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RELATIONAL CALCULUS AS A FORMAL SYSTEM

By

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Abstract

In model theory, function symbols for formal systems are usually assumed to be totally defined. This paper reports that category theory is defined and discussed as a formal system with partial operations in first order predicate logic. We also review a formal definition of relational calculus with a local boolean structure and satisfying de Morgan-Schröder equivalences.

Key Words and Phrases: Category theory, Formal system, Relational calculus, de Morgan-Schröder equivalence.

1. Introduction

Relational calculus (theory of binary relations) was founded by [de Morgan (1864)], [Peirce (1883)] and [Schröder (1895)]. [Freyd and Scedrov (1990)] presented theory of allegories as a categorical relational calculus improving relation algebras by [Tarski (1941)]. Recently [Benzmüller and Scott (2018)] and [Benzmüller et al. (2020)] investigate automatic consistency checking of axiom systems for category theory.

Relational calculus has been applied to a maximum principle [Berge (1963)] in mathematical economy, Hoare logic [Hoare and Jifeng (1986)] verifying programs, graph transformations [Ehrig et al. (2010), Mizoguchi and Kawahara (1995)] by graph grammars, continuous transition relations [Furusawa et al. (2012)] of non-deterministic cellular automata, and formal concept analysis [Ganter and Wille (1999)]. Our research aims development of softwares for relational calculus, formal verification of small-scale but important programs for medical devices concerned with human life, analysis of graph structures from big data, and so on.

In model theory, function symbols for formal systems are usually assumed to be totally defined. For example, [Burmeister (1986)] and [Robinson (1989)] have been studied general theory of partial algebras. The present paper reports that category theory [Mac Lane (1971)] is defined and discussed as a formal system with partial operations (partial function symbols) in first-order predicate logic. At the end of the paper, as a generalization of relation algebras [Tarski (1941)], we will mention a foundation of relational calculus with a local boolean structure and satisfying de Morgan-Schröder equivalences.

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2. Category theory as a formal system

First of all we assume (classical) first order predicate logic with logical symbols $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence), $\forall$ (universal quantifier) and $\exists$ (existential quantifier). Of course we use the equality symbol $=$ as a primitive predicate. Category theory as a formal system [Freyd and Scedrov (1990), Scott (1979)] has two unary operations $^{\ominus}$ (source) and $^{\oplus}$ (target), and a binary operation $\cdot$ (composition).

As usual, *terms* in category theory are recursively defined as follows:

(a) Each variable x and constant c are terms.

(b) If t is a term, then so are $t^{\ominus}$ and $t^{\oplus}$.

(c) If t and s are terms, then so are $t \cdot s$.

Also, formulas are recursively defined as follows.

(a) If t and s are terms, then the equation $\langle\, t = s \,\rangle$ is a formula.

(b) If ϕ and ψ are formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$.

(c) If x is a variable and ϕ is a formula, then $\forall x.\ \phi$ and $\exists x.\ \phi$ are formulas.

The first axioms in the paper are axioms for equality.

AXIOM 2.1 AXIOMS FOR EQUALITY.

[eq1] $\langle\, x = y \,\rangle \rightarrow \langle\, y = x \,\rangle$, (symmetric law)

[eq2] $\langle\, x = y \,\rangle \wedge \langle\, y = z \,\rangle \rightarrow \langle\, x = z \,\rangle$. (transitive law) □

We do not assume the reflexive law $\langle\, x = x \,\rangle$. The equation $\langle\, t = t \,\rangle$ will be used to suggest the existence (or, definability) of a term t, and $\langle\, t = t \,\rangle$ is abbreviated as $\langle\, t \,\rangle$. Clearly [eq0] $\langle\, x = y \,\rangle \rightarrow \langle\, x \,\rangle \wedge \langle\, y \,\rangle$ follows [eq1] and [eq2]. This means that the expression $\langle\, t = s \,\rangle$ presents not only the equality but also the equational existence of terms t and s. Kleene equality $\langle\, x \simeq y \,\rangle$ is an abbreviation [Scott (1979)] of a formula $\langle\, x \,\rangle \vee \langle\, y \,\rangle \rightarrow \langle\, x = y \,\rangle$. Obviously $\langle\, x \simeq y \,\rangle$ and $(\langle\, x \,\rangle \leftrightarrow \langle\, y \,\rangle) \wedge (\langle\, x \,\rangle \rightarrow \langle\, x = y \,\rangle) \rightarrow \langle\, x \simeq y \,\rangle$ hold.

3. Axioms for categories

Category theory satisfies the following axioms. In what follows the composition symbol $\cdot$ will be omitted.

AXIOM 3.1 AXIOMS FOR CATEGORIES.

[sbn$^{\ominus}$] $\langle\, x = y \,\rangle \rightarrow \langle\, x^{\ominus} = y^{\ominus} \,\rangle$, (substitution)

[sbn$^{\oplus}$] $\langle\, x = y \,\rangle \rightarrow \langle\, x^{\oplus} = y^{\oplus} \,\rangle$,

[def^{C}] $\langle\, xy \,\rangle \rightarrow \langle\, x^{\oplus} = y^{\ominus} \,\rangle$, (definability)

[sbn^{C}] $\langle\, x = y \,\rangle \wedge \langle\, xz \,\rangle \rightarrow \langle\, xz = yz \,\rangle$, (substitution)

$\langle\, x = y \,\rangle \wedge \langle\, zx \,\rangle \rightarrow \langle\, zx = zy \,\rangle$,

[id$^{\ominus}$] $\langle\, x^{\ominus} \,\rangle \rightarrow \langle\, x^{\ominus}x = x \,\rangle$, (identity)

[id$^{\oplus}$] $\langle\, x^{\oplus} \,\rangle \rightarrow \langle\, xx^{\oplus} = x \,\rangle$,

[ass^{C}] $\langle\, xy \,\rangle \wedge \langle\, yz \,\rangle \rightarrow \langle\, (xy)z = x(yz) \,\rangle$. (associative) □

The above axioms slightly modify ones of [Freyd and Scedrov (1990)]. Cf. [Benzmüller and Scott (2018), Benzmüller et al. (2020)].

As usual, a dual t^* of a term t is recursively defined as follows: $x^* \equiv x$ for a variable x, $c^* \equiv c$ for a constant symbol c, and $(t^{\ominus})^* \equiv (t^*)^{\oplus}$, $(t^{\oplus})^* \equiv (t^*)^{\ominus}$ and $(ts)^* \equiv s^*t^*$ for terms t and s. Also a dual ϕ^* of a formula ϕ is recursively defined as follows: $\langle\, t = s \,\rangle^* \equiv \langle\, t^* = s^* \,\rangle$ for terms t and s, and $(\phi \wedge \psi)^* \equiv \phi^* \wedge \psi^*$, $(\phi \vee \psi)^* \equiv \phi^* \vee \psi^*$, $(\neg\phi)^* \equiv \neg\phi^*$, $(\phi \to \psi)^* \equiv \phi^* \to \psi^*$ and $(\phi \leftrightarrow \psi)^* \equiv \phi^* \leftrightarrow \psi^*$ for formulas ϕ and ψ. Where the symbol $\equiv$ denotes the syntax equality for terms and formulas.

The axioms 3.1 for category theory consist of dual pairs of formulas. For example, $[\mathrm{sbn}^{\oplus}]^* \leftrightarrow [\mathrm{sbn}^{\ominus}]$, $[\mathrm{def}^{C}]^* \leftrightarrow [\mathrm{def}^{C}]$ and $[\mathrm{ass}^{C}]^* \leftrightarrow [\mathrm{ass}^{C}]$. Therefore, if a formula ϕ is provable (using 2.1 and 3.1), then so is the dual ϕ^*.

Proposition 3.2. In category theory the following hold.

$$
\begin{array}{cl}
(\mathrm{def}^{\ominus}) & \langle\, x^{\ominus} \,\rangle \leftrightarrow \langle\, x \,\rangle, \\
(\mathrm{def}^{\oplus}) & \langle\, x^{\oplus} \,\rangle \leftrightarrow \langle\, x \,\rangle, \\
(^{\ominus\oplus}) & \langle\, x^{\ominus\oplus} \simeq x^{\ominus} \,\rangle, \\
(^{\oplus\ominus}) & \langle\, x^{\oplus\ominus} \simeq x^{\oplus} \,\rangle, \\
(^{\ominus\ominus}) & \langle\, x^{\ominus\ominus} \simeq x^{\ominus} \,\rangle, \\
(^{\oplus\oplus}) & \langle\, x^{\oplus\oplus} \simeq x^{\oplus} \,\rangle, \\
(\mathrm{reg}^{C}) & \langle\, xy \,\rangle \to \langle\, x \,\rangle \wedge \langle\, y \,\rangle, \\
(\mathrm{C}^{\ominus}) & \langle\, xy \,\rangle \to \langle\, (xy)^{\ominus} = x^{\ominus} \,\rangle, \\
(\mathrm{C}^{\oplus}) & \langle\, xy \,\rangle \to \langle\, (xy)^{\oplus} = y^{\oplus} \,\rangle, \\
(\mathrm{id}^{\ominus}_{*}) & \langle\, y^{\ominus}x \,\rangle \to \langle\, y^{\ominus}x = x \,\rangle, \quad \langle\, y^{\oplus}x \,\rangle \to \langle\, y^{\oplus}x = x \,\rangle, \\
(\mathrm{id}^{\oplus}_{*}) & \langle\, xy^{\oplus} \,\rangle \to \langle\, xy^{\oplus} = x \,\rangle, \quad \langle\, xy^{\ominus} \,\rangle \to \langle\, xy^{\ominus} = x \,\rangle, \\
(\mathrm{def}^{C}_{*}) & \langle\, xy \,\rangle \leftrightarrow \langle\, x^{\oplus} = y^{\ominus} \,\rangle, \\
(\mathrm{sbn}^{C}_{*}) & \langle\, x = y \,\rangle \wedge \langle\, z = w \,\rangle \to \langle\, xz \simeq yw \,\rangle, \\
 & \langle\, x = y \,\rangle \wedge \langle\, z = w \,\rangle \to \langle\, zx \simeq wy \,\rangle, \\
(\mathrm{ass}^{C}_{*}) & \langle\, (xy)z \simeq x(yz) \,\rangle.
\end{array}
$$

Proof. $(\mathrm{def}^{\ominus})$ $\langle\, x^{\ominus} \,\rangle \leftrightarrow \langle\, x \,\rangle$:

$$
\begin{array}{cll}
\langle\, x^{\ominus} \,\rangle & \to \langle\, x^{\ominus}x = x \,\rangle & \{\ [\mathrm{id}^{\ominus}]\ \} \\
 & \to \langle\, x = x \,\rangle & \{\ [\mathrm{eq0}]\ \} \\
 & \to \langle\, x^{\ominus} = x^{\ominus} \,\rangle & \{\ [\mathrm{sbn}^{\ominus}]\ \} \\
 & \to \langle\, x^{\ominus} \,\rangle. &
\end{array}
$$

Note that $(\mathrm{def}^{\ominus})$ and $(\mathrm{def}^{\oplus})$ imply the following equivalences.

$$\langle\, x \,\rangle \leftrightarrow \langle\, x^{\ominus} \,\rangle \leftrightarrow \langle\, x^{\oplus} \,\rangle \leftrightarrow \langle\, x^{\ominus\ominus} \,\rangle \leftrightarrow \langle\, x^{\ominus\oplus} \,\rangle \leftrightarrow \langle\, x^{\oplus\ominus} \,\rangle \leftrightarrow \langle\, x^{\oplus\oplus} \,\rangle.$$

For example, $\langle\, x^{\ominus} \,\rangle \leftrightarrow \langle\, x^{\ominus\oplus} \,\rangle$ can be deduced by substituting $x^{\ominus}$ for x in $\langle\, x \,\rangle \leftrightarrow \langle\, x^{\oplus} \,\rangle$.

$(^{\ominus\oplus})$ $\langle\, x^{\ominus\oplus} \simeq x^{\ominus} \,\rangle$:

$$\begin{array}{llll} \langle\, x \,\rangle & \to & \langle\, x^{\ominus} \,\rangle & \{\ (\mathrm{def}^{\ominus})\ \} \\ & \to & \langle\, x^{\ominus}x = x \,\rangle & \{\ [\mathrm{id}^{\ominus}]\ \} \\ & \to & \langle\, x^{\ominus}x \,\rangle & \{\ [\mathrm{eq0}]\ \} \\ & \to & \langle\, x^{\ominus\oplus} = x^{\ominus} \,\rangle. & \{\ [\mathrm{def}^{C}]\ \} \end{array}$$

$(^{\ominus\ominus})$ $\langle\, x^{\ominus\ominus} \simeq x^{\ominus} \,\rangle$:

$$\begin{array}{llll} \langle\, x \,\rangle & \to & \langle\, x^{\ominus} = x^{\ominus\oplus} \,\rangle \wedge \langle\, x^{\ominus\oplus\ominus} = x^{\ominus\oplus} \,\rangle & \{\ (^{\ominus\oplus}),\ (^{\oplus\ominus})\ \} \\ & \to & \langle\, x^{\ominus\ominus} = x^{\ominus\oplus\ominus} \,\rangle \wedge \langle\, x^{\ominus} = x^{\ominus\oplus} \,\rangle \wedge \langle\, x^{\ominus\oplus\ominus} = x^{\ominus\oplus} \,\rangle & \{\ [\mathrm{sbn}^{\ominus}]\ \} \\ & \to & \langle\, x^{\ominus\ominus} = x^{\ominus} \,\rangle. & \end{array}$$

(reg^{C}) $\langle\, xy \,\rangle \to \langle\, x \,\rangle \wedge \langle\, y \,\rangle$:

$$\begin{array}{llll} \langle\, xy \,\rangle & \to & \langle\, x^{\oplus} = y^{\ominus} \,\rangle & \{\ [\mathrm{def}^{C}]\ \} \\ & \to & \langle\, x^{\oplus} \,\rangle \wedge \langle\, y^{\ominus} \,\rangle & \{\ [\mathrm{eq0}]\ \} \\ & \to & \langle\, x \,\rangle \wedge \langle\, y \,\rangle. & \{\ (\mathrm{def}^{\oplus}),\ (\mathrm{def}^{\ominus})\ \} \end{array}$$

$(\mathrm{C}^{\ominus})$ $\langle\, xy \,\rangle \to \langle\, (xy)^{\ominus} = x^{\ominus} \,\rangle$:

$$\begin{array}{llll} \langle\, xy \,\rangle & \to & \langle\, x \,\rangle \wedge \langle\, xy \,\rangle & \{\ (\mathrm{reg}^{C})\ \} \\ & \to & \langle\, x^{\ominus}x \,\rangle \wedge \langle\, xy \,\rangle & \{\ (\mathrm{def}^{\ominus}),\ [\mathrm{id}^{\ominus}]\ \} \\ & \to & \langle\, (x^{\ominus}x)y = x^{\ominus}(xy) \,\rangle & \{\ [\mathrm{ass}^{C}]\ \} \\ & \to & \langle\, x^{\ominus}(xy) \,\rangle & \{\ [\mathrm{eq0}]\ \} \\ & \to & \langle\, x^{\ominus\oplus} = (xy)^{\ominus} \,\rangle & \{\ [\mathrm{def}^{C}]\ \} \\ & \to & \langle\, x^{\ominus} = (xy)^{\ominus} \,\rangle. & \{\ (^{\ominus\oplus})\ \} \end{array}$$

$(\mathrm{id}^{\ominus}_{*})$ $\langle\, y^{\ominus}x \,\rangle \to \langle\, y^{\ominus}x = x \,\rangle$:

$$\begin{array}{llll} \langle\, y^{\ominus}x \,\rangle & \to & \langle\, y^{\ominus\oplus} = x^{\ominus} \,\rangle \wedge \langle\, y^{\ominus}x \,\rangle & \{\ [\mathrm{def}^{C}]\ \} \\ & \to & \langle\, y^{\ominus} = x^{\ominus} \,\rangle \wedge \langle\, y^{\ominus}x \,\rangle & \{\ (^{\ominus\oplus})\ \} \\ & \to & \langle\, y^{\ominus}x = x^{\ominus}x \,\rangle & \{\ [\mathrm{sbn}^{C}]\ \} \\ & \to & \langle\, y^{\ominus}x = x \,\rangle. & \{\ [\mathrm{id}^{\ominus}]\ \} \end{array}$$

Note that $\langle\, y^{\oplus}x \,\rangle \to \langle\, y^{\oplus}x = x \,\rangle$ follows $(^{\oplus\ominus})$ and $(\mathrm{id}^{\ominus}_{*})$.

(def^{C}_{*}) $\langle\, xy \,\rangle \leftrightarrow \langle\, x^{\oplus} = y^{\ominus} \,\rangle$:
An implication $\langle\, xy \,\rangle \to \langle\, x^{\oplus} = y^{\ominus} \,\rangle$ has been given as $[\mathrm{def}^{C}]$. We need to show the converse implication $\langle\, x^{\oplus} = y^{\ominus} \,\rangle \to \langle\, xy \,\rangle$.

$$\begin{array}{llll} \langle\, x^{\oplus} = y^{\ominus} \,\rangle & \to & \langle\, x = xx^{\oplus} \,\rangle \wedge \langle\, y^{\ominus}y = y \,\rangle \wedge \langle\, x^{\oplus} = y^{\ominus} \,\rangle & \{\ [\mathrm{id}^{\oplus}],\ [\mathrm{id}^{\ominus}]\ \} \\ & \to & \langle\, x = xx^{\oplus} \,\rangle \wedge \langle\, x^{\oplus}y = y^{\ominus}y \,\rangle & \{\ [\mathrm{sbn}^{C}]\ \} \\ & \to & \langle\, x = xx^{\oplus} \,\rangle \wedge \langle\, (xx^{\oplus})y = x(x^{\oplus}y) \,\rangle & \{\ [\mathrm{ass}^{C}]\ \} \\ & \to & \langle\, xy = (xx^{\oplus})y \,\rangle & \{\ [\mathrm{sbn}^{C}]\ \} \\ & \to & \langle\, xy \,\rangle. & \{\ [\mathrm{eq0}]\ \} \end{array}$$

(sbn^C_*) $\langle x = y \rangle \wedge \langle z = w \rangle \to \langle xz \simeq yw \rangle$:

$$\begin{array}{lll} & \langle xz \rangle \wedge \langle x = y \rangle \wedge \langle z = w \rangle & \\ \to & \langle xz = yz \rangle \wedge \langle z = w \rangle & \{ [\mathrm{sbn}^C] \} \\ \to & \langle xz = yz \rangle \wedge \langle yz = yw \rangle & \{ [\mathrm{sbn}^C] \} \\ \to & \langle xz = yw \rangle. & \{ [\mathrm{eq2}] \} \end{array}$$

(ass^C_*) $\langle (xy)z \simeq x(yz) \rangle$:

$$\begin{array}{llll} \langle (xy)z \rangle & \to & \langle (xy)^\oplus = z^\ominus \rangle & \{ [\mathrm{def}^C] \} \\ & \to & \langle xy \rangle \wedge \langle (xy)^\oplus = z^\ominus \rangle & \{ (\mathrm{def}^\oplus) \} \\ & \to & \langle xy \rangle \wedge \langle y^\oplus = z^\ominus \rangle & \{ (\mathrm{C}^\oplus) \} \\ & \to & \langle xy \rangle \wedge \langle yz \rangle. & \{ (\mathrm{def}^C_*) \} \end{array}$$

In a similar way we prove $\langle x(yz) \rangle \to \langle xy \rangle \wedge \langle yz \rangle$. Hence the statement is true by $[\mathrm{ass}^C]$. □

In category theory a term t is called a *morphism* if $\langle t \rangle$ holds. By $(\mathrm{def}^\ominus)$ a term t is a morphism if and only if so is $t^\ominus$ (or $t^\oplus$). Thus the source $^\ominus$ and the target $^\oplus$ are totally defined on morphisms. On the other hands, a term rs is a morphism (or, terms r and s are composable) if and only if $\langle r^\oplus = s^\ominus \rangle$. The fact indicates that the composition $\cdot$ may be partially defined. A morphism t is called an *identity morphism* (or object) if $\langle t = t^\ominus \rangle$. It is trivial that both of $t^\ominus$ and $t^\oplus$ are identity morphisms for all morphisms t. A morphism t is an *endomorphism* if $\langle t^\ominus = t^\oplus \rangle$. All identity morphisms are endomorphisms, since $\langle t^{\ominus\ominus} = t^\ominus = t^{\ominus\oplus} \rangle$ and $\langle t^{\oplus\ominus} = t^\oplus = t^{\oplus\oplus} \rangle$.

We will abbreviate a formula $\langle r^\ominus = s^\ominus \rangle \wedge \langle r^\oplus = s^\oplus \rangle$ as $\langle r \parallel s \rangle$. A pair of morphisms r and s satisfying $\langle r \parallel s \rangle$ is called *parallel*.

PROPOSITION 3.3. In category theory the following hold.

$$\begin{array}{lll} (\mathrm{reg}^\parallel) & \langle x \parallel y \rangle \to \langle x \rangle \wedge \langle y \rangle, & \text{(regular)} \\ (\mathrm{ref}^\parallel) & \langle x = y \rangle \to \langle x \parallel y \rangle, & \text{(reflexive)} \\ (\mathrm{sym}^\parallel) & \langle x \parallel y \rangle \to \langle y \parallel x \rangle, & \text{(symmetric)} \\ (\mathrm{trn}^\parallel) & \langle x \parallel y \rangle \wedge \langle y \parallel z \rangle \to \langle x \parallel z \rangle, & \text{(transitive)} \\ (\mathrm{id}^\parallel) & \langle x^\ominus \parallel y^\ominus \rangle \to \langle x^\ominus = y^\ominus \rangle, & \\ (\mathrm{sbn}^\parallel) & \langle x \parallel y \rangle \wedge \langle xz \rangle \leftrightarrow \langle xz \parallel yz \rangle, & \text{(substitution)} \\ & \langle x \parallel y \rangle \wedge \langle zx \rangle \leftrightarrow \langle zx \parallel zy \rangle. & \end{array}$$

Proof. $(\mathrm{reg}^\parallel)$ $\langle x \parallel y \rangle \to \langle x \rangle \wedge \langle y \rangle$:

$$\begin{array}{llll} \langle x \parallel y \rangle & \to & \langle x^\ominus = y^\ominus \rangle \wedge \langle x^\oplus = y^\oplus \rangle & \\ & \to & \langle x^\ominus \rangle \wedge \langle y^\oplus \rangle & \{ [\mathrm{eq0}] \} \\ & \to & \langle x \rangle \wedge \langle y \rangle. & \{ (\mathrm{def}^\ominus), (\mathrm{def}^\oplus) \} \end{array}$$

$(\mathrm{ref}^\parallel)$ $\langle x = y \rangle \to \langle x \parallel y \rangle$:

$$\begin{array}{llll} \langle x = y \rangle & \to & \langle x^\ominus = y^\ominus \rangle \wedge \langle x^\oplus = y^\oplus \rangle & \{ [\mathrm{sbn}^{\ominus,\oplus}] \} \\ & \to & \langle x \parallel y \rangle. & \{ \text{def of } \parallel \} \end{array}$$

(sym$^{\|}$) $\langle\, x \parallel y \,\rangle \to \langle\, y \parallel x \,\rangle$:

$$\begin{array}{lll} \langle\, x \parallel y \,\rangle & \to \langle\, x^{\ominus} = y^{\ominus} \,\rangle \wedge \langle\, x^{\oplus} = y^{\oplus} \,\rangle & \{\ \parallel\ \} \\ & \to \langle\, y^{\ominus} = x^{\ominus} \,\rangle \wedge \langle\, y^{\oplus} = x^{\oplus} \,\rangle & \{\ [\mathrm{eq1}]\ \} \\ & \to \langle\, y \parallel x \,\rangle. & \{\ \parallel\ \} \end{array}$$

(trn$^{\|}$) $\langle\, x \parallel y \,\rangle \wedge \langle\, y \parallel z \,\rangle \to \langle\, x \parallel z \,\rangle$:

$$\begin{array}{lll} & \langle\, x \parallel y \,\rangle \wedge \langle\, y \parallel z \,\rangle & \\ \to & \langle\, x^{\ominus} = y^{\ominus} \,\rangle \wedge \langle\, x^{\oplus} = y^{\oplus} \,\rangle \wedge \langle\, y^{\ominus} = z^{\ominus} \,\rangle \wedge \langle\, y^{\oplus} = z^{\oplus} \,\rangle & \{\ \parallel\ \} \\ \to & \langle\, x^{\ominus} = z^{\ominus} \,\rangle \wedge \langle\, x^{\oplus} = z^{\oplus} \,\rangle & \{\ [\mathrm{eq2}]\ \} \\ \to & \langle\, x \parallel z \,\rangle. & \{\ \parallel\ \} \end{array}$$

(id$^{\|}$) $\langle\, x^{\ominus} \parallel y^{\ominus} \,\rangle \to \langle\, x^{\ominus} = y^{\ominus} \,\rangle$:

$$\begin{array}{lll} \langle\, x^{\ominus} \parallel y^{\ominus} \,\rangle & \to \langle\, x^{\ominus\ominus} = y^{\ominus\ominus} \,\rangle & \{\ \parallel\ \} \\ & \to \langle\, x^{\ominus} = y^{\ominus} \,\rangle. & \{\ (^{\ominus\ominus})\ \} \end{array}$$

(sbn$^{\|}$) $\langle\, x \parallel y \,\rangle \wedge \langle\, xz \,\rangle \leftrightarrow \langle\, xz \parallel yz \,\rangle$:

$$\begin{array}{lll} & \langle\, x \parallel y \,\rangle \wedge \langle\, xz \,\rangle & \\ \to & \langle\, x \parallel y \,\rangle \wedge \langle\, xz \,\rangle \wedge \langle\, x^{\oplus} = y^{\oplus} \,\rangle \wedge \langle\, x^{\oplus} = z^{\ominus} \,\rangle & \{\ \parallel, [\mathrm{def}^{\,\mathrm{C}}]\ \} \\ \to & \langle\, x \parallel y \,\rangle \wedge \langle\, xz \,\rangle \wedge \langle\, y^{\oplus} = z^{\ominus} \,\rangle & \{\ [\mathrm{eq2}]\ \} \\ \to & \langle\, x^{\ominus} = y^{\ominus} \,\rangle \wedge \langle\, xz \,\rangle \wedge \langle\, yz \,\rangle & \{\ \parallel, (\mathrm{def}^{\,\mathrm{C}}_{\,*})\ \} \\ \to & \langle\, (xz)^{\ominus} = (yz)^{\ominus} \,\rangle \wedge \langle\, (xz)^{\oplus} = (yz)^{\oplus} \,\rangle & \{\ (\mathrm{C}^{\ominus}), (\mathrm{C}^{\oplus})\ \} \\ \to & \langle\, xz \parallel yz \,\rangle. & \{\ \parallel\ \} \end{array}$$

$$\begin{array}{lll} & \langle\, xz \parallel yz \,\rangle & \\ \to & \langle\, xz \,\rangle \wedge \langle\, yz \,\rangle \wedge \langle\, (xz)^{\ominus} = (yz)^{\ominus} \,\rangle & \{\ (\mathrm{reg}^{\|}), \parallel\ \} \\ \to & \langle\, xz \,\rangle \wedge \langle\, x^{\oplus} = z^{\ominus} \,\rangle \wedge \langle\, y^{\oplus} = z^{\ominus} \,\rangle \wedge \langle\, x^{\ominus} = y^{\ominus} \,\rangle & \{\ (\mathrm{def}^{\,C}_{\,*}), (\mathrm{C}^{\ominus})\ \} \\ \to & \langle\, xz \,\rangle \wedge \langle\, x^{\ominus} = y^{\ominus} \,\rangle \wedge \langle\, x^{\oplus} = y^{\oplus} \,\rangle & \{\ [\mathrm{eq2}]\ \} \\ \to & \langle\, xz \,\rangle \wedge \langle\, x \parallel y \,\rangle. & \{\ \parallel\ \} \end{array}$$

□

The parallelism $\langle\, x \parallel y \,\rangle$ is an equivalence relation.

4. Axioms for boolean operations

In relational calculus we will use four unary operations $^{\ominus}$ (source), $^{\oplus}$ (target), $^{-}$ (complement) and $^{\sharp}$ (converse), and three binary operations $\cdot$ (composition), $\sqcap$ (meet) and $\sqcup$ (join).

Relational terms in relational calculus are recursively defined as follows:

(a) Each variable x and constant c are relational terms,

(b) If t is a relational term, then so are $t^{\ominus}$, $t^{\oplus}$, t^{-} and $t^{\sharp}$.

(c) If t and s are relational terms, then so are ts, $t \sqcap s$ and $t \sqcup s$.

A relational term t will be called a *relation* if $\langle t \rangle$ holds. The duality for categorical terms may be extended to one for relational terms as follows: $(t^-)^* = (t^*)^-$, $(t^\sharp)^* = (t^*)^\sharp$, $(t \sqcap s)^* = t^* \sqcup s^*$ and $(t \sqcup s)^* = t^* \sqcap s^*$ for terms t and s.

Next we will state axioms imposed on $\sqcap$, $\sqcup$ and $^-$. The axioms supply category theory with a locally boolean structure.

AXIOM 4.1 AXIOMS FOR BOOLEAN OPERATIONS.

[def$^\sqcap$]	$\langle x \sqcap y \rangle \to \langle x \parallel x \sqcap y \rangle$,	(definability)
[def$^\sqcup$]	$\langle x \sqcup y \rangle \to \langle x \parallel x \sqcup y \rangle$,	
[sbn$^\sqcap$]	$\langle x = y \rangle \to \langle x \sqcap z \simeq y \sqcap z \rangle$,	(substitution)
[sbn$^\sqcup$]	$\langle x = y \rangle \to \langle x \sqcup z \simeq y \sqcup z \rangle$,	
[com$^\sqcap$]	$\langle x \sqcap y \simeq y \sqcap x \rangle$,	(commutative)
[com$^\sqcup$]	$\langle x \sqcup y \simeq y \sqcup x \rangle$,	
[dst$^\sqcap$]	$\langle x \sqcap (y \sqcup z) \simeq (x \sqcap y) \sqcup (x \sqcap z) \rangle$,	(distributive)
[dst$^\sqcup$]	$\langle x \sqcup (y \sqcap z) \simeq (x \sqcup y) \sqcap (x \sqcup z) \rangle$,	
[reg$^-$]	$\langle x^- \rangle \to \langle x \rangle$,	(regular)
[∇]	$\langle x \parallel y \rangle \to \langle x \sqcap (y \sqcup y^-) = x \rangle$,	
[$\emptyset$]	$\langle x \parallel y \rangle \to \langle x \sqcup (y \sqcap y^-) = x \rangle$.	□

Terms $y \sqcup y^-$ and $y \sqcap y^-$ will be abbreviated as y^∇ and $y^\emptyset$, respectively.

The axioms 4.1 for boolean operations consist of dual pairs of formulas. For example, $[\mathrm{def}^\sqcap]^* \leftrightarrow [\mathrm{def}^\sqcup]$, $[\nabla]^* \leftrightarrow [\emptyset]$ and $[\mathrm{reg}^-]^* \leftrightarrow [\mathrm{reg}^-]$. Therefore, if a formula ϕ is provable (using 2.1, 3.1 and 4.1), then so is the dual ϕ^*.

PROPOSITION 4.2. In relational calculus the following hold.

(def$^\sqcap_*$)	$\langle x \sqcap y \rangle \leftrightarrow \langle x \parallel y \rangle$,	(definability)
(def$^\sqcup_*$)	$\langle x \sqcup y \rangle \leftrightarrow \langle x \parallel y \rangle$,	
(idm$^\sqcap$)	$\langle x \rangle \to \langle x \sqcap x = x \rangle$,	(idempotent)
(idm$^\sqcup$)	$\langle x \rangle \to \langle x \sqcup x = x \rangle$,	
($\parallel^\nabla$)	$\langle x \parallel y \rangle \leftrightarrow \langle x \parallel y^\nabla \rangle \leftrightarrow \langle x^\nabla = y^\nabla \rangle$,	
($\parallel^\emptyset$)	$\langle x \parallel y \rangle \leftrightarrow \langle x \parallel y^\emptyset \rangle \leftrightarrow \langle x^\emptyset = y^\emptyset \rangle$,	
($\parallel^-$)	$\langle x \parallel y \rangle \to \langle x \parallel y^- \rangle$,	
($^{\nabla \sqcup}$)	$\langle x \parallel y \rangle \leftrightarrow \langle x \sqcup y^\nabla = y^\nabla \rangle$,	
($^{\emptyset \sqcap}$)	$\langle x \parallel y \rangle \leftrightarrow \langle x \sqcap y^\emptyset = y^\emptyset \rangle$,	
(abs$^\sqcap$)	$\langle x \parallel y \rangle \to \langle x \sqcap (x \sqcup y) = x \rangle$,	(absorption)
(abs$^\sqcup$)	$\langle x \parallel y \rangle \to \langle x \sqcup (x \sqcap y) = x \rangle$,	
($\sqsubseteq$)	$\langle x \sqcup y = y \rangle \leftrightarrow \langle x \sqcap y = x \rangle$,	
(L$^=$)	$\langle x \sqcup y = x \sqcup z \rangle \wedge \langle x \sqcap y = x \sqcap z \rangle \to \langle y = z \rangle$,	
(L$^=_0$)	$\langle x \sqcup y = x^\nabla \rangle \wedge \langle x \sqcap y = x^\emptyset \rangle \to \langle y = x^- \rangle$,	

$(^{\nabla\,-})$ $\langle\, x\,\rangle \to \langle\, x^{\nabla\,-} = x^{\emptyset}\,\rangle$,
$(^{\emptyset\,-})$ $\langle\, x\,\rangle \to \langle\, x^{\emptyset\,-} = x^{\nabla}\,\rangle$,
$(^{-\,-})$ $\langle\, x\,\rangle \to \langle\, x^{-\,-} = x\,\rangle$,
(sbn^{-}) $\langle\, x = y\,\rangle \to \langle\, x^{-} = y^{-}\,\rangle$,
$(^{\nabla\,C})$ $\langle\, xy\,\rangle \to \langle\, xy^{\nabla}\,\rangle \wedge \langle\, x^{\nabla}y\,\rangle$,
$(^{\emptyset\,C})$ $\langle\, xy\,\rangle \to \langle\, xy^{\emptyset}\,\rangle \wedge \langle\, x^{\emptyset}y\,\rangle$.

Proof. $(\mathrm{def}^{\sqcap}_{*})$ $\langle\, x \sqcap y\,\rangle \leftrightarrow \langle\, x \parallel y\,\rangle$:

$$
\begin{array}{lll}
 & \langle\, x \sqcap y\,\rangle & \\
\to & \langle\, x \parallel x \sqcap y\,\rangle \wedge \langle\, x \sqcap y = y \sqcap x\,\rangle & \{\ [\mathrm{def}^{\sqcap}],\ [\mathrm{com}^{\sqcap}]\ \} \\
\to & \langle\, x \parallel x \sqcap y\,\rangle \wedge \langle\, x \sqcap y \parallel y \sqcap x\,\rangle \wedge \langle\, y \sqcap x\,\rangle & \{\ (\mathrm{ref}^{\parallel}),\ [\mathrm{eq0}]\ \} \\
\to & \langle\, x \parallel x \sqcap y\,\rangle \wedge \langle\, x \sqcap y \parallel y \sqcap x\,\rangle \wedge \langle\, y \parallel y \sqcap x\,\rangle & \{\ [\mathrm{def}^{\sqcap}]\ \} \\
\to & \langle\, x \parallel y\,\rangle. & \{\ (\mathrm{sym}^{\parallel}),\ (\mathrm{trn}^{\parallel})\ \}
\end{array}
$$

$$
\begin{array}{llll}
\langle\, x \parallel y\,\rangle & \to & \langle\, x \sqcap (y \sqcup y^{-}) = x\,\rangle & \{\ [\,\nabla\,]\ \} \\
 & \to & \langle\, (x \sqcap y) \sqcup (x \sqcap y^{-}) = x\,\rangle & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\
 & \to & \langle\, (x \sqcap y) \sqcup (x \sqcap y^{-})\,\rangle & \{\ [\mathrm{eq0}]\ \} \\
 & \to & \langle\, x \sqcap y \parallel (x \sqcap y) \sqcup (x \sqcap y^{-})\,\rangle & \{\ [\mathrm{def}^{\sqcup}]\ \} \\
 & \to & \langle\, x \sqcap y\,\rangle. & \{\ (\mathrm{reg}^{\parallel})\ \}
\end{array}
$$

Note that $(\mathrm{reg}^{\sqcap})$ $\langle\, x \sqcap y\,\rangle \to \langle\, x\,\rangle \wedge \langle\, y\,\rangle$ follows $(\mathrm{def}^{\sqcap}_{*})$ and $(\mathrm{reg}^{\parallel})$.

$(\mathrm{idm}^{\sqcap})$ $\langle\, x\,\rangle \to \langle\, x \sqcap x = x\,\rangle$:

$$
\begin{array}{llll}
\langle\, x\,\rangle & \to & \langle\, x \parallel x\,\rangle & \{\ (\mathrm{ref}^{\parallel})\ \} \\
 & \to & \langle\, x \sqcap x\,\rangle & \{\ (\mathrm{def}^{\sqcap}_{*})\ \} \\
 & \to & \langle\, x \parallel x \sqcap x\,\rangle & \{\ [\mathrm{def}^{\sqcap}]\ \} \\
 & \to & \langle\, x \sqcap x \parallel x\,\rangle. & \{\ (\mathrm{sym}^{\parallel})\ \} \\
 & \to & \langle\, x \sqcap x = (x \sqcap x) \sqcup (x \sqcap x^{-})\,\rangle & \{\ [\,\emptyset\,]\ \} \\
 & \to & \langle\, x \sqcap x = x \sqcap (x \sqcup x^{-})\,\rangle & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\
 & \to & \langle\, x \sqcap x = x\,\rangle. & \{\ [\,\nabla\,]\ \}
\end{array}
$$

$(\parallel^{\nabla})$ $\langle\, x \parallel y\,\rangle \leftrightarrow \langle\, x \parallel y^{\nabla}\,\rangle \leftrightarrow \langle\, x^{\nabla} = y^{\nabla}\,\rangle$:
(1) $\langle\, x \parallel y\,\rangle \to \langle\, x \parallel y^{\nabla}\,\rangle$:

$$
\begin{array}{llll}
\langle\, x \parallel y\,\rangle & \to & \langle\, x \sqcap y^{\nabla} = x\,\rangle & \{\ [\,\nabla\,]\ \} \\
 & \to & \langle\, x \sqcap y^{\nabla}\,\rangle & \{\ [\mathrm{eq0}]\ \} \\
 & \to & \langle\, x \parallel y^{\nabla}\,\rangle. & \{\ (\mathrm{def}^{\sqcap}_{*})\ \}
\end{array}
$$

(2) $\langle\, x \parallel y^{\nabla}\,\rangle \to \langle\, x^{\nabla} = y^{\nabla}\,\rangle$:

$$
\begin{array}{llll}
\langle\, x \parallel y^{\nabla}\,\rangle & \to & \langle\, x \parallel y^{\nabla}\,\rangle \wedge \langle\, x\,\rangle \wedge \langle\, y^{\nabla}\,\rangle & \{\ (\mathrm{reg}^{\parallel})\ \} \\
 & \to & \langle\, x \parallel y^{\nabla}\,\rangle \wedge \langle\, x \parallel x^{\nabla}\,\rangle \wedge \langle\, y \parallel y^{\nabla}\,\rangle & \{\ (1),\ [\mathrm{def}^{\sqcup}]\ \} \\
 & \to & \langle\, y^{\nabla} \parallel x\,\rangle \wedge \langle\, x^{\nabla} \parallel y\,\rangle & \{\ (\mathrm{trn}^{\parallel})\ \} \\
 & \to & \langle\, y^{\nabla} \sqcap x^{\nabla} = y^{\nabla}\,\rangle \wedge \langle\, x^{\nabla} \sqcap y^{\nabla} = x^{\nabla}\,\rangle & \{\ [\,\nabla\,]\ \} \\
 & \to & \langle\, x^{\nabla} = y^{\nabla}\,\rangle. & \{\ [\mathrm{com}^{\sqcap}]\ \}
\end{array}
$$

(3) $\langle x^\nabla = y^\nabla \rangle \to \langle x \parallel y \rangle$:

$$\begin{array}{rcll} \langle x^\nabla = y^\nabla \rangle & \to & \langle x^\nabla \parallel y^\nabla \rangle \wedge \langle x^\nabla \rangle \wedge \langle y^\nabla \rangle & \{\ (\mathrm{ref}^{\parallel})\ \} \\ & \to & \langle x^\nabla \parallel y^\nabla \rangle \wedge \langle x \parallel x^\nabla \rangle \wedge \langle y \parallel y^\nabla \rangle & \{\ [\mathrm{def}^{\sqcup}]\ \} \\ & \to & \langle x \parallel y \rangle. & \{\ (\mathrm{trn}^{\parallel})\ \} \end{array}$$

$(\parallel^-)$ $\langle x \parallel y \rangle \to \langle x \parallel y^- \rangle$:

$$\begin{array}{rcll} \langle x \parallel y \rangle & \to & \langle x \sqcap (y \sqcup y^-) = x \rangle & \{\ [\nabla]\ \} \\ & \to & \langle (x \sqcap y) \sqcup (x \sqcap y^-) = x \rangle & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\ & \to & \langle (x \sqcap y) \sqcup (x \sqcap y^-) \rangle & \{\ [\mathrm{eq0}]\ \} \\ & \to & \langle x \sqcap y \parallel x \sqcap y^- \rangle & \{\ (\mathrm{def}^{\sqcup}_*)\ \} \\ & \to & \langle x \sqcap y^- \rangle & \{\ (\mathrm{reg}^{\parallel})\ \} \\ & \to & \langle x \parallel y^- \rangle. & \{\ (\mathrm{def}^{\sqcap}_*)\ \} \end{array}$$

Note that $(\parallel^\nabla)$, $(\parallel^\emptyset)$ and $(\parallel^-)$ imply

$$\langle x \parallel y \rangle \to \langle x^\nabla = y^{\nabla\,\nabla} = y^{\emptyset\,\nabla} = y^{-\,\nabla} \rangle \wedge \langle x^\emptyset = y^{\emptyset\,\emptyset} = y^{\nabla\,\emptyset} = y^{-\,\emptyset} \rangle.$$

$(^{\emptyset\,\sqcap})$ $\langle x \parallel y \rangle \to \langle x \sqcap y^\emptyset = y^\emptyset \rangle$:
(1) $\langle x \rangle \to \langle x \sqcap x^\emptyset = x^\emptyset \rangle$:

$$\begin{array}{rcll} \langle x \rangle & \to & \langle x \parallel x^\emptyset \rangle & \{\ (\parallel^\emptyset)\ \} \\ & \to & \langle x \sqcap x^\emptyset \rangle & \{\ (\mathrm{def}^{\sqcap}_*)\ \} \\ & \to & \langle x \parallel x \sqcap x^\emptyset \rangle & \{\ [\mathrm{def}^{\sqcap}]\ \} \\ & \to & \langle (x \sqcap x^\emptyset) \sqcup (x \sqcap x^-) = x \sqcup x^\emptyset \rangle & \{\ [\emptyset]\ \} \\ & \to & \langle x \sqcap (x^\emptyset \sqcup x^-) = x \sqcap x^\emptyset \rangle & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\ & \to & \langle x \sqcap x^- = x \sqcap x^\emptyset \rangle & \{\ [\emptyset]\ \} \\ & \to & \langle x^\emptyset = x \sqcap x^\emptyset \rangle. & \end{array}$$

(2) $\langle x \parallel y \rangle \to \langle x \sqcap y^\emptyset = y^\emptyset \rangle$:

$$\begin{array}{rcll} \langle x \parallel y \rangle & \to & \langle x^\emptyset = y^\emptyset \rangle \wedge \langle x \sqcap x^\emptyset = x^\emptyset \rangle & \{\ (\parallel^\emptyset),\ (1)\ \} \\ & \to & \langle x \sqcap y^\emptyset = y^\emptyset \rangle. & \{\ [\mathrm{sbn}^{\sqcap}]\ \} \end{array}$$

$(\mathrm{abs}^{\sqcup})$ $\langle x \parallel y \rangle \to \langle x \sqcup (x \sqcap y) = x \rangle$:

$$\begin{array}{rcll} \langle x \parallel y \rangle & \to & \langle x \sqcap y \rangle & \{\ (\mathrm{def}^{\sqcap}_*)\ \} \\ & \to & \langle x \parallel x \sqcap y \rangle & \{\ [\mathrm{def}^{\sqcap}]\ \} \\ & \to & \langle x \sqcup (x \sqcap y) = x \sqcup (x \sqcap y) \rangle & \{\ (\mathrm{def}^{\sqcup}_*)\ \} \\ & \to & \langle x \sqcup (x \sqcap y) = (x \sqcap y^\nabla) \sqcup (x \sqcap y) \rangle & \{\ [\nabla],\ [\mathrm{sbn}^{\sqcup}]\ \} \\ & \to & \langle x \sqcup (x \sqcap y) = x \sqcap (y^\nabla \sqcup y) \rangle & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\ & \to & \langle x \sqcup (x \sqcap y) = x \sqcap y^\nabla \rangle & \{\ (^{\nabla\,\sqcup})\ \} \\ & \to & \langle x \sqcup (x \sqcap y) = x \rangle. & \{\ [\nabla]\ \} \end{array}$$

($\sqsubseteq$) $\langle\, x \sqcup y = y \,\rangle \leftrightarrow \langle\, x \sqcap y = x \,\rangle$:

$$\begin{array}{rcll} \langle\, x \sqcup y = y \,\rangle & \to & \langle\, x = x \sqcap (x \sqcup y) \,\rangle \wedge \langle\, x \sqcup y = y \,\rangle & \{\ (\mathrm{abs}^{\sqcap})\ \} \\ & \to & \langle\, x = x \sqcap y \,\rangle. & \{\ [\mathrm{sbn}^{\sqcap}]\ \} \end{array}$$

$$\begin{array}{rcll} \langle\, x \sqcap y = x \,\rangle & \to & \langle\, y = y \sqcup (y \sqcap x) \,\rangle \wedge \langle\, x \sqcap y = x \,\rangle & \{\ (\mathrm{abs}^{\sqcup})\ \} \\ & \to & \langle\, y = y \sqcup x \,\rangle. & \{\ [\mathrm{sbn}^{\sqcup}]\ \} \end{array}$$

($\mathrm{L}^{=}$) $\langle\, x \sqcap y = x \sqcap z \,\rangle \wedge \langle\, x \sqcup y = x \sqcup z \,\rangle \to \langle\, y = z \,\rangle$:

$$\begin{array}{rcll} y & = & y \sqcap (y \sqcup x) & \{\ (\mathrm{abs}^{\sqcap})\ \} \\ & = & y \sqcap (z \sqcup x) & \{\ x \sqcup y = x \sqcup z\ \} \\ & = & (y \sqcap z) \sqcup (y \sqcap x) & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\ & = & (z \sqcap y) \sqcup (y \sqcap x) & \{\ [\mathrm{com}^{\sqcap}]\ \} \\ & = & (z \sqcap y) \sqcup (z \sqcap x) & \{\ x \sqcap y = x \sqcap z\ \} \\ & = & z \sqcap (y \sqcup x) & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\ & = & z \sqcap (z \sqcup x) & \{\ x \sqcup y = x \sqcup z\ \} \\ & = & z. & \{\ (\mathrm{abs}^{\sqcap})\ \} \end{array}$$

Note. $\langle\, x \sqcap y = a \,\rangle \wedge \langle\, x \sqcup y = b \,\rangle \to \langle\, y = a \sqcup (b \sqcap x^-) \,\rangle$:

$$\begin{array}{rcll} y & = & y \sqcap (x \sqcup x^-) & \{\ [\nabla]\ \} \\ & = & (y \sqcap x) \sqcup (y \sqcap x^-) & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\ & = & a \sqcup (y \sqcap x^-) & \{\ x \sqcap y = a\ \} \\ & = & a \sqcup ((y \sqcap x^-) \sqcup (x \sqcap x^-)) & \{\ [\emptyset]\ \} \\ & = & a \sqcup ((y \sqcup x) \sqcap x^-) & \{\ [\mathrm{dst}^{\sqcap}]\ \} \\ & = & a \sqcup (b \sqcap x^-). & \{\ x \sqcup y = b\ \} \end{array}$$

($\mathrm{L}_0^{=}$) $\langle\, x \sqcap y = x^{\emptyset} \,\rangle \wedge \langle\, x \sqcup y = x^{\nabla} \,\rangle \to \langle\, y = x^- \,\rangle$:
It is a particular case of ($\mathrm{L}^{=}$).

($^{\nabla\,-}$) $\langle\, x \,\rangle \to \langle\, x^{\nabla\,-} = x^{\emptyset} \,\rangle$:
First note that $\langle\, x \,\rangle \to \langle\, x \parallel x^- \,\rangle \wedge \langle\, x \parallel x^{\nabla} \,\rangle$ and so $\langle\, x \,\rangle \to \langle\, x^{\nabla} = x^{\nabla\,-\,\nabla} \,\rangle$ by ($\parallel^{\nabla}$) and ($\parallel^-$). Hence

$$\begin{array}{rcll} x^{\nabla\,-} & = & x^{\nabla\,-} \sqcap x^{\nabla\,-\,\nabla} & \{\ [\nabla]\ \} \\ & = & x^{\nabla\,-} \sqcap x^{\nabla} & \{\ x^{\nabla\,-\,\nabla} = x^{\nabla}\ \} \\ & = & x^{\nabla\,\emptyset} & \{\ [\mathrm{com}^{\sqcap}],\ [\emptyset]\} \\ & = & x^{\emptyset}. & \{\ x \parallel x^{\nabla},\ (\parallel^{\emptyset})\ \} \end{array}$$

($^{-\,-}$) $\langle\, x \,\rangle \to \langle\, x^{-\,-} = x \,\rangle$:
Assume that $\langle\, x \,\rangle$. Then $x^- \sqcap x = x^{\emptyset}$ and $x^- \sqcup x = x^{\nabla}$. Hence $x = x^{-\,-}$ holds by ($\mathrm{L}_0^{=}$).

(sbn^-) $\langle\, x = y \,\rangle \to \langle\, x^- = y^- \,\rangle$:
If $\langle\, x = y \,\rangle$, then

$$\begin{array}{rcll} x \sqcap y^- & = & y \sqcap y^- & \{\ [\emptyset],\ [\mathrm{sbn}^{\sqcap}]\ \} \\ & = & y^{\emptyset} & \\ & = & x^{\emptyset}, & \{\ (\mathrm{ref}^{\parallel}),\ (\parallel^{\emptyset})\ \} \end{array}$$

and

$$\begin{array}{rcll} x \sqcup y^- & = & y \sqcup y^- & \{\ [\nabla],\ [\mathrm{sbn}^{\sqcup}]\ \} \\ & = & y^{\nabla} & \\ & = & x^{\nabla}. & \{\ (\mathrm{ref}^{\|}),\ (\|^{\nabla})\ \} \end{array}$$

Hence we have $\langle\, y^- = x^- \,\rangle$ by ($\mathrm{L}_0^{=}$).

$({}^{\nabla\, C})$ $\langle\, xy \,\rangle \to \langle\, xy^{\nabla} \,\rangle \wedge \langle\, x^{\nabla} y \,\rangle$:

$$\begin{array}{rcll} \langle\, xy \,\rangle & \to & \langle\, x^{\oplus} = y^{\ominus} \,\rangle & \{\ (\mathrm{def}^{C})\ \} \\ & \to & \langle\, x^{\oplus} = y^{\nabla\,\ominus} \,\rangle & \{\ (\|^{\nabla})\ \} \\ & \to & \langle\, xy^{\nabla} \,\rangle. & \{\ (\mathrm{def}_*^{C})\ \}. \end{array}$$

□

The join $\sqcup$ and the meet $\sqcap$ are partially defined on relations, and the complement $^-$ is totally defined on relations.

A formula $\langle\, x \sqcap y = x \,\rangle$ will be abbreviated as $\langle\, x \sqsubseteq y \,\rangle$. Recall 4.2 ($\sqsubseteq$) $\langle\, x \sqcap y = x \,\rangle \leftrightarrow \langle\, x \sqcup y = y \,\rangle$.

Proposition 4.3. In relational calculus the following hold.

$(^{\nabla})$	$\langle\, x \parallel y \,\rangle \to \langle\, x \sqsubseteq y^{\nabla} \,\rangle$,	
$(^{\emptyset})$	$\langle\, x \parallel y \,\rangle \to \langle\, y^{\emptyset} \sqsubseteq x \,\rangle$,	
$(\mathrm{ref}^{\sqsubseteq})$	$\langle\, x = y \,\rangle \to \langle\, x \sqsubseteq y \,\rangle$,	(reflexive)
$(\mathrm{def}^{\sqsubseteq})$	$\langle\, x \sqsubseteq y \,\rangle \to \langle\, x \parallel y \,\rangle$,	
$(\mathrm{trn}^{\sqsubseteq})$	$\langle\, x \sqsubseteq y \,\rangle \wedge \langle\, y \sqsubseteq z \,\rangle \to \langle\, x \sqsubseteq z \,\rangle$,	(transitive)
$(\mathrm{ant}^{\sqsubseteq})$	$\langle\, x \sqsubseteq y \,\rangle \wedge \langle\, y \sqsubseteq x \,\rangle \to \langle\, x = y \,\rangle$,	(antisymmetric)
(inf)	$\langle\, z \sqsubseteq x \sqcap y \,\rangle \leftrightarrow \langle\, z \sqsubseteq x \,\rangle \wedge \langle\, z \sqsubseteq y \,\rangle$,	
(sup)	$\langle\, x \sqcup y \sqsubseteq z \,\rangle \leftrightarrow \langle\, x \sqsubseteq z \,\rangle \wedge \langle\, y \sqsubseteq z \,\rangle$,	
$(\mathrm{ass}^{\sqcap})$	$\langle\, (x \sqcap y) \sqcap z \simeq x \sqcap (y \sqcap z) \,\rangle$,	(associative)
$(\mathrm{ass}^{\sqcup})$	$\langle\, (x \sqcup y) \sqcup z \simeq x \sqcup (y \sqcup z) \,\rangle$,	
$(\sqsubseteq^{\sqcap})$	$\langle\, x \sqsubseteq x' \,\rangle \wedge \langle\, y \sqsubseteq y' \,\rangle \to \langle\, x \sqcap y \sqsubseteq x' \sqcap y' \,\rangle$,	(monotonic)
$(\sqsubseteq^{\sqcup})$	$\langle\, x \sqsubseteq x' \,\rangle \wedge \langle\, y \sqsubseteq y' \,\rangle \to \langle\, x \sqcup y \sqsubseteq x' \sqcup y' \,\rangle$,	
$(\sqsubseteq^{-})$	$\langle\, x \sqsubseteq y^- \,\rangle \leftrightarrow \langle\, x \sqcap y = x^{\emptyset} \,\rangle \leftrightarrow \langle\, y \sqsubseteq x^- \,\rangle$,	
$(\sqsupseteq^{-})$	$\langle\, x^- \sqsubseteq y \,\rangle \leftrightarrow \langle\, x \sqcup y = x^{\nabla} \,\rangle \leftrightarrow \langle\, y^- \sqsubseteq x \,\rangle$,	
$(\forall^{\sqsubseteq})$	$\forall z.\ (\langle\, z \sqsubseteq x \,\rangle \leftrightarrow \langle\, z \sqsubseteq y \,\rangle) \to \langle\, x = y \,\rangle$,	
$(\forall^{\sqsupseteq})$	$\forall z.\ (\langle\, x \sqsubseteq z \,\rangle \leftrightarrow \langle\, y \sqsubseteq z \,\rangle) \to \langle\, x = y \,\rangle$,	
$(\forall^{\emptyset})$	$\forall z.\ (\langle\, x \sqcap z = z^{\emptyset} \,\rangle \leftrightarrow \langle\, y \sqcap z = z^{\emptyset} \,\rangle) \to \langle\, x = y \,\rangle$,	
$(\sqcap^{-})$	$\langle\, x \parallel y \,\rangle \to \langle\, (x \sqcap y)^- = x^- \sqcup y^- \,\rangle$,	(de Morgan)
$(\sqcup^{-})$	$\langle\, x \parallel y \,\rangle \to \langle\, (x \sqcup y)^- = x^- \sqcap y^- \,\rangle$,	
$(\sqsubseteq_*^{-})$	$\langle\, x \sqcap y \sqsubseteq z \sqcup w \,\rangle \leftrightarrow \langle\, x \sqcap z^- \sqsubseteq y^- \sqcup w \,\rangle$,	(contraposition)
	$\langle\, x \sqsubseteq y \,\rangle \leftrightarrow \langle\, y^- \sqsubseteq x^- \,\rangle$.	

Proof. ($^\nabla$) $\langle\, x \parallel y \,\rangle \to \langle\, x \sqsubseteq y^\nabla \,\rangle$:

$$\begin{array}{rcll} \langle\, x \parallel y \,\rangle & \to & \langle\, x \sqcap y^\nabla = x \,\rangle & \{ \, [\,\nabla\,] \, \} \\ & \to & \langle\, x \sqsubseteq y^\nabla \,\rangle. & \end{array}$$

(ref$^\sqsubseteq$) $\langle\, x = y \,\rangle \to \langle\, x \sqsubseteq y \,\rangle$:

$$\begin{array}{rcll} \langle\, x = y \,\rangle & \to & \langle\, x = y \,\rangle \wedge \langle\, x \sqcap x = x \,\rangle & \{ \, (\mathrm{idm}^\sqcap) \, \} \\ & \to & \langle\, x \sqcap y = x \,\rangle. & \{ \, [\mathrm{sbn}^\sqcap] \, \} \\ & \to & \langle\, x \sqsubseteq y \,\rangle. & \end{array}$$

(def$^\sqsubseteq$) $\langle\, x \sqsubseteq y \,\rangle \to \langle\, x \parallel y \,\rangle$:

$$\begin{array}{rcll} \langle\, x \sqsubseteq y \,\rangle & \to & \langle\, x \sqcap y = x \,\rangle & \\ & \to & \langle\, x \sqcap y \,\rangle & \{ \, [\mathrm{eq0}] \, \} \\ & \to & \langle\, x \parallel y \,\rangle. & \{ \, (\mathrm{def}^\sqcap_*) \, \} \end{array}$$

(trn$^\sqsubseteq$) $\langle\, x \sqsubseteq y \,\rangle \wedge \langle\, y \sqsubseteq z \,\rangle \to \langle\, x \sqsubseteq z \,\rangle$:
Assume $\langle\, x \sqsubseteq y \,\rangle$ and $\langle\, y \sqsubseteq z \,\rangle$. Then

$$\begin{array}{rcll} x & = & x \sqcup (x \sqcap z) & \{ \, (\mathrm{abs}^\sqcup) \, \} \\ & = & (x \sqcap y) \sqcup (x \sqcap z) & \{ \, x \sqsubseteq y \, \} \\ & = & x \sqcap (y \sqcup z) & \{ \, [\mathrm{dst}^\sqcap] \, \} \\ & = & x \sqcap z, & \{ \, y \sqsubseteq z \, \} \end{array}$$

which implies $\langle\, x \sqsubseteq z \,\rangle$.

(ant$^\sqsubseteq$) $\langle\, x \sqsubseteq y \,\rangle \wedge \langle\, y \sqsubseteq x \,\rangle \to \langle\, x = y \,\rangle$:

$$\begin{array}{rcll} \langle\, x \sqsubseteq y \,\rangle \wedge \langle\, y \sqsubseteq x \,\rangle & \to & \langle\, x = x \sqcap y \,\rangle \wedge \langle\, y = y \sqcap x \,\rangle & \\ & \to & \langle\, x = y \,\rangle. & \{ \, [\mathrm{com}^\sqcap] \, \} \end{array}$$

(inf) $\langle\, z \sqsubseteq x \sqcap y \,\rangle \leftrightarrow \langle\, z \sqsubseteq x \,\rangle \wedge \langle\, z \sqsubseteq y \,\rangle$:

$$\begin{array}{rll} & \langle\, z \sqsubseteq x \sqcap y \,\rangle & \\ \to & \langle\, x \sqcap y \sqsubseteq x \,\rangle \wedge \langle\, x \sqcap y \sqsubseteq y \,\rangle \wedge \langle\, z \sqsubseteq x \sqcap y \,\rangle & \{ \, (\mathrm{abs}^\sqcap) \, \} \\ \to & \langle\, z \sqsubseteq x \,\rangle \wedge \langle\, z \sqsubseteq y \,\rangle. & \{ \, (\mathrm{trn}^\sqsubseteq) \, \} \end{array}$$

$$\begin{array}{rll} & \langle\, z \sqsubseteq x \,\rangle \wedge \langle\, z \sqsubseteq y \,\rangle & \\ \to & \langle\, x = x \sqcap z \,\rangle \wedge \langle\, y = y \sqcap z \,\rangle \wedge \langle\, x \parallel y \,\rangle & \{ \, (\mathrm{def}^\sqcap_*) \, \} \\ \to & \langle\, x \sqcap y = (x \sqcup z) \sqcap (y \sqcup z) \,\rangle & \{ \, (\mathrm{def}^\sqcap_*), [\mathrm{sbn}^\sqcap] \, \} \\ \to & \langle\, x \sqcap y = (x \sqcap y) \sqcup z) \,\rangle & \{ \, [\mathrm{dst}^\sqcup] \, \} \\ \to & \langle\, z \sqsubseteq x \sqcap y \,\rangle. & \end{array}$$

(ass$^\sqcap$) $\langle\, (x \sqcap y) \sqcap z \simeq x \sqcap (y \sqcap z) \,\rangle$:
First observe that

$$\langle\, (x \sqcap y) \sqcap z \,\rangle \leftrightarrow \langle\, x \parallel y \,\rangle \wedge \langle\, y \parallel z \,\rangle \leftrightarrow \langle\, x \sqcap (y \sqcap z) \,\rangle.$$

$$\begin{array}{llll}
\langle\,(x\sqcap y)\sqcap z\,\rangle & \leftrightarrow & \langle\,x\sqcap y\parallel z\,\rangle & \{\ (\mathrm{def}^{\sqcap}_{*})\ \}\\
 & \to & \langle\,x\sqcap y\parallel z\,\rangle\wedge\langle\,x\sqcap y\,\rangle & \{\ (\mathrm{reg}^{\parallel})\ \}\\
 & \to & \langle\,x\sqcap y\parallel z\,\rangle\wedge\langle\,x\parallel y\,\rangle\wedge\langle\,x\parallel x\sqcap y\,\rangle & \{\ (\mathrm{def}^{\sqcap}_{*})\ \}\\
 & \to & \langle\,x\parallel y\,\rangle\wedge\langle\,y\parallel z\,\rangle. & \{\ (\mathrm{trn}^{\parallel})\ \}
\end{array}$$

Assume $\langle\,(x\sqcap y)\sqcap z\,\rangle$, or equivalently $\langle\,x\parallel y\,\rangle\wedge\langle\,y\parallel z\,\rangle$ by the above observation. Then using (inf) and $(\mathrm{trn}^{\sqsubseteq})$ we have

$$\begin{array}{ll}
\to & \langle\,x\sqcap(y\sqcap z)\sqsubseteq x\,\rangle\wedge\langle\,y\sqcap z\sqsubseteq y\,\rangle\\
 & \quad\wedge\langle\,x\sqcap(y\sqcap z)\sqsubseteq y\sqcap z\,\rangle\wedge\langle\,y\sqcap z\sqsubseteq z\,\rangle\\
\to & \langle\,x\sqcap(y\sqcap z)\sqsubseteq x\sqcap y\,\rangle\wedge\langle\,x\sqcap(y\sqcap z)\sqsubseteq z\,\rangle\\
\to & \langle\,x\sqcap(y\sqcap z)\sqsubseteq(x\sqcap y)\sqcap z\,\rangle.
\end{array}$$

By the similar way the converse inclusion $(x\sqcap y)\sqcap z\sqsubseteq x\sqcap(y\sqcap z)$ holds.

$(\sqsubseteq^{\sqcup})$ $\langle\,x\sqsubseteq x'\,\rangle\wedge\langle\,y\sqsubseteq y'\,\rangle\to\langle\,x\sqcup y\sqsubseteq x'\sqcup y'\,\rangle$:

$$\begin{array}{lll}
 & \langle\,x\sqsubseteq x'\,\rangle\wedge\langle\,y\sqsubseteq y'\,\rangle & \\
\to & \langle\,x\sqsubseteq x'\,\rangle\wedge\langle\,x'\sqsubseteq x'\sqcup y'\,\rangle\wedge\langle\,y\sqsubseteq y'\,\rangle\wedge\langle\,y'\sqsubseteq x'\sqcup y'\,\rangle & \{\ (\mathrm{sup})\ \}\\
\to & \langle\,x\sqsubseteq x'\sqcup y'\,\rangle\wedge\langle\,y\sqsubseteq x'\sqcup y'\,\rangle & \{\ (\mathrm{trn}^{\sqsubseteq})\ \}\\
\to & \langle\,x\sqcup y\sqsubseteq x'\sqcup y'\,\rangle. & \{\ (\mathrm{sup})\ \}
\end{array}$$

$(\sqsubseteq^{-})$ $\langle\,x^{-}\sqsubseteq y\,\rangle\leftrightarrow\langle\,x\sqcup y=x^{\nabla}\,\rangle\leftrightarrow\langle\,y^{-}\sqsubseteq x\,\rangle$:
By the commutative law $[\mathrm{com}^{\sqcup}]$ it suffices to prove the first equivalence.

$$\begin{array}{llll}
\langle\,x^{-}\sqsubseteq y\,\rangle & \to & \langle\,x\sqcup y=x\sqcup y\,\rangle\wedge\langle\,x^{-}\sqcup y=y\,\rangle & \{\ x\parallel y\ \}\\
 & \to & \langle\,x\sqcup y=x\sqcup(x^{-}\sqcup y)\,\rangle & \{\ [\mathrm{sbn}^{\sqcup}]\ \}\\
 & \to & \langle\,x\sqcup y=(x\sqcup x^{-})\sqcup y\,\rangle & \{\ (\mathrm{ass}^{\sqcup})\ \}\\
 & \to & \langle\,x\sqcup y=x^{\nabla}\sqcup y\,\rangle & \\
 & \to & \langle\,x\sqcup y=x^{\nabla}\,\rangle. & \{\ ({}^{\nabla}{}^{\sqcup})\ \}
\end{array}$$

$$\begin{array}{lll}
 & \langle\,x\sqcup y=x^{\nabla}\,\rangle & \\
\to & \langle\,y=y\sqcup(x\sqcap x^{-})\,\rangle\wedge\langle\,x\sqcup y=x^{\nabla}\,\rangle & \{\ [\emptyset]\ \}\\
\to & \langle\,y=(y\sqcup x)\sqcap(y\sqcup x^{-})\,\rangle\wedge\langle\,x\sqcup y=x^{\nabla}\,\rangle & \{\ (\mathrm{dst}^{\sqcup})\ \}\\
\to & \langle\,y=x^{\nabla}\sqcap(y\sqcup x^{-})\,\rangle & \{\ [\mathrm{sbn}^{\sqcup}]\ \}\\
\to & \langle\,y=y\sqcup x^{-}\,\rangle. & \{\ [\nabla]\ \}
\end{array}$$

$(\forall^{\sqsubseteq})$ $\forall z.\ (\langle\,x\sqsubseteq z\,\rangle\leftrightarrow\langle\,y\sqsubseteq z\,\rangle)\to\langle\,x=y\,\rangle$:

$$\begin{array}{lll}
 & \forall z.\ (\langle\,x\sqsubseteq z\,\rangle\leftrightarrow\langle\,y\sqsubseteq z\,\rangle) & \\
\to & \langle\,y\sqsubseteq x\,\rangle\wedge\langle\,x\sqsubseteq y\,\rangle & \{\ x\sqsubseteq x,\ y\sqsubseteq y\ \}\\
\to & \langle\,x=y\,\rangle. & \{\ (\mathrm{ant}^{\sqsubseteq})\ \}
\end{array}$$

$(\forall^{\sqsupseteq})$ $\forall z.\ (\langle\,z\sqsubseteq x\,\rangle\leftrightarrow\langle\,z\sqsubseteq y\,\rangle)\to\langle\,x=y\,\rangle$: dual of $(\forall^{\sqsubseteq})$.

$(\forall^{\emptyset})$ $\forall z.\ (\langle x \sqcap z = z^{\emptyset} \rangle \leftrightarrow \langle y \sqcap z = z^{\emptyset} \rangle) \to \langle x = y \rangle$:

$$\begin{array}{lll} & \forall z.\ (\langle x \sqcap z = z^{\emptyset} \rangle \leftrightarrow \langle y \sqcap z = z^{\emptyset} \rangle) & \\ \to & \forall z.\ (\langle z \sqsubseteq x^{-} \rangle \leftrightarrow \langle z \sqsubseteq y^{-} \rangle) & \{\ (\sqsubseteq^{-})\ \} \\ \to & \langle x^{-} = y^{-} \rangle & \{\ (\forall^{\sqsubseteq})\ \} \\ \to & \langle x = y \rangle. & \{(\mathrm{sbn}^{-}),\ (^{--})\} \end{array}$$

$(\sqcap^{-})$ $\langle x \parallel y \rangle \to \langle (x \sqcup y)^{-} = x^{-} \sqcap y^{-} \rangle$:

$$\begin{array}{llll} \langle z \sqsubseteq (x \sqcup y)^{-} \rangle & \leftrightarrow & \langle x \sqcup y \sqsubseteq z^{-} \rangle & \{\ (\sqsubseteq^{-})\ \} \\ & \leftrightarrow & \langle x \sqsubseteq z^{-} \rangle \wedge \langle y \sqsubseteq z^{-} \rangle & \{\ (\mathrm{sup})\ \} \\ & \leftrightarrow & \langle z \sqsubseteq x^{-} \rangle \wedge \langle z \sqsubseteq y^{-} \rangle & \{\ (\sqsubseteq^{-})\ \} \\ & \leftrightarrow & \langle z \sqsubseteq x^{-} \sqcap y^{-} \rangle. & \{\ (\mathrm{inf})\ \} \end{array}$$

$(\sqsubseteq^{-}_{*})$ $\langle x \sqcap y \sqsubseteq z \sqcup w \rangle \leftrightarrow \langle x \sqcap z^{-} \sqsubseteq y^{-} \sqcup w \rangle$:

$$\begin{array}{lll} & \langle x \sqcap y \sqsubseteq z \sqcup w \rangle & \\ \leftrightarrow & \langle (x \sqcap y) \sqcap (z \sqcup w)^{-} = x^{\emptyset} \rangle & \{\ (\sqsubseteq^{-})\ \} \\ \leftrightarrow & \langle (x \sqcap y) \sqcap (z^{-} \sqcap w^{-}) = x^{\emptyset} \rangle & \{\ (\text{de Morgan})\ \} \\ \leftrightarrow & \langle (x \sqcap z^{-}) \sqcap (y \sqcap w^{-}) = x^{\emptyset} \rangle & \{\ [\mathrm{com}^{\sqcap}],\ (\mathrm{ass}^{\sqcap})\ \} \\ \leftrightarrow & \langle (z \sqcap z^{-}) \sqcap (y^{--} \sqcap w^{-}) = x^{\emptyset} \rangle & \{\ (^{--})\ \} \\ \leftrightarrow & \langle (x \sqcap z^{-}) \sqcap (y^{-} \sqcup w)^{-} = x^{\emptyset} \rangle & \{\ (\text{de Morgan})\ \} \\ \leftrightarrow & \langle x \sqcap z^{-} \sqsubseteq y^{-} \sqcup w \rangle. & \{\ (\sqsubseteq^{-})\ \} \end{array}$$

In particular, $\langle x \sqsubseteq y \rangle \leftrightarrow \langle y^{-} \sqsubseteq x^{-} \rangle$ holds. □

A substitution law $\langle x = y \rangle \wedge \langle x \sqsubseteq z \rangle \to \langle y \sqsubseteq z \rangle$ immediately follows $(\mathrm{ref}^{\sqsubseteq})$ and $(\mathrm{trn}^{\sqsubseteq})$.

5. Axioms for converse

We may say that the converse $^{\sharp}$ is a particular operation characterizing relational calculus.

AXIOM 5.1 AXIOMS FOR CONVERSE.

$[\mathrm{reg}^{\sharp}]$ $\langle x^{\sharp} \rangle \to \langle x \rangle$,

$[\mathrm{def}^{\sharp}]$ $\langle x \parallel y \rangle \to \langle xy^{\sharp} \rangle \wedge \langle x^{\sharp}y \rangle$,

[MS] $\langle xy \sqcap z = z^{\emptyset} \rangle \leftrightarrow \langle x^{\sharp}z \sqcap y = y^{\emptyset} \rangle \leftrightarrow \langle zy^{\sharp} \sqcap x = x^{\emptyset} \rangle$.

(de Morgan-Schröder equivalences) □

de Morgan-Schröder equivalences [MS] are often given as

$$\langle xy \sqsubseteq z \rangle \leftrightarrow \langle x^{\sharp}z^{-} \sqsubseteq y^{-} \rangle \leftrightarrow \langle z^{-}y^{\sharp} \sqsubseteq x^{-} \rangle,$$

and it is equivalent to Dedekind formula

$$\langle xy \parallel z \rangle \to \langle xy \sqcap z \sqsubseteq (x \sqcap zy^{\sharp})(y \sqcap x^{\sharp}z) \rangle.$$

A formal system satisfying all axioms axioms 2.1, 3.1, 4.1 and 5.1 is sometimes called *elementary relational calculus*. Historically, almost all relational calculus till relation algebras [Tarski (1941)] might be studied within elementary relational calculus except for categorical setting.

Proposition 5.2. In relational calculus the following hold.

$(\mathrm{reg}^{\sharp}_{*})$	$\langle x^{\sharp} \rangle \leftrightarrow \langle x \rangle$,	
$({}^{\sharp\,\ominus})$	$\langle x \rangle \to \langle x^{\sharp\,\ominus} = x^{\oplus} \rangle$,	
$({}^{\sharp\,\oplus})$	$\langle x \rangle \to \langle x^{\sharp\,\oplus} = x^{\ominus} \rangle$,	
$(\mathrm{def}^{\,\sharp}_{*})$	$\langle x \rangle \to \langle xx^{\sharp} \parallel x^{\ominus} \rangle \wedge \langle x^{\sharp}x \parallel x^{\oplus} \rangle$,	
$(\parallel^{\sharp})$	$\langle x \parallel y \rangle \to \langle x^{\sharp} \parallel y^{\sharp} \rangle$.	
$(\mathrm{sbn}^{\sharp})$	$\langle x = y \rangle \to \langle x^{\sharp} = y^{\sharp} \rangle$,	(substitution)
$({}^{\sharp\,\sharp})$	$\langle x \rangle \to \langle x^{\sharp\,\sharp} = x \rangle$,	(involutive)
$(\mathrm{C}^{\sharp})$	$\langle xy \rangle \to \langle (xy)^{\sharp} = y^{\sharp}x^{\sharp} \rangle$,	(contravariant)
$(\mathrm{id}^{\sharp})$	$\langle x \rangle \to \langle x^{\ominus\,\sharp} = x^{\ominus} \rangle \wedge \langle x^{\oplus\,\sharp} = x^{\oplus} \rangle$,	
$(\mathrm{C}^{\emptyset})$	$\langle xy \rangle \to \langle xy^{\emptyset} = (xy)^{\emptyset} \rangle \wedge \langle x^{\emptyset}y = (xy)^{\emptyset} \rangle$,	(zero law)
$(\sqcup^{C})$	$\langle (x \sqcup y)z \simeq xz \sqcup yz \rangle$,	(distributive)
	$\langle x(y \sqcup z) \simeq xy \sqcup xz \rangle$,	
$(\sqsubseteq^{C})$	$\langle x \sqsubseteq y \rangle \wedge \langle z \sqsubseteq w \rangle \wedge \langle xz \rangle \to \langle xz \sqsubseteq yw \rangle$,	(monotonic)
$(\sqcap^{C})$	$\langle x(y \sqcap z) \rangle \to \langle x(y \sqcap z) \sqsubseteq xy \sqcap xz \rangle$,	(subdistributive)
	$\langle (x \sqcap y)z) \rangle \to \langle (x \sqcap y)z \sqsubseteq xz \sqcap yz \rangle$,	
$(\sqcup^{\sharp})$	$\langle x \parallel y \rangle \to \langle (x \sqcup y)^{\sharp} = x^{\sharp} \sqcup y^{\sharp} \rangle$,	
$(\sqcap^{\sharp})$	$\langle x \parallel y \rangle \to \langle (x \sqcap y)^{\sharp} = x^{\sharp} \sqcap y^{\sharp} \rangle$,	
$(\sqsubseteq^{\sharp})$	$\langle x \sqsubseteq y \rangle \to \langle x^{\sharp} \sqsubseteq y^{\sharp} \rangle$,	(monotonic)
$({}^{\nabla\,\sharp})$	$\langle x \rangle \to \langle x^{\nabla\,\sharp} = x^{\sharp\,\nabla} \rangle$,	
$({}^{\emptyset\,\sharp})$	$\langle x \rangle \to \langle x^{\emptyset\,\sharp} = x^{\sharp\,\emptyset} \rangle$,	
$({}^{-\,\sharp})$	$\langle x \rangle \to \langle x^{-\,\sharp} = x^{\sharp\,-} \rangle$.	

Proof. $(\mathrm{reg}^{\sharp}_{*})$ $\langle x^{\sharp} \rangle \leftrightarrow \langle x \rangle$:

$$
\begin{aligned}
\langle x^{\sharp} \rangle &\to \langle x \rangle && \{\ [\mathrm{reg}^{\sharp}]\ \} \\
&\to \langle x^{\sharp}x \rangle && \{\ [\mathrm{def}^{\,\sharp}]\ \} \\
&\to \langle x^{\sharp} \rangle. && \{\ (\mathrm{reg}^{C})\ \}
\end{aligned}
$$

$({}^{\sharp\,\ominus})$ $\langle x \rangle \to \langle x^{\sharp\,\ominus} = x^{\oplus} \rangle$:

$$
\begin{aligned}
\langle x \rangle &\to \langle xx^{\sharp} \rangle && \{\ [\mathrm{def}^{\,\sharp}]\ \} \\
&\to \langle x^{\oplus} = x^{\sharp\,\ominus} \rangle. && \{\ [\mathrm{def}^{\,C}]\ \}
\end{aligned}
$$

(def$^\sharp_*$) $\langle\, x\,\rangle \to \langle\, xx^\sharp \parallel x^\ominus\,\rangle \wedge \langle\, x^\sharp x \parallel x^\oplus\,\rangle$:

$$\begin{array}{lcll}
\langle\, x\,\rangle & \to & \langle\, x \parallel x\,\rangle & \{\ (\mathrm{ref}^{\parallel})\ \} \\
 & \to & \langle\, xx^\sharp\,\rangle & \{\ [\mathrm{def}^{\sharp}]\ \} \\
 & \to & \langle\,(xx^\sharp)^\ominus = x^\ominus\,\rangle \wedge \langle\,(xx^\sharp)^\oplus = x^{\sharp\,\oplus}\,\rangle & \{\ (\mathrm{C}^\ominus),\ (\mathrm{C}^\oplus)\ \} \\
 & \to & \langle\,(xx^\sharp)^\ominus = x^\ominus\,\rangle \wedge \langle\,(xx^\sharp)^\oplus = x^\ominus\,\rangle & \{\ (^{\sharp\,\oplus})\ \} \\
 & \to & \langle\,(xx^\sharp)^\ominus = x^{\ominus\,\ominus}\,\rangle \wedge \langle\,(xx^\sharp)^\oplus = x^{\ominus\,\oplus}\,\rangle & \{\ (^{\ominus\,\ominus}),\ (^{\ominus\,\oplus})\ \} \\
 & \to & \langle\, xx^\sharp \parallel x^\ominus\,\rangle. & \{\ \parallel\ \ \}
\end{array}$$

($\parallel^\sharp$) $\langle\, x \parallel y\,\rangle \to \langle\, x^\sharp \parallel y^\sharp\,\rangle$:

$$\begin{array}{lcll}
\langle\, x \parallel y\,\rangle & \to & \langle\, x^\ominus = y^\ominus\,\rangle \wedge \langle\, x^\oplus = y^\oplus\,\rangle & \{\ \parallel\ \} \\
 & \to & \langle\, x^{\sharp\,\oplus} = y^{\sharp\,\oplus}\,\rangle \wedge \langle\, x^{\sharp\,\ominus} = y^{\sharp\,\ominus}\,\rangle & \{\ (^{\sharp\,\oplus}),\ (^{\sharp\,\ominus})\ \} \\
 & \to & \langle\, x^\sharp \parallel y^\sharp\,\rangle. & \{\ \parallel\ \}
\end{array}$$

(sbn$^\sharp$) $\langle\, x = y\,\rangle \to \langle\, x^\sharp = y^\sharp\,\rangle$:
Assume $\langle\, x = y\,\rangle$. For all z we have

$$\begin{array}{lcll}
\langle\, x^\sharp \sqcap z = z^\emptyset\,\rangle & \leftrightarrow & \langle\, x^\sharp u \sqcap z = z^\emptyset\,\rangle & \{\ u = z^\oplus,\ (\mathrm{id}^\oplus_*)\ \} \\
 & \leftrightarrow & \langle\, xz \sqcap u = u^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \} \\
 & \leftrightarrow & \langle\, yz \sqcap u = u^\emptyset\,\rangle & \{\ [\mathrm{sbn}^C],\ [\mathrm{sbn}^\sqcap]\ \} \\
 & \leftrightarrow & \langle\, y^\sharp u \sqcap z = z^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \} \\
 & \leftrightarrow & \langle\, y^\sharp \sqcap z = z^\emptyset\,\rangle, & \{\ u = z^\oplus,\ (\mathrm{id}^\oplus_*)\ \}
\end{array}$$

which implies $\langle\, x^\sharp = y^\sharp\,\rangle$.

($^{\sharp\,\sharp}$) $\langle\, x\,\rangle \to \langle\, x^{\sharp\,\sharp} = x\,\rangle$:
Note that $\langle\, x^{\sharp\,\sharp}\,\rangle \leftrightarrow \langle\, x\,\rangle$ holds by (reg$^\sharp_*$). Assume $\langle\, x\,\rangle$. For all z we have

$$\begin{array}{lcll}
\langle\, x \sqcap z = z^\emptyset\,\rangle & \leftrightarrow & \langle\, xu \sqcap z = z^\emptyset\,\rangle & \{\ u = z^\oplus,\ (\mathrm{id}^\oplus_*)\ \} \\
 & \leftrightarrow & \langle\, x^\sharp z \sqcap u = u^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \} \\
 & \leftrightarrow & \langle\, x^{\sharp\,\sharp} u \sqcap z = z^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \} \\
 & \leftrightarrow & \langle\, x^{\sharp\,\sharp} \sqcap z = z^\emptyset\,\rangle. & \{\ u = z^\oplus,\ (\mathrm{id}^\oplus_*)\ \}
\end{array}$$

Hence $\langle\, x^{\sharp\,\sharp} = x\,\rangle$ holds by ($\forall^{\sqsubseteq}$).

(C$^\sharp$) $\langle\, xy\,\rangle \to \langle\,(xy)^\sharp = y^\sharp x^\sharp\,\rangle$:
Assume $\langle\, xy\,\rangle$. For all z it follows that

$$\begin{array}{lcll}
\langle\,(xy)^\sharp \sqcap z = z^\emptyset\,\rangle & \leftrightarrow & \langle\,(xy)^\sharp u \sqcap z = z^\emptyset\,\rangle & \{\ u = z^\oplus,\ (\mathrm{id}^\oplus_*)\ \} \\
 & \leftrightarrow & \langle\,(xy)z \sqcap u = u^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \} \\
 & \leftrightarrow & \langle\, x(yz) \sqcap u = u^\emptyset\,\rangle & \{\ [\mathrm{ass}^C]\ \} \\
 & \leftrightarrow & \langle\, x^\sharp u \sqcap yz = (yz)^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \} \\
 & \leftrightarrow & \langle\, yz \sqcap x^\sharp = x^{\sharp\,\emptyset}\,\rangle & \{\ u = z^\oplus,\ (\mathrm{id}^\oplus_*)\ ,\ \mathrm{com}^\sqcap\ \} \\
 & \leftrightarrow & \langle\, y^\sharp x^\sharp \sqcap z = z^\emptyset\,\rangle, & \{\ [\mathrm{MS}]\ \}
\end{array}$$

which implies $\langle\, (xy)^\sharp = y^\sharp x^\sharp \,\rangle$.

(id$^\sharp$) $\langle\, x \,\rangle \to \langle\, x^{\ominus\,\sharp} = x^\ominus \,\rangle \wedge \langle\, x^{\oplus\,\sharp} = x^\oplus \,\rangle$:

$$\begin{array}{lll} \langle\, x \,\rangle & \to \langle\, x^\ominus \,\rangle & \{\ [\mathrm{sbn}^\ominus]\ \} \\ & \to \langle\, x^\ominus x^{\ominus\,\sharp} \,\rangle & \{\ [\mathrm{def}^\sharp]\ \} \\ & \to \langle\, x^\ominus x^{\ominus\,\sharp} = x^{\ominus\,\sharp} \,\rangle & \{\ (\mathrm{id}^\ominus_*)\ \} \\ & \to \langle\, x^{\ominus\,\sharp\sharp} x^{\ominus\,\sharp} = x^{\ominus\,\sharp} \,\rangle & \{\ (^{\sharp\sharp}),\ [\mathrm{sbn}^C]\ \} \\ & \to \langle\, (x^\ominus x^{\ominus\,\sharp})^\sharp = x^{\ominus\,\sharp} \,\rangle & \{\ (\mathrm{C}^\sharp)\ \} \\ & \to \langle\, x^{\ominus\,\sharp\sharp} = x^{\ominus\,\sharp} \,\rangle & \{\ (\mathrm{id}^\ominus_*),\ (\mathrm{sbn}^\sharp)\ \} \\ & \to \langle\, x^\ominus = x^{\ominus\,\sharp} \,\rangle. & \{\ (^{\sharp\sharp})\ \} \end{array}$$

(C$^\emptyset$) $\langle\, xy \,\rangle \to \langle\, xy^\emptyset = (xy)^\emptyset \,\rangle$:

$$\begin{array}{lll} \langle\, xy \,\rangle & \to \langle\, x^\sharp x \parallel x^\oplus \,\rangle \wedge \langle\, xy \,\rangle & \{\ (\mathrm{def}^\sharp_*)\ \} \\ & \to \langle\, (x^\sharp x)y = x^\sharp(xy) \,\rangle \wedge \langle\, x^\sharp x \parallel x^\oplus \,\rangle & \{\ (\mathrm{ass}^C)\ \} \\ & \to \langle\, (x^\sharp x)y \parallel x^\sharp(xy) \,\rangle \wedge \langle\, (x^\sharp x)y \parallel x^\oplus y \,\rangle & \{\ (\mathrm{sbn}^\sharp)\ \} \\ & \to \langle\, x^\sharp(xy) \parallel y \,\rangle \wedge \langle\, xy \parallel (xy)^\nabla \,\rangle & \{\ (\mathrm{id}^\oplus_*)\ \} \\ & \to \langle\, x^\sharp(xy)^\nabla \parallel y \,\rangle & \{\ (\mathrm{sbn}^\parallel)\ \} \\ & \to \langle\, x^\sharp(xy)^\nabla \sqsubseteq y^\nabla \,\rangle & \{\ (^\nabla)\ \} \\ & \to \langle\, xy^{\nabla\,-} \sqsubseteq (xy)^{\nabla\,-} \,\rangle & \{\ [\mathrm{MS}]\ \} \\ & \to \langle\, xy^\emptyset \sqsubseteq (xy)^\emptyset \,\rangle & \{\ (^{\nabla\,-})\ \} \\ & \to \langle\, xy^\emptyset = xy^\emptyset \sqcap (xy)^\emptyset \,\rangle & \\ & \to \langle\, xy^\emptyset = (xy)^\emptyset \,\rangle. & \{\ (^{\emptyset\,\sqcap})\ \} \end{array}$$

Remark that $\langle\, xy \,\rangle \to \langle\, xy^\nabla = (xy)^\nabla \,\rangle$ fails in general. For example, if $\langle\, e = e^\ominus = e^\nabla \,\rangle$ and $\langle\, e \neq e^\emptyset \,\rangle$ (Cf. Axiom 6.1), then $\langle\, e^\emptyset e^\nabla = e^\emptyset e = e^\emptyset \,\rangle$ and $\langle\, (e^\emptyset e)^\nabla = e^{\emptyset\,\nabla} = e^\nabla = e \,\rangle$.

($\sqcup^C$) $\langle\, (x \sqcup y)z \simeq xz \sqcup yz \,\rangle$:

$$\begin{array}{lll} \langle\, (x \sqcup y)z \sqsubseteq w \,\rangle & \leftrightarrow \langle\, w^- z^\sharp \sqsubseteq (x \sqcup y)^- \,\rangle & \{\ [\mathrm{MS}]\ \} \\ & \leftrightarrow \langle\, w^- z^\sharp \sqsubseteq x^- \sqcap y^- \,\rangle & \{\ (\text{de Morgan})\ \} \\ & \leftrightarrow \langle\, w^- z^\sharp \sqsubseteq x^- \wedge w^- z^\sharp \sqsubseteq y^- \,\rangle & \{\ (\mathrm{inf})\ \} \\ & \leftrightarrow \langle\, xz \sqsubseteq w \,\rangle \wedge \langle\, yz \sqsubseteq w \,\rangle & \{\ [\mathrm{MS}]\ \} \\ & \leftrightarrow \langle\, xz \sqcup yz \sqsubseteq w \,\rangle. & \{\ (\mathrm{sup})\ \} \end{array}$$

($\sqsubseteq^C$) $\langle\, x \sqsubseteq y \,\rangle \wedge \langle\, z \sqsubseteq w \,\rangle \wedge \langle\, xz \,\rangle \to \langle\, xz \sqsubseteq yw \,\rangle$:
It suffices to show that $\langle\, x \sqsubseteq y \,\rangle \wedge \langle\, xz \,\rangle \to \langle\, xz \sqsubseteq yz \,\rangle$.

$$\begin{array}{lll} \langle\, yz \sqsubseteq yz \,\rangle & \leftrightarrow \langle\, (yz)^- z^\sharp \sqsubseteq y^- \,\rangle & \{\ [\mathrm{MS}]\ \} \\ & \to \langle\, (yz)^- z^\sharp \sqsubseteq x^- \,\rangle & \{\ (\sqsubseteq^-_*)\ y^- \sqsubseteq x^-\ \} \\ & \leftrightarrow \langle\, xz \sqsubseteq yz \,\rangle. & \{\ [\mathrm{MS}]\ \} \end{array}$$

Tarski [Tarski (1941)] remarked that the assertion deduces [sbn^C].

$(\sqcap^C)$ $\langle\,(x\sqcap y)z)\,\rangle \to \langle\,(x\sqcap y)z \sqsubseteq xz\sqcap yz\,\rangle$:

$$\begin{array}{lll}\langle\,(x\sqcap y)z\,\rangle & \to & \langle\,(x\sqcap y)z)\,\rangle \wedge \langle\, x\sqcap y\sqsubseteq x\,\rangle \wedge \langle\, x\sqcap y \sqsubseteq y\,\rangle \quad \{\text{ (inf) }\}\\ & \to & \langle\,(x\sqcap y)z\sqsubseteq xz\,\rangle \wedge \langle\,(x\sqcap y)z\sqsubseteq yz\,\rangle \quad \{\ (\sqsubseteq^C)\ \}\\ & \to & \langle\,(x\sqcap y)z\sqsubseteq xz\sqcap yz\,\rangle. \quad \{\text{ (inf) }\}\end{array}$$

$(\sqcup^\sharp)$ $\langle\, x\parallel y\,\rangle \to \langle\,(x\sqcup y)^\sharp = x^\sharp\sqcup y^\sharp\,\rangle$: For the proof we use 4.3 $(\forall^\emptyset)$. Assume $\langle\, x\parallel y\,\rangle$.

$$\begin{array}{lll} & \langle\,(x\sqcup y)^\sharp\sqcap z = z^\emptyset\,\rangle & \\ \leftrightarrow & \langle\,(x\sqcup y)^\sharp u\sqcap z = z^\emptyset\,\rangle & \{\ u = z^\oplus,\ (\mathrm{id}_*^\oplus)\ \}\\ \leftrightarrow & \langle\,(x\sqcup y)z\sqcap u = u^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \}\\ \leftrightarrow & \langle\,(xz\sqcup yz)\sqcap u = u^\emptyset\,\rangle & \{\ (\sqcup^C)\ \}\\ \leftrightarrow & \langle\,(xz\sqcap u)\sqcup(yz\sqcap u) = u^\emptyset\,\rangle & \{\ [\mathrm{dst}^\sqcap]\ \}\\ \leftrightarrow & \langle\, xz\sqcap u = u^\emptyset\,\rangle\wedge\langle\, yz\sqcap u = u^\emptyset\,\rangle & \{\ (^\emptyset),\ (\mathrm{idm}^\sqcup)\ \}\\ \leftrightarrow & \langle\, x^\sharp u\sqcap z = z^\emptyset\,\rangle\wedge\langle\, y^\sharp u\sqcap z = z^\emptyset\,\rangle & \{\ [\mathrm{MS}]\ \}\\ \leftrightarrow & \langle\,(x^\sharp u\sqcap z)\sqcup(y^\sharp u\sqcap z) = z^\emptyset\,\rangle & \{\ (\mathrm{idm}^\sqcup),\ (^\emptyset)\ \}\\ \leftrightarrow & \langle\,(x^\sharp u\sqcup y^\sharp u)\sqcap z = z^\emptyset\,\rangle & \{\ [\mathrm{dst}^\sqcap]\ \}\\ \leftrightarrow & \langle\,(x^\sharp\sqcup y^\sharp)\sqcap z = z^\emptyset\,\rangle. & \{\ (\mathrm{id}_*^\oplus)\ \}\end{array}$$

Note that $\langle\,(x\sqcup y)^\sharp\,\rangle \leftrightarrow \langle\, x\parallel y\,\rangle \leftrightarrow \langle\, x^\sharp\sqcup y^\sharp\,\rangle$.

$(\sqsubseteq^\sharp)$ $\langle\, x\sqsubseteq y\,\rangle \to \langle\, x^\sharp\sqsubseteq y^\sharp\,\rangle$:

$$\begin{array}{llll}\langle\, x\sqsubseteq y\,\rangle & \to & \langle\, x\parallel y\,\rangle & \\ & \to & \langle\, x^\sharp\sqcup y^\sharp = (x\sqcup y)^\sharp\,\rangle & \{\ (\sqcup^\sharp)\ \}\\ & \to & \langle\, x^\sharp\sqcup y^\sharp = y^\sharp\,\rangle & \{\ x\sqcup y = y\ \}\\ & \to & \langle\, x^\sharp\sqsubseteq y^\sharp\,\rangle. & \end{array}$$

$(^{\nabla\,\sharp})$ $\langle\, x\,\rangle \to \langle\, x^{\nabla\,\sharp} = x^{\sharp\,\nabla}\,\rangle$:
Assume $\langle\, x\,\rangle$.

$$\begin{array}{llll} x^{\nabla\,\sharp} & = & x^{\sharp\,\nabla}\sqcap x^{\nabla\,\sharp} & \{\ (^\nabla)\ \}\\ & = & x^{\sharp\,\nabla\,\sharp\,\sharp}\sqcap x^{\nabla\,\sharp} & \{\ (^{\sharp\,\sharp})\ \}\\ & = & (x^{\sharp\,\nabla\,\sharp}\sqcap x^{\nabla})^\sharp & \{\ (\sqcap^\sharp)\ \}\\ & = & x^{\sharp\,\nabla\,\sharp\,\sharp} & \{\ (^\nabla)\ \}\\ & = & x^{\sharp\,\nabla}. & \{\ (^{\sharp\,\sharp})\ \}\end{array}$$

$(^{-\,\sharp})$ $\langle\, x\,\rangle \to \langle\, x^{-\,\sharp} = x^{\sharp\,-}\,\rangle$:
Assume $\langle\, x\,\rangle$.

$$\begin{array}{llll} x^\sharp\sqcap x^{-\,\sharp} & = & (x\sqcap x^-)^\sharp & \{\ (\sqcap^\sharp)\ \}\\ & = & x^{\emptyset\,\sharp} & \{\ x^\emptyset\ \}\\ & = & x^{\sharp\,\emptyset}, & \{\ (^{\emptyset\,\sharp})\ \}\end{array}$$

$$\begin{array}{llll} x^\sharp\sqcup x^{-\,\sharp} & = & (x\sqcup x^-)^\sharp & \{\ (\sqcup^\sharp)\ \}\\ & = & x^{\nabla\,\sharp} & \{\ x^\nabla\ \}\\ & = & x^{\sharp\,\nabla}. & \{\ (^{\nabla\,\sharp})\ \}\end{array}$$

Hence $\langle\, x^{-\,\sharp} = x^{\sharp\,-} \,\rangle$ follows ($\mathrm{L}^{=}$). □

A relation x is called a (*total*) *function* (tfn, for short) if $\langle\, x^{\sharp}x \sqsubseteq x^{\oplus} \,\rangle$ (univalent) and $\langle\, x^{\ominus} \sqsubseteq xx^{\sharp} \,\rangle$ (total). We will abbreviate a formula

$$\langle\, x^{\sharp}x \sqsubseteq x^{\oplus} \,\rangle \wedge \langle\, x^{\ominus} \sqsubseteq xx^{\sharp} \,\rangle$$

as $\langle\, x : \mathrm{tfn} \,\rangle$. Every identity relation is a function. Likewise, injection, surjection and bijection are defined in relational calculus.

PROPOSITION 5.3. *In relational calculus the following hold.*

(a) $\langle\, x, y : \mathrm{tfn} \,\rangle \to (\langle\, zy \sqsubseteq xw \,\rangle \leftrightarrow \langle\, x^{\sharp}z \sqsubseteq wy^{\sharp} \,\rangle)$,

(b) $\langle\, x, y : \mathrm{tfn} \,\rangle \wedge \langle\, x \sqsubseteq y \,\rangle \to \langle\, x = y \,\rangle$,

(c) $\langle\, x, y : \mathrm{tfn} \,\rangle \to \langle\, x(z \sqcap w)y^{\sharp} \simeq xzy^{\sharp} \sqcap xwy^{\sharp} \,\rangle$,

(d) $\langle\, x, y : \mathrm{tfn} \,\rangle \to \langle\, (xzy^{\sharp})^{-} \simeq xz^{-}y^{\sharp} \,\rangle$.

Proof. (a) $\langle\, x, y : \mathrm{tfn} \,\rangle \to (\langle\, zy \sqsubseteq xw \,\rangle \leftrightarrow \langle\, x^{\sharp}z \sqsubseteq wy^{\sharp} \,\rangle)$:
Assume $\langle\, x, y : \mathrm{tfn} \,\rangle$.

$$\begin{array}{rcll}
\langle\, zy \sqsubseteq xw \,\rangle & \to & \langle\, x^{\sharp}zyy^{\sharp} \sqsubseteq x^{\sharp}xwy^{\sharp} \,\rangle & \{\ (\sqsubseteq^{C})\ \} \\
 & \to & \langle\, x^{\sharp}zy^{\ominus} \sqsubseteq x^{\oplus}wy^{\sharp} \,\rangle & \{\ x, y : \mathrm{tfn}\ \} \\
 & \to & \langle\, x^{\sharp}z \sqsubseteq wy^{\sharp} \,\rangle & \{\ (\mathrm{id}^{\oplus}_{*}),\ (\mathrm{id}^{\ominus}_{*})\ \} \\
 & \to & \langle\, xx^{\sharp}zy \sqsubseteq xwy^{\sharp}y \,\rangle & \{\ (\sqsubseteq^{C})\ \} \\
 & \to & \langle\, x^{\ominus}zy \sqsubseteq xwy^{\ominus} \,\rangle & \{\ x, y : \mathrm{tfn}\ \} \\
 & \to & \langle\, zy \sqsubseteq xw \,\rangle. & \{\ (\mathrm{id}^{\ominus}_{*}),\ (\mathrm{id}^{\oplus}_{*})\ \}
\end{array}$$

(b) $\langle\, x, y : \mathrm{tfn} \,\rangle \wedge \langle\, x \sqsubseteq y \,\rangle \to \langle\, x = y \,\rangle$:
Assume $\langle\, x, y : \mathrm{tfn} \,\rangle$.

$$\begin{array}{rcll}
\langle\, x \sqsubseteq y \,\rangle & \to & \langle\, x^{\ominus}x \sqsubseteq yy^{\oplus} \,\rangle & \{\ (\mathrm{id}^{\ominus}_{*}),\ (\mathrm{id}^{\oplus}_{*})\ \} \\
 & \to & \langle\, y^{\sharp}x^{\ominus} \sqsubseteq y^{\oplus}x^{\sharp} \,\rangle & \{\ x, y : \mathrm{tfn}, \text{(a)}\ \} \\
 & \to & \langle\, y^{\sharp} \sqsubseteq x^{\sharp} \,\rangle & \{\ (\mathrm{id}^{\oplus}_{*}),\ (\mathrm{id}^{\ominus}_{*})\ \} \\
 & \to & \langle\, y \sqsubseteq x \,\rangle & \{\ (\sqsubseteq^{\sharp}),\ (^{\sharp\,\sharp})\ \} \\
 & \to & \langle\, x = y \,\rangle. &
\end{array}$$

(c) $\langle\, x, y : \mathrm{tfn} \,\rangle \to \langle\, x(z \sqcap w)y^{\sharp} \simeq xzy^{\sharp} \sqcap xwy^{\sharp} \,\rangle$:
Assume $\langle\, x, y : \mathrm{tfn} \,\rangle$. For all u it follows that

$$\begin{array}{rcll}
u \sqsubseteq x(z \sqcap w)y^{\sharp} & \leftrightarrow & x^{\sharp}uy \sqsubseteq z \sqcap w & \{\ \text{(b)}\ \} \\
 & \leftrightarrow & x^{\sharp}uy \sqsubseteq z \wedge x^{\sharp}uy \sqsubseteq w & \{\ \text{(inf)}\ \} \\
 & \leftrightarrow & u \sqsubseteq xzy^{\sharp} \wedge u \sqsubseteq xwy^{\sharp} & \{\ \text{(b)}\ \} \\
 & \leftrightarrow & u \sqsubseteq xzy^{\sharp} \sqcap xwy^{\sharp}, & \{\ \text{(inf)}\ \}
\end{array}$$

which implies $\langle\, x(z \sqcap w)y^{\sharp} = xzy^{\sharp} \sqcap xwy^{\sharp} \,\rangle$.

(d) $\langle\, x,\, y : \mathrm{tfn}\,\rangle \to \langle\, xz^{\nabla}y^{\sharp} \simeq (xzy^{\sharp})^{\nabla}\,\rangle$:
Assume $\langle\, x,\, y : \mathrm{tfn}\,\rangle$ and $\langle\, xzy^{\sharp}\,\rangle$.

$$\begin{array}{lll}\to & \langle\, x^{\sharp}(xzy^{\sharp})^{\nabla}y \sqsubseteq z^{\nabla}\,\rangle & \{\, x^{\sharp}(xzy^{\sharp})^{\nabla}y \parallel z,\ (^{\nabla})\,\} \\ \to & \langle\, (xzy^{\sharp})^{\nabla} \sqsubseteq xz^{\nabla}y^{\sharp}\,\rangle & \{\, x,\, y : \mathrm{tfn},\, \text{(a)}\,\} \\ \to & \langle\, (xzy^{\sharp})^{\nabla} = xz^{\nabla}y^{\sharp}\,\rangle. & \{\, (^{\nabla})\,\}\end{array}$$

(e) $\langle\, x,\, y : \mathrm{tfn}\,\rangle \to \langle\, (xzy^{\sharp})^{-} \simeq xz^{-}y^{\sharp}\,\rangle$:
Assume $\langle\, x,\, y : \mathrm{tfn}\,\rangle$ and $\langle\, xzy^{\sharp}\,\rangle$.

$$\begin{array}{rcll} xzy^{\sharp} \sqcup xz^{-}y^{\sharp} & = & x(z \sqcup z^{-})y^{\sharp} & \{\, (\sqcup^{C})\,\} \\ & = & xz^{\nabla}y^{\sharp} & \\ & = & (xzy^{\sharp})^{\nabla}, & \{\, \text{(d)}\,\} \end{array}$$

$$\begin{array}{rcll} xzy^{\sharp} \sqcap xz^{-}y^{\sharp} & = & x(z \sqcap z^{-})y^{\sharp} & \{\, \text{(c)}\,\} \\ & = & xz^{\emptyset}y^{\sharp} & \\ & = & (xzy^{\sharp})^{\emptyset}. & \{\, (\mathrm{C}^{\emptyset})\,\} \end{array}$$

Hence $\langle\, (xzy^{\sharp})^{-} = xz^{-}y^{\sharp}\,\rangle$ holds by $(\mathrm{L}_0^{=})$. □

In some applications of relations one prefers to use residual compositions (due by Peirce [Peirce (1883)]):

$$x \rhd y = (xy^{-})^{-} \quad \text{and} \quad x \lhd y = (x^{-}y)^{-}.$$

de Morgan-Schröder equivalences [MS] yield the residual equivalences

$$\langle\, z \sqsubseteq x \rhd y\,\rangle \leftrightarrow \langle\, x^{\sharp}z \sqsubseteq y\,\rangle \quad \text{and} \quad \langle\, z \sqsubseteq x \lhd y\,\rangle \leftrightarrow \langle\, zy^{\sharp} \sqsubseteq x\,\rangle.$$

It is well-known that a lot of formulas concerned with residual compositions can be derived using the residual equivalences.

6. Relational calculus $\mathbf{Rel}_*$

Finally we mention additional axioms for relational calculus below.

AXIOM 6.1.

(Unit) $\exists e.\ (\langle\, e = e^{\ominus}\,\rangle \wedge \langle\, e = e^{\nabla}\,\rangle \wedge \langle\, e \neq e^{\emptyset}\,\rangle)$,

(Tot) $\forall x.\ (\langle\, x^{\ominus} \sqsubseteq xx^{\sharp}\,\rangle \vee \langle\, x^{\oplus} \sqsubseteq x^{-\,\sharp}x^{-}\,\rangle)$,

(AC_0) $\forall x.\ (\langle\, x^{\ominus} \sqsubseteq xx^{\sharp}\,\rangle \to \exists y.\ (\langle\, y \sqsubseteq x\,\rangle \wedge \langle\, y : \mathrm{tfn}\,\rangle))$,

(Tab) $\forall x\, \exists f\, \exists g.\ (\langle\, f, g : \mathrm{tfn}\,\rangle \wedge \langle\, x = f^{\sharp}g\,\rangle \wedge \langle\, ff^{\sharp} \sqcap gg^{\sharp} = f^{\ominus}\,\rangle)$,

(Comp) For all relations t and all formulas ϕ the following hold.

$$\exists z\, \forall y.\ (\langle\, y \sqsubseteq z\,\rangle \leftrightarrow \forall x.\ (\phi \to \langle\, y \sqsubseteq t\,\rangle)),$$

and

$$\exists z\, \forall y.\ (\langle\, z \sqsubseteq y\,\rangle \leftrightarrow \forall x.\ (\phi \to \langle\, t \sqsubseteq y\,\rangle)).$$ □

The axiom of unit (Unit) due to [Freyd and Scedrov (1990)] guarantees the existence of an object e (called a unit) analogous to a singleton set. Tarski [Tarski (1941)] introduced the axiom of totality (Tot). The axiom of relational choice (AC_0) may be equivalent to the existence of Skolem functions in predicate logic. The axiom of tabulation (Tab) by [Freyd and Scedrov (1990)] indicates that relations are included in cartesian products. The axiom of completeness (Comp) indicates that there exist the infimum and the supremum for a collection $\{t = t(x) \mid \phi(x)\}$ of relations. It simply generalizes the existence of hereditary closures due to Frege [Frege (1879)]. For example, let a be an endorelation ($a^{\ominus} = a^{\oplus}$), x a term $t(x)$ and $\langle\, xa \sqcup a^{\ominus} \sqsubseteq x \,\rangle$ a formula $\phi(x)$. Then the reflexive and transitive closure a^* of a is characterized by a formula

$$\langle\, y \sqsubseteq a^* \,\rangle \leftrightarrow \forall x.\ (\langle\, ax \sqcup a^{\ominus} \sqsubseteq x \,\rangle \rightarrow \langle\, y \sqsubseteq x \,\rangle).$$

Definition 6.2. A formal system satisfying all axioms 2.1, 3.1, 4.1, 5.1 and 6.1 is called *relational calculus* Rel_*. □

The relational calculus Rel_* given in the paper is a formal system of algebraic logic, completely independent of set theory. We are aware of many properties on points (a function from the unit e), Cantor's diagonal argument for comprehensive relations, Bernstein-Schröder theorem on the cardinality of relations, elementary theory of Dedekind finiteness and so on. In particular, Lyndon's formulas known as an evidence of the incompleteness of relation algebras [Tarski (1941)] are provable in Rel_*. Also we can develop a modification of Morse-Kelley set theory [Kelley (1955)] based on Rel_*. These interesting discussion on the basic properties of Rel_* will be publish elsewhere.

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