Toyoji KUMAI*

The paper shows theoretical study on the bending of rectangular plate with variable cross sections of concave in both directions and that of convex in one direction and remark on the plate with concave-convex in respective section are given. A numerical example for the plate of convex section with free-free edges is illustrated to corroborate the results of stress measurements of a marine propeller blade.

1. Introduction

The equation of bending of the rectangular plate with variable stiffness has presented by Olsson [1]. He investigated into the bending of the plate with linearly varying stiffness in one direction. The simpler method for evaluating Olsson's equation was given by Reissner [2].

Present paper shows analytical solutions of the bending of the rectangular plate with variable stiffnesses in two directions along principal axes. The variational functions of the stiffness are assumed to be of exponential in both directions, of trigonometric in one direction and of exponential-linear in respective direction. Some remarks on the equation presented by Olsson are given in the last section. The object of the present investigation is theoretical confirmation of the results of stress measurements of a marine propeller blade.

2. Fundamental Equation

The flexural bending moments in x and y directions and twisting moment of a rectangular plate are respectively written by

$$\begin{aligned} M_x &= -N\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right), \\ M_y &= -N\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right), \\ H &= -(1-\nu)\, N\frac{\partial^2 w}{\partial x \partial y}, \end{aligned} \tag{1}$$

*Professor, Research Institute for Applied Mechanics, Kyushu University, Fukuoka, Japan.

where, N is the stiffness which varying in both directions, w is the deflection of the plate and ν is Poisson's ratio.

The equation of equilibrium of the rectangular plate is presented by

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 H}{\partial x \partial y}+\frac{\partial^2 M_y}{\partial y^2}=-p(xy), \tag{2}$$

where, $p\,(xy)$ is the normal load distributed in x and y directions.

For obtaining the fundamental equation of the bending of the plate considered, substitute (1) into (2), then we have the following equation.

$$\begin{aligned} &N\Delta\Delta w+2\frac{\partial N}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+2\frac{\partial N}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\\ &+\frac{\partial^2 N}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+2\,(1-\nu)\frac{\partial^2 N}{\partial x\partial y}\,\frac{\partial^2 w}{\partial x \partial y}+\frac{\partial^2 N}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\\ &=p(xy), \end{aligned} \tag{3}$$

where, $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

3. Solutions of Equation

3.1. Stiffness varying with exponential functions in both directions

Now we assume that the stiffness distributions of the plate is presented by exponential functions that,

$$N=N_0\ exp(kx+ly), \tag{4}$$

where, N_0 is the stiffness of the plate at the origine of the co-ordinate and k and l are the rates of increase of stiffnesses along x and y axes respectively.

It becomes clear that the variation of thickness of the plate along axes are proportional to the cube root of the function of stiffness of the plate that,

$$h=h_0\ exp\left(\frac{kx+ly}{3}\right), \tag{5}$$

where, h_0 is the thickness of the plate at the origine, thus the cross sections of two directions are both concave.

For evaluating equation (3), two opposit edges $x=0$ and $x=a_1$ are assumed to be of free support, so the deflection of the plate is written by

$$w=\sum_n X_n Y_n=\sum_n exp\ (i\alpha_n x)\ Y_n, \tag{6}$$

where, $\alpha_n=\frac{n\pi}{a_1}$,

a_1=length of span between $x=0$ and $x=a_1$.

For obtaining the ordinary differential equation with respect to Y_n, substitute from (4) and (6) into (3), if so the two equations are presented, in which one is that obtained from real part and another imaginary part, these are respectively written by

$$Y_n^{IV}+2lY_n'''-\{2\alpha_n^2-(\nu k^2+l^2)\}Y_n''-2l\nu\alpha_n^2Y_n' \\ +\alpha_n^2\{\alpha_n^2-(k^2+\nu l^2)\}Y_n=0, \tag{7}$$

$$\nu Y''_n+(1-\nu)lY_n'-\alpha_n^2Y_n=0. \tag{8}$$

By the use of (8), equation (7) is simplified as follows;

$$Y_n^{IV}+2lY_n'''-(2\alpha_n^2-l^2)Y_n''-l\{(1-\nu)k^2+2\nu\alpha_n^2\}Y_n' \\ +\alpha_n^2(\alpha_n^2-\nu l^2)Y_n=0. \tag{9}$$

We may easily evaluate the above differential equation by obtaining four roots of the following quadratic equation,

$$\lambda^4+a\lambda^3+b\lambda^2+c\lambda+d=0, \tag{10}$$

where,
$$a=2l,$$
$$b=-(2\alpha_n^2-l^2),$$
$$c=-l\{(1-\nu)k^2+2\nu\alpha_n^2\},$$
$$d=\alpha_n^2(\alpha_n^2-\nu l^2).$$

For calculating the roots of equation (10), we may firstly inspect the discriminant of the roots of the quadratic equation in general case. The

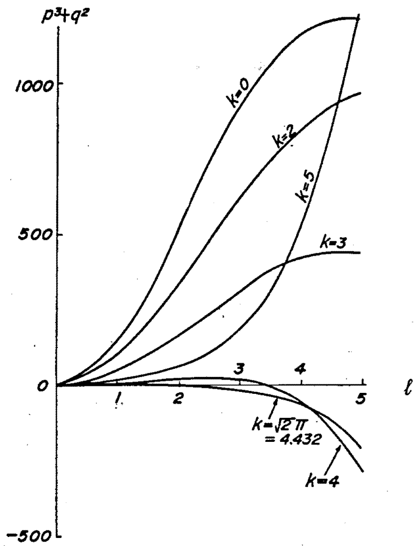

Fig. 1 Discriminants of quadratic equation (10) versus l with parameter k in $n=1$ and $\nu=1/6$.

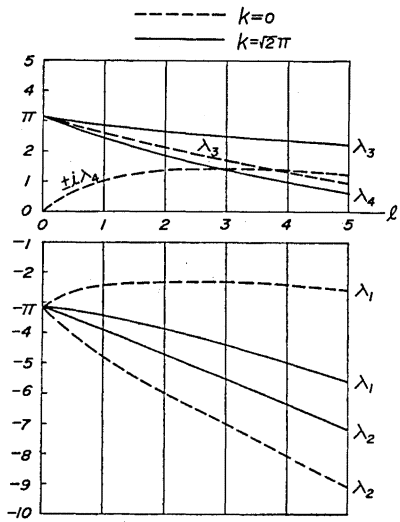

Fig. 2 Variation of four roots of equation (10) versus l with $k=0$ and $k=\sqrt{2}\,\pi$.

numerical values of the discriminant for given data, $n=1$ and $\nu=1/6$, are shown in Figure 1 as an example. If the sign of the discriminant takes positive for given values of k and l, two real roots and conjugate complex roots are obtained, and the solutions of equation (9) is thus written by

$$Y_n=Ae^{\lambda_1 y}+Be^{\lambda_2 y}+Ce^{\lambda_3 y}\ cos\lambda_4 y+De^{\lambda_3 y}\ sin\lambda_4 y+Y_p=0, \tag{11}$$

where, Y_p is the particular solution of equation (3).

The λ-values for $k=0$ in the above equation with several values l are shown in Figure 2.

When the sign of the discriminant becomes negative, for instance, $k=\sqrt{2}\pi$ for any vlues of l, all the roots of equation (10) become real values, the solutions are thus presented by

$$Y_n=Ae^{\lambda_1 y}+Be^{\lambda_2 y}+Ce^{\lambda_3 y}+De^{\lambda_4 y}+Y_p. \tag{12}$$

The λ-values in such a case as mentioned above are also shown in Figure 2.

As was seen in Figure 2, when l approaches zero, the real values λ approach to $\pm\pi$ and the imaginary numbers tend to zero in all values of k, then in the extreme case $l=0$, we have the well known solutions [3] that,

$$Y_n=(A+By)e^{-\pi y}+(C+Dy)e^{\pi y}+Y_p. \tag{13}$$

The above solutions are that of the bending of plate with a uniform thickness in the case $n=1$. It is to be noted that k-value is independent of the solutions when l takes zero.

A particuler case for obtaining four real roots is presented here as follows; For eliminating the second term of equation (10), put

$$\lambda = z - \frac{a}{4}, \tag{a}$$

and substitute (a) into (10), we have,

$$z^4 + \alpha z^2 + \beta z + \gamma = 0, \tag{b}$$

where,
$$\alpha = -2\left(\alpha_n{}^2 + \frac{l^2}{4}\right),$$
$$\beta = (1-\nu)\,l\,(2\alpha_n{}^2 - k^2),$$
$$\gamma = \left(\alpha_n{}^2 - \frac{l^2}{4}\right)^2 + \frac{(1-\nu)}{2}\,l^2k^2.$$

In the above equation, if k is given equal to $\sqrt{2}\,\alpha_n$, β vanishes from equation (b), so we have

$$z^4 + \alpha z^2 + \gamma = 0. \tag{c}$$

The four real roots of the above equation are easily obtained that,

$$\lambda_{1\sim4} = \pm\sqrt{\alpha_n{}^2 + \frac{l^2}{4} \pm 2\sqrt{\nu}\,\alpha_n l} - \frac{l}{2}. \tag{d}$$

The numerical values of λ obtained from (d) for given values $n=1$ and $\nu=1/6$ of four real roots are shown in Figure 2 as an example.

3.2. Stiffness varying with trigonometric function in one direction

The stiffness distributions in this case are assumed to be as follows;

$$N = N_0 exp(kx + i\beta y), \tag{14}$$

where, $\beta = \dfrac{\pi}{l_1}$, $\quad l_1 > b$

N_0 is the stiffness at the origine of the co-ordinate provided that the origine is taken at the centre of figure.

Now, we assume that the deflection of the plate is the same expression as (6) in the previous section. Thus, substitute from (6) and (14) into (3), the equations obtained from real part and imaginary part are respectively written by

$$Y_n{}^{IV} - (2\alpha_n{}^2 - \nu k^2 + \beta^2)\,Y_n'' + 2(1-\nu)\,\alpha_n\beta k Y_n' + \alpha_n{}^2(\alpha_n{}^2 - k^2 + \nu\beta^2)\,Y_n = 0, \tag{15}$$

$$k\alpha_n(\nu Y_n''-\alpha_n^2 Y_n)-\beta(Y_n'''-\nu\alpha_n^2 Y_n')=0. \tag{16}$$

By substitution from (16) into (15), the equation becomes the following form,

$$Y_n^{IV}+\frac{\beta k}{\alpha_n}Y_n'''-(2\alpha_n^2+\beta^2)Y_n''+(2-3\nu)\alpha_n\beta kY_n'+\alpha_n^2(\alpha_n^2+\nu\beta^2)Y_n=0. \tag{17}$$

The above equation is solved by the similar method as that in the previous section.

As a particular case, if we consider that k takes zero, the equation becomes,

$$Y_n^{IV}-(2\alpha_n^2+\beta^2)Y_n''+\alpha_n^2(\alpha_n^2+\nu\beta^2)Y_n=0. \tag{18}$$

The solutions of the above equation is easily obtained as follows;

$$Y_n=ACos\lambda_1y+BSin\lambda_1y+CCos\lambda_2y+DSin\lambda_2y+Yp, \tag{19}$$

where, $$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\}=\sqrt{\frac{1}{2}\left\{2+\left(\frac{\beta}{\alpha_n}\right)^2\pm\frac{\beta}{\alpha_n}\sqrt{4(1-\nu)+\left(\frac{\beta}{\alpha_n}\right)^2}\right\}}\ \alpha_n.$$

When the cross section of the plate and also the load distribution are both assumed to be of symmetric about $y=0$, then, odd functions are vanish, the solutions are thus represented by

$$Y_n=ACos\lambda_1y+CCos\lambda_2y+Y_p. \tag{20}$$

On the other hand, the thickness distribution of the plate considered is presented by the cube root of cosine curve with a half wave length l_1, in which the thickness ratio of edges to that at origine is given by

$$\underset{y=\pm\frac{b}{2}}{h}=h_0\left(cos\frac{\pi b}{2l_1}\right)^{\frac{1}{3}}. \tag{21}$$

If l_1 is assumed closely near to b, the cross section shows convex such as a symmetric aerofoil section as seen in Figure 4.

3.3. Stiffness varying with linear and exponential in respective direction

Stiffness of the plate under consideration is assumed as follows;

$$N=N_0\left(1+\frac{N_1}{N_0}y\right)\ exp(kx). \tag{22}$$

Substitute (6) and (22) into the homogeneous equation of (3) and using the equation obtained from the imaginary part, we have

$$N(Y_n^{IV}-2\alpha_n^2Y_n''+\alpha_n^4Y_n)+2\frac{dN}{dy}\left\{Y_n'''-\left(\nu\alpha_n^2+\frac{(1-\nu)k^2}{2}\right)Y_n'\right\}=0. \tag{23}$$

As a particular case, if we assume $k=0$ in the above equation, the result should be agree with that in Olsson's investigation. However, it should be noted that his equation does not agree with that obtained from the fundamental equation or (23) in the last term, that

$$N\Delta\Delta w+2\frac{\partial N}{\partial y}\frac{\partial}{\partial y}\Delta w=p\,(xy)\,; \qquad \text{given by Olsson [1]}, \tag{24}$$

$$N\Delta\Delta w+2\frac{\partial N}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)=p\,(xy)\,; \quad \text{otained from (3)}. \tag{25}$$

The above two equations are identical provided that $\nu=1$. The solutions of equation (23) in general case or even in the case of $k=0$ will be not so easily obtained such as that evaluated by Reissner [2].

When k-value is assumed to be $k=\sqrt{2}\,\pi$, ($\alpha_n=\pi$, when $n=1$), as a particular case, the coefficient of the last term in equation (23) becomes

$$\left\{\nu\alpha_n^2+\frac{(1-\nu)k^2}{2}\right\}_{k=\sqrt{2}\alpha_n}=\alpha_n^2. \tag{26}$$

Substitute (26) into (23), we have the same equation as that investigated by Olsson [1]. It is to be noted that k-value should be given by $k=\sqrt{2}\,\pi$, but not equal to zero, so that the distributions of stiffness in Olsson's investigation are exponential-linear in respective direction, but not constant stiffness in x-axis as seen in Figure 3.

The method of evaluation of the equation in the case $k=\sqrt{2}\pi$ mentioned above is clearly shown by Reissner [2][3], and solutions of the above equation are discussed in detail by Olsson [1].

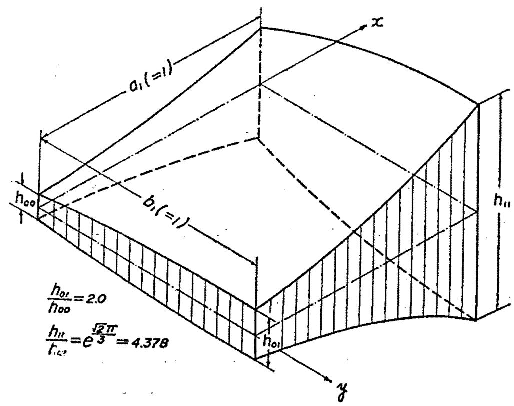

Fig. 3 Plate of concave-convex cross sections discussed in Section 3.3.

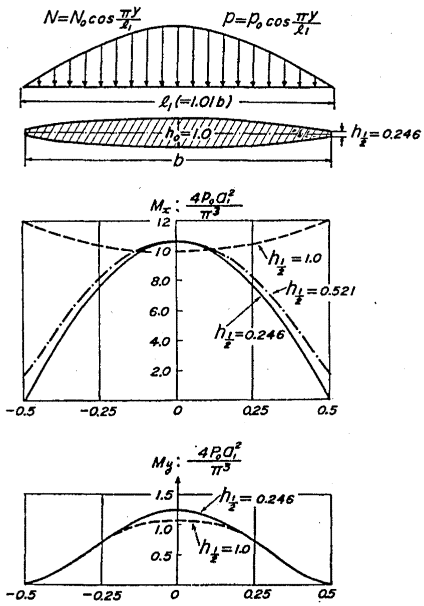

Fig. 4 Distributions of load and bending moments of plate with stiffness varying by cosine curve with boundary condition of free-free edges.

4. Numerical Example

An illustration of numerical calculations on the bending moments of the plate of stiffness varying with trigonometric function in y-axis and constant stiffness in x-axis are shown. In this example, the normal load is assumed to be the same distribution as that of the stiffness and the boundary condition is assumed both edges free. The results are compared with that of the plate of uniform thickness as seen in Figure 4. So far as concern to the distributions of M_x, there seems clearly that M_x distribute almost proportional to the stiffness distribution in the present case. The above result is theoretically verified the measured results of stresses on the surface of a marine propeller blade to some extent.

5. Conclusions

The bending of rectangular plate, of which stiffness varying with exponential functions in both directions and with trigonometric function in one direction are investigated, and some remarks are given to the plate of liniealy varying stiffness investigated by Olsson.

As a conclusive evidence, when the plate with the convex section such as an aerofoil is under the condition of free-free edges, the bending moment distribution along the chord is almost proportional to the stiffness of the plate, whereas the corresponding moment is assumed to be constant in the beam theory. The above result will be theoretically corroborated to some extent the stress measurements of a marine propeller blade.

Further study on the bending of the curved plate with variable stiffness required for verifying the experimental results of propeller blade stresses.

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