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KAWATATE, Kazuo
Research Institute for Applied Mechanics, Kyushu University : Associate Professor

https://doi.org/10.5109/7170232

出版情報：Reports of Research Institute for Applied Mechanics. 17 (58), pp.123-149, 1969. 九州大学応用力学研究所
バージョン：
権利関係：

Reports of Research Institute for Applied Mechananics
Vol. XVII. No. 58, 1969

STRESS DISTRIBUTION ON THE COLLISION SURFACE OF A SEMI-INFINITE ELASTIC PLANE WITH A RIGID BEAM

By Kazuo KAWATATE*

Dynamic contact problem is studied. Dual integral equations are derived as usual for the mixed boundary value problem: the shear stress vanishes on the entire surface of an elastic half plane, an impinged rigid beam moves with the elastic plane in the direction of right angle to the contact surface between them, and the normal stress disappears on the free surface. By considering the stress distribution on the contact surface, the dual integral equations are reduced to the Fredholm integral equation of the first kind. The transition of stress distribution is found through the numerical procedure.

At the instance of a collision of a rigid beam with an elastic plane, the stress is uniformly distributed on the entire contact surface. While, as time passes, the stress becomes higher around the contact surface than in the middle region. The total force which interacts between two bodies decays monotonously.

1. Introduction

The contact phenomenon which accompanied with the local deformation of bodies at the vicinity of impact point was first incorporated by Timoshenko[1)] in the problem of a collision of one object with another. With his historically famous and physically well defined problem as a momentum, various attempts[2] followed to solve impact problems in which local indentation took place. In these dynamic problems, however, contact stresses were assumed given by the Hertz laws which are established and valid in the static theory of elasticity.

Although the contact phenomenon in the collision problem will be again considered, the transition of stress distribution on the collision surface between two bodies will be especially sought in this report where we shall start from the theory of dynamic elasticity more appropriate for the problem at hand.

2. Basic Equations and Helmholz Functions

Now, if we take Cartesian coordinate system (x, y, z) and express displa-

* Associate Professor, Member of Research Institute for Applied Mechanics, Kyushu University, Hakozaki, Fukuoka, Japan 812.

cement by $u=(u_x, u_y, u_z)$, stress by σ_{xx} etc., mass density by ρ, time by t, and Lame's constants by λ and μ, the equations of motion are given as follows[3]:

$$\left.\begin{aligned}\sigma\frac{\partial^2 u_x}{\partial t^2}&=\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z},\\ \rho\frac{\partial^2 u_y}{\partial t^2}&=\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z},\\ \rho\frac{\partial^2 u_z}{\partial t^2}&=\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z},\end{aligned}\right\} \tag{2.1}$$

and Hooke's law is written[3]

$$\left.\begin{aligned}\sigma_{xx}&=\lambda\Delta+2\mu\frac{\partial u_x}{\partial x},\\ \sigma_{yy}&=\lambda\Delta+2\mu\frac{\partial u_y}{\partial y},\\ \sigma_{zz}&=\lambda\Delta+2\mu\frac{\partial u_z}{\partial z},\\ \sigma_{yz}&=\mu\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right),\\ \sigma_{zx}&=\mu\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right),\\ \sigma_{xy}&=\mu\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right),\end{aligned}\right\} \tag{2.2}$$

in which Δ is a dilatation given by

$$\Delta=\mathrm{div}\,\boldsymbol{u}=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}. \tag{2.3}$$

When stress is eliminated from the above, the basic equation written on displacement is obtained as follows[3]:

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{u}}{\partial t^2}=\beta^2\ \mathrm{grad.\ div}\ \boldsymbol{u}-\mathrm{curl.\ curl}\ \boldsymbol{u}, \tag{2.4}$$

where c represents the velocity of transverse wave that propagates through an extended media

$$c^2=\frac{\mu}{\rho}, \tag{2.5}$$

and β the ratio of the velocity of longitudinal and transverse waves

$$\beta^2=\frac{\lambda+2\mu}{\mu}=\frac{2(1-\nu)}{1-2\nu}, \tag{2.6}$$

ν being Poisson's ratio.

Putting Laplasian operator ∇^2, as is frequently done,

$$\left.\begin{aligned}&\nabla^2=\text{div. grad}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\ \text{(for scalar quantity)}\\&\text{grad. div}-\text{curl. curl}=\nabla^2,\ \text{(the same for vector quantity)}\end{aligned}\right\}\tag{2.7}$$

we shall hereafter use wave operators defined by

$$\Box_1=\nabla^2-\frac{1}{\beta^2}\frac{\partial^2}{\partial T^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial T^2},$$

(for longitudinal wave) (2.8)

and
$$\Box_2=\nabla^2-\frac{\partial^2}{\partial T^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial T^2},$$

(for transverse wave) (2.9)

where $T=ct$. (2.10)

It is easily shown from (2.4) that

$$\Box_1\ \text{div}\ \boldsymbol{u}=0,\tag{2.11}$$

and
$$\Box_2\ \text{curl}\ \boldsymbol{u}=0.\tag{2.12}$$

When we use the curvlinear coordinate system, the tensor calculus is consulted. In order to consider the boundary condition, it is useful to rewrite the results in the physical components. By using the operators grad, div, and curl defined for physical components of the coordinate system adopted[4], the Laplasian operators for scalar and vector quantities are denoted respectively by ∇^2 and ☆:

$$\nabla^2 f=\text{div. grad}\ f,$$
$$☆\,\boldsymbol{F}=\text{grad. div}\ \boldsymbol{F}-\text{curl. curl}\ \boldsymbol{F},$$

so that we may apply them, for example, to (2.11). It happens that $\nabla^2=$☆ in the Cartesian coordinate system. As for general curvilinear coordinate system, ∇^2 is not the same as ☆. However, the above conclusions and the following considerations also hold true for these coordinate systems.

Again depending on the Cartesian coordinate system, we express the displacement in terms of Helmholz functions ϕ and ψ as follows[5]:

$$\boldsymbol{u}=\text{grad}\,\phi+\text{curl}\,\psi,\quad\text{where div}\,\psi=0.\tag{2.13}$$

Substituting the relations

$$\text{div}\ \boldsymbol{u}=\nabla^2\phi,\tag{2.14}$$

and
$$\text{curl}\ \boldsymbol{u}=-\nabla^2\psi\tag{2.15}$$

into the basic equation (2.4), we obtain that

$$\beta^2\ \text{grad}(\Box_1\phi)+\text{curl}(\Box_2\psi)=0.\tag{2.6}$$

By operating div and curl on (2.16), we have

$$\nabla^2\Box_1\phi=0,\tag{2.17}$$

and $$\nabla^2 \Box_2 \psi=0. \tag{2.18}$$

Now, it is easily shown that (2.16) through (2.18) are satisfied if ϕ and ψ are solutions of the equations

$$\Box_1 \phi=0, \tag{2.19}$$

and $$\Box_2 \psi=0, \tag{2.20}$$

which are wave equations and are frequently used in solving the problems of stress wave propagation. Substituting (2.19) and (2.20) into (2.14) and (2.15), we obtain that

$$\text{div } \boldsymbol{u}=\frac{1}{\beta^2}\frac{\partial^2\phi}{\partial T^2}, \quad \text{and} \quad \text{curl } \boldsymbol{u}=\frac{\partial^2\psi}{\partial T^2}.$$

When we examine the limiting case of the present problem as a static state, the last two equations become

$$\text{div } \boldsymbol{u}=0, \text{ and curl } \boldsymbol{u}=0.$$

To avoid this formal difficulty of using the Helmholz functions, we may consult the Galerkin or the Boussinesq-Papkovich-Neuber functions which will be considered in the next two articles. As far as we consider the dynamic boundary value problem, however, it is easily shown that any one of these three kinds of fuctions can be used to reach the same results.

3. Galerkin Function

If we express displacement in terms of vector $\chi=(\chi_x, \chi_y, \chi_z)$ as follows:

$$\boldsymbol{u}=2(1-\nu)\Box_1\chi-\text{grad. div}\chi, \tag{3.1}$$

it become obvious that χ is the dynamic version of Galerkin type potential which is already well known in the static theory of elasticity.[5),6)] Substituting (3.1) into the basic equation (2.4), we have that

$$\Box_1 \Box_2 \chi=0. \tag{3.2}$$

We shall call (3.2) the biwave equation similarly to the biharmonic equation in the static theory. It is reduced from (3.1) that

$$\text{div } \boldsymbol{u}=(1-2\nu) \Box_2 \text{ div } \chi, \tag{3.3}$$

and $$\text{curl } \boldsymbol{u}=2(1-\nu) \Box_1 \text{ curl } \chi. \tag{3.4}$$

Thus, as long as the Galerkin function is governed by the biwave equation, the relations which were derived in the preceding article

$$\Box_1 \text{ div}\boldsymbol{u}=0, \text{ and } \Box_2 \text{ curl } \boldsymbol{u}=0,$$

are satisfied.

It is shown that one of these three components χ_x, χ_y, and χ_z can be put zero. Taking the function f which satisfies the biwave equation

$$\Box_1 \Box_2 \boldsymbol{f}=0, \tag{3.5}$$

and putting

$$\boldsymbol{\chi}'=(1-2\nu)\Box_2\boldsymbol{f}+\text{grad. div}\boldsymbol{f}, \tag{3.6}$$

we find that

$$\Box_1 \Box_2 \boldsymbol{\chi}'=0, \tag{3.7}$$

and

$$\boldsymbol{u}=2(1-\nu)\Box_1\boldsymbol{\chi}'-\text{grad. div}\boldsymbol{\chi}'=0. \tag{3.8}$$

Then let us put $\boldsymbol{f}=(f, 0, 0)$ and write

$$\begin{bmatrix} \chi_x \\ \\ \chi_y \\ \\ \chi_z \end{bmatrix} = \begin{bmatrix} \omega_x \\ \\ \omega_y \\ \\ 0 \end{bmatrix} + \begin{bmatrix} (1-2\nu)\Box_2 f+\dfrac{\partial^2 f}{\partial x^2} \\ \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial z \partial x} \end{bmatrix} \tag{3.9}$$

We can in principle perform this procedure because: it is possible to determine f which satisfies $\chi_z=\dfrac{\partial^2 f}{\partial z\partial x}$; then ω_x and ω_y can be obtained according to (3.9). It is also considered that f, ω_x, and ω_y just obtained satisfy the biwave equation. By virture of (3.8), we substitute (3.9) into (3.1) to obtain that

$$\boldsymbol{u}=2(1-\nu)\Box_1\boldsymbol{\omega}-\text{grad. div}\boldsymbol{\omega}, \tag{3.10}$$

where $\boldsymbol{\omega}=(\omega_x, \omega_y, 0)$. This is equivalent to the result of putting $\chi_z=0$ in (3.1).

It is clear by the foregoing consideration that the Galerkin function satisfies the biwave equation, the partial differential equation of fourth order. The displacement and the stress fields are given by the second and third derivatives of the Galerkin function, respectively. We must generally face much difficulties in solving higher order differential equation than dealing with lower order equation. One way of overcoming this difficulty is to adopt the function to be due to Boussinesq, Papkovich, and Neuber which will satisfy an usual wave equation and be considered in the following article. The other method is to use the integral transformation technique which will be utilized in the fifth and the subsequent articles.

4. Boussinesq-Papkovich-Neuber Functions

Using the Galerkin function $\boldsymbol{\chi}$, we define that

$$\Phi=\text{div}\ \boldsymbol{\chi}, \tag{4.1}$$

and

$$\boldsymbol{\Psi}=\Box_1\ \boldsymbol{\chi}. \tag{4.2}$$

Then displacement field is given by the sum of first and zeroth derivatives of them,

$$\boldsymbol{u} = -\operatorname{grad} \Phi + 2(1-\nu)\,\boldsymbol{\Psi}. \tag{4.3}$$

The functions Φ and $\boldsymbol{\Psi}=(\Psi_x, \Psi_y, \Psi_z)$ are the dynamic version of Boussinesq-Papkovich-Neuber potential already known in the static theory.[5),6),7)] Since χ satisfies the biwave equation, it is shown that

$$\Box_1 \Box_2 \Phi = 0, \tag{4.4}$$

and

$$\Box_2 \boldsymbol{\Psi} = 0. \tag{4.5}$$

Also it can be reduced that

$$\operatorname{div} \boldsymbol{u} = (1-2\nu) \Box_2 \Phi, \tag{4.6}$$

and

$$\operatorname{curl} \boldsymbol{u} = 2(1-\nu) \Box_1 \operatorname{curl} \boldsymbol{\Psi}. \tag{4.7}$$

Thus, it is easy to show that $\Box_1 \operatorname{div}\boldsymbol{u}=0$ and $\Box_2\operatorname{curl}\boldsymbol{u}=0$, which were derived in the second article. Further, eliminating χ from (4.1) and (4.2), we obtain the relation between Φ and $\boldsymbol{\Psi}$

$$\Box_1 \Phi = \operatorname{div} \boldsymbol{\Psi}. \tag{4.8}$$

In the preceding discussions, Φ and $\boldsymbol{\Psi}$ are functions of x, y, z, and T. To make clear the dependence of T, it is convenient to write

$$\Phi = \Phi(x, y, z, T) = \Phi(T), \tag{4.9}$$

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}(x, y, z, T) = \boldsymbol{\Psi}(T), \tag{4.10}$$

and rewrite (4. 3), (4.4), (4.5), and (4.8) as follows:

$$\boldsymbol{u} = -\operatorname{grad} \Phi(T) + 2(1-\nu)\boldsymbol{\Psi}(T), \tag{4.11}$$

$$\Box_1 \Box_2 \Phi(T) = 0, \tag{4.12}$$

$$\Box_2 \boldsymbol{\Psi}(T) = 0, \tag{4.13}$$

and

$$\Box_1 \Phi(T) = \operatorname{div} \boldsymbol{\Psi}(T). \tag{4.14}$$

Putting βT for T in (4.13), we find that

$$\Box_1 \boldsymbol{\Psi}(\beta T) = 0, \tag{4.15}$$

which can be utilized to solve (4.14) with respect to Φ. Through considering the relations

$$\Box_1 \left(\frac{r}{2} \boldsymbol{\Psi}(\beta T)\right) = \operatorname{div} \boldsymbol{\Psi}(\beta T), \tag{4.16}$$

$$\Box_1 \left(\int_T^{\beta T} (\tau - \boldsymbol{\Psi})^2 \frac{\partial \operatorname{div} \boldsymbol{\Psi}(\tau)}{\partial \tau} d\tau\right) = -2\left(1 - \frac{1}{\beta^2}\right) \cdot$$

$$\cdot\left[\operatorname{div}\boldsymbol{\Psi}(T) - \operatorname{div}\boldsymbol{\Psi}(\beta T) + 2\frac{1-\dfrac{1}{\beta}}{1+\dfrac{1}{\beta}} \frac{\partial \operatorname{div} \boldsymbol{\Psi}(\beta T)}{\partial \beta T}\right], \tag{4.17}$$

$$\Box_1 \left(\frac{r}{2} \beta T \frac{\partial \boldsymbol{\Psi}(\beta T)}{\partial \beta T}\right) = \beta T \frac{\partial \operatorname{div} \boldsymbol{\Psi}(\beta T)}{\partial \beta T} - 2\frac{r}{2} \frac{\partial^2 \boldsymbol{\Psi}(\beta T)}{\partial \beta T^2}, \tag{4.18}$$

and
$$\Box_1\left\{\frac{\boldsymbol{r}}{2}\left(\frac{1}{2}\boldsymbol{\Psi}(\beta T)+2\left(\frac{\boldsymbol{r}}{2}\nabla\right)\boldsymbol{\Psi}(\beta T)-\frac{\boldsymbol{r}}{2}\operatorname{div}\boldsymbol{\Psi}(\beta T)\right)\right\}=2\frac{\boldsymbol{r}}{2}\frac{\partial^2\boldsymbol{\Psi}(\beta T)}{\partial\beta T^2}, \tag{4.19}$$

it is confirmed after some reductions that $\Phi(T)$, the solution of (4.14), is given in the form

$$\Phi(T)=F(T)+\frac{\boldsymbol{r}}{2}\boldsymbol{\Psi}(\beta T)-(1-\nu)\int_T^{\beta T}(\tau-T)^2\frac{\partial\operatorname{div}\boldsymbol{\Psi}(\tau)}{\partial\tau}d\tau-$$
$$-2\frac{1-\frac{1}{\beta}}{1+\frac{1}{\beta}}\frac{\boldsymbol{r}}{2}\left[\beta T\frac{\partial\boldsymbol{\Psi}(\beta T)}{\partial\beta T}+\frac{1}{2}\boldsymbol{\Psi}(\beta T)+\right.$$
$$\left.+2\left(\frac{\boldsymbol{r}}{2}\nabla\right)\boldsymbol{\Psi}(\beta T)-\frac{\boldsymbol{r}}{2}\operatorname{div}\boldsymbol{\Psi}(\beta T)\right], \tag{4.20}$$

where
$$\Box_1 F(T)=0, \tag{4.21}$$
$$\left.\begin{aligned}&\boldsymbol{r}=(x,\ y,\ z),\qquad \nabla=\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right),\\&\boldsymbol{r}\nabla=\left(x\frac{\partial}{\partial x},\ y\frac{\partial}{\partial y},\ z\frac{\partial}{\partial z}\right).\end{aligned}\right\} \tag{4.22}$$

By substitution of $\Phi(T)$ just obtained into (4.11), the displacement is expressed in terms of the functions F and $\boldsymbol{\Psi}$ which satisfy the wave equation given in (4.21) and (4.13).

One of the four functions F, Ψ_x, Ψ_y, and Ψ_z can be put zero[7] in (4.20). Let us write

$$\boldsymbol{\Psi}(T)=\operatorname{grad}\,\theta(T)+\boldsymbol{\omega}(T), \tag{4.23}$$

in which
$$\boldsymbol{\omega}=(\omega_x,\ \omega_y,\ 0), \tag{4.24}$$
$$\Box_2\ \boldsymbol{\omega}(T)=0, \tag{4.25}$$

and
$$\Box_2\ \theta(T)=0. \tag{4,26}$$

Through the ordinary quadrature, θ can be determined from $\Psi_z=\frac{\partial\theta}{\partial z}$ given by a part of (4.23). Then, according to the remainders of (4.23), $\boldsymbol{\omega}$ can be obtained. We calculate

$$\Theta(T)=2(1-\nu)\theta(\beta T)-\left\{\frac{\boldsymbol{r}}{2}\operatorname{grad}\theta(\beta T)+\frac{\beta T}{2}\frac{\partial\theta(\beta T)}{\partial\beta T}\right\}+$$
$$+\frac{1-\frac{1}{\beta}}{1+\frac{1}{\beta}}\frac{1}{2}\left\{(\beta T)^2\frac{\partial^2\theta(\beta T)}{\partial\beta T^2}-\beta T\frac{\partial\theta(\beta T)}{\partial\beta T}\right\}+$$

$$+2\frac{1-\frac{1}{\beta}}{1+\frac{1}{\beta}}\frac{\boldsymbol{r}}{2}\left\{\beta T\frac{\partial \mathrm{grad}\theta(\beta T)}{\partial\beta T}+\frac{1}{2}\mathrm{grad}\theta(\beta T)+\right.$$

$$\left.+2\left(\frac{\boldsymbol{r}}{2}\nabla\right)\mathrm{grad}\theta(\beta T)-\frac{\boldsymbol{r}}{2}\frac{\partial^2\theta(\beta T)}{\partial\beta T^2}\right\}, \tag{4.27}$$

and put

$$F(T)=F'(T)+\Theta(T). \tag{4.28}$$

If we use the relations grven by

$$\Box_1\left(\beta T\frac{\partial\theta(\beta T)}{\partial\beta T}\right)=-2\frac{\partial^2\theta(\beta T)}{\partial\beta T^2}, \tag{4.29}$$

$$\Box_1\left((\beta T)^2\frac{\partial^2\theta(\beta T)}{\partial\beta T^2}\right)=-2\frac{\partial^2\theta(\beta T)}{\partial\beta T^2}-4\beta T\frac{\partial^3\theta(\beta T)}{\partial\beta T^3} \tag{4.30}$$

in addition to (4.16), (4.18), (4.19), and (4.26), it is shown that

$$\Box_1\,\Theta(T)=0. \tag{4.31}$$

Thus, it is reduced by virture of (4.21) that

$$\Box_1 F'(T)=0. \tag{4.32}$$

Substituting F and $\boldsymbol{\Psi}$ given in (4.28) and (4.23) respectively into (4.20), and considering the relation

$$\int_T^{\beta T}(\tau-T)^2\frac{\partial\nabla^2\theta(\tau)}{\partial\tau}d\tau=\int_T^{\beta T}(\tau-T)^2\frac{\partial^3\theta(\tau)}{\partial\tau^3}d\tau=$$

$$=\left(1-\frac{1}{\beta}\right)^2(\beta T)^2\frac{\partial^2\theta(\beta T)}{\partial\beta T^2}-2\left(1-\frac{1}{\beta}\right)\beta T\frac{\partial\theta(\beta T)}{\partial\beta T}+2\theta(\beta T)-2\theta(T), \tag{4.33}$$

we have that

$$\Phi(T)=2(1-\nu)\theta(T)+\Phi', \tag{4.34}$$

in which Φ' is given by (4.20) with F' and $\boldsymbol{\omega}$ instead of F and $\boldsymbol{\Psi}$, respectinely. Inserting (4.23) and (4.34) into (4.11), we obtain the displacement in the form

$$\boldsymbol{u}(T)=-\mathrm{grad}\,\Phi'(T)+2(1-\nu)\boldsymbol{\omega}(T). \tag{4.35}$$

This expression is essentially equal to that of making $\Psi_z=0$ in (4.20).

Again we take θ which satisfy (4.26), calculate Θ given by (4.27), and assume for this time that

$$F(T)=\Theta(T). \tag{4.36}$$

Still (4.21) holds by (4.31). Putting

$$\boldsymbol{\Psi}(T)=\mathrm{grad}\;\theta(T)+\boldsymbol{\Omega}(T), \tag{4.37}$$

$$\boldsymbol{\Omega}=(\Omega_x,\ \Omega_y,\ \Omega_z), \tag{4.38}$$

we find by virture of (4.5) and (4.26) that

$$\Box_2 \, \Omega(T)=0. \tag{4.39}$$

If (4.36) and (4.37) are substituted into (4.20), it is found after the similar reduction to previous one that

$$\Phi(T)=2(1-\nu)\theta(T)+\Phi''(T), \tag{4.40}$$

where Φ'' is given by (4.20) with 0 and Ω in place of F and Ψ, reepectively. Hence, the displacement is given by

$$\boldsymbol{u}(T)=-\text{grad } \Phi''(T)+2(1-\nu)\Omega(\text{T}), \tag{4.41}$$

which is effectively the same result as that of putting $F=0$ in (4.20).

We shall consider the static state in which $\partial/\partial T=0$. Putting

$$G=\frac{\boldsymbol{r}}{2}\left(\frac{1}{2}\Psi+2\left(\frac{\boldsymbol{r}}{2}\nabla\right)\Psi-\frac{\boldsymbol{r}}{2}\text{div}\,\Psi\right), \tag{4.42}$$

we are able to confirm that $\nabla^2 G=0$. Then writing F in place of $F+G$, we find that (4.20) and (4.21) respecively become

$$\Phi=F+\frac{\boldsymbol{r}}{2}\Psi, \tag{4.43}$$

and

$$\nabla^2 F=0. \tag{4.44}$$

Naturally these are the results established independently by Boussinesq, Papkovich, and Neuber.

5. Plane Strain Problem

Let us consider the plane strain problem in which the z-component of displacement u_z vanishes everywhere and the stress field as well as the another components of displacement are dependent upon x, y, and T. If we use the Galerkin function, we are able to put $\chi_z=0$. By the similar reasoning to the derivation of (3.10), one of the two remainders χ_x and χ_y can be put zero. Let, for example, χ_y be zero.

Writing χ_x instead of $2\mu\chi_x$, we find from (3.1) that

$$2\mu u_x=2(1-\nu)\left(\frac{\partial^2\chi_x}{\partial x^2}+\frac{\partial^2\chi_x}{\partial y^2}-\frac{1}{\beta^2}\frac{\partial^2\chi_x}{\partial T^2}\right)-\frac{\partial^2\chi_x}{\partial x^2}, \tag{5.1}$$

and

$$2\mu u_y= \qquad -\frac{\partial^2\chi_x}{\partial x\partial y}, \tag{5.2}$$

Inserting (5.1) and (5.2) into the Hooke's relations (2.2) and considering the relation

$$\frac{\lambda}{2\mu}=\frac{\nu}{1-2\nu}, \tag{5.3}$$

we obtain after a little reduction that

$$\sigma_{xx}=\nu\frac{\partial}{\partial x}\left(\frac{\partial^2\chi_x}{\partial x^2}+\frac{\partial^2\chi_x}{\partial y^2}-\frac{\partial^2\chi_x}{\partial T^2}\right)+$$

$$+2(1-\nu)\frac{\partial}{\partial x}\left(\frac{\partial^2\chi_x}{\partial x^2}+\frac{\partial^2\chi_x}{\partial y^2}-\frac{1\partial^2\chi_x}{\beta^2\partial T^2}\right)-\frac{\partial^3\chi_x}{\partial x^3}, \tag{5.4}$$

$$\sigma_{yy}=\nu\frac{\partial}{\partial x}\left(\frac{\partial^2\chi_x}{\partial x^2}+\frac{\partial^2\chi_x}{\partial y^2}-\frac{\partial^2\chi_x}{\partial T^2}\right)-\frac{\partial^3\chi_x}{\partial x\partial y^2}, \tag{5.5}$$

$$\sigma_{xy}=\frac{\partial}{\partial y}\left[(1-\nu)\left(\frac{\partial^2\chi_x}{\partial x^2}+\frac{\partial^2\chi_x}{\partial y^2}-\frac{1}{\beta^2}\frac{\partial^2\chi_x}{\partial T^2}\right)-\frac{\partial^2\chi_x}{\partial x^2}\right]. \tag{5.6}$$

According to (3.2), χ_x ought to satisfy the biwave equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{\beta^2}\frac{\partial^2}{\partial T^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial T^2}\right)\chi_x=0. \tag{5.7}$$

Let $\hat{\chi}_x$ be the Laplase transform of χ_x as to T, and X be the Fourier transform of $\hat{\chi}_x$ with respect to y. By definition, it is shown that

$$\hat{\chi}_x(x,y,s)=\int_0^\infty \chi_x(x,y,T)\,e^{-sT}dT, \tag{5.8}$$

$$X(x,\eta,s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\chi}_x(x,y,s)e^{i\eta y}dy, \tag{5.9}$$

$$\chi_x(x,y,T)=\frac{1}{2\pi i}\int_{Br.}\hat{\chi}_x(x,y,s)e^{Ts}ds, \tag{5.10}$$

and

$$\hat{\chi}_x(x,y,s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}X(x,\eta,s)e^{-iy\eta}d\eta. \tag{5.11}$$

The biwave equation (5.7) becomes that

$$\left(\frac{\partial^2}{\partial x^2}-\eta^2-\frac{s^2}{\beta^2}\right)\left(\frac{\partial^2}{\partial x^2}-\eta^2-s^2\right)X(x,\eta,s)=0. \tag{5.12}$$

The solution is given by

$$X(x,\eta,s)=A(\eta,s)\exp(-\sqrt{\eta^2+s^2/\beta^2}\,x)+B(\eta,s)\exp(-\sqrt{\eta^2+s^2}\,x)+$$
$$+C(\eta,s)\exp(\sqrt{\eta^2+s^2/\beta^2}\,x)+C(\eta,s)\exp(\sqrt{\eta^2+s^2}\,x). \tag{5.13}$$

where A, B, C, and D are to be determined from the boundary conditions.

In a static state, where $\partial/\partial T=0$, the solution becomes

$$X(x,\eta,s)=\frac{1}{s}\left[\{A'(\eta)+xB'(\eta)\}e^{-\eta y}+\{C'(\eta)+xD'(\eta)\}e^{\eta x}\right]. \tag{5.14}$$

It is interesting to point out here that (5.14) may be suggested from the theory of anisotropic elasticity in which the degenerated solution also appears when an orthotropic body shifts to quasi-isotropic one.[8)]

6. Collision of a Rigid Beam with an Elastic Half Plane

Let us consider the problem in which a rigid beam of line mass density m with initial velocity v impinges on an elastic semi-infinite plane of which free surface is denoted by $x=0$ as shown in Fig. 1. We shall denote the width of the beam by $2a$, take an instance of contact between the rigid beam and the half plane as an origin of time, and express the displacement of the beam in x direction by $u(T)$. Further, if we choose a midpoint of initial contact surface as $y=0$, the present problem becomes symmetrical with respect to y. Thus, Fourier exponential integral given in (5.9) or (5.11) can be rewritten in terms of Fourier cosine integral.

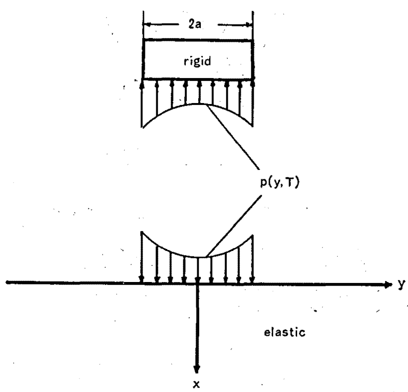

Fig. 1 Collision of a Rigid Beam with a Semi-Infinite Elastic Plane

Using the space-like time T defined by (2.10), we have the equation of motion of a beam

$$\frac{\partial^2 u(T)}{\partial T^2} = -\frac{f(T)}{mc^2}. \tag{6.1}$$

in which $f(T)$ is expressed in terms of the compressive stress $p(y, T)$ on the contact surface as follows:

$$f(T) = \int_{-a}^{a} p(y, T)dy = 2\int_0^a p(y, T)dy. \tag{6.2}$$

The last eqation (6.2) expresses the force per unit length excerted to a rigid beam by the elastic plane. The initial conditions are given by

$$\left.\frac{\partial u(T)}{\partial T}\right|_{T=0} = \frac{v}{c}, \tag{6.3}$$

and

$$u(T)\,|_{T=0} = 0. \tag{6.4}$$

Neglecting, for simplicity, the friction* on the contact surface, we find that the semi-infinite elastic plane must satisfy the following boundary conditions at the surface $x=0$:

$$\sigma_{xy}(0, y, T)=0, \qquad -\infty<y<\infty, \tag{6.5}$$

$$\sigma_{xx}(0, y, T)=0, \qquad a<|y|, \tag{6.6}$$

and

$$u_x(0, y. T)=u(T), \qquad |y|<a. \tag{6.7}$$

We shall perform the Laplace transformation on the quanties thus far appeared and express the transformed functions by. ^ similarly to (5.8). Thus, we have the equation for the beam

$$\hat{u}(s)=\frac{1}{s^2}\left(\frac{v}{c}-\frac{\hat{f}(s)}{mc^2}\right), \tag{6.8}$$

where

$$f(s)=2\int^a \hat{p}(y, s)\,dy, \tag{6.9}$$

and the boundary conditions of the elastic half plane as follows:

$$\hat{\sigma}_{xy}(0, y, s)=0, \qquad -\infty<y<\infty, \tag{6.10}$$

$$\hat{\sigma}_{xx}(0, y, s)=0, \qquad a<|y|, \tag{6.11}$$

and

$$\hat{u}_x(0, y, s)=\hat{u}(s), \qquad |y|<a. \tag{6.12}$$

Since it can be considered that the displacement and the stress field of the elastic half plane vanish in the place infinitely far from the surface, we can put $C(\eta,s)=D(\eta,s)=0$ in (5.13). Using (5.13) and (5.11) in addition to (5.1), (5.2), (5.4), (5.5) and (5.6), we find that the displacement and the stress are given as follows:

$$2\mu\hat{u}_x(x, y, s)=-\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\left(\eta^2+\frac{s^2}{\beta^2}\right)A(\eta,s)\exp\left(-\sqrt{\eta^2+\frac{s^2}{\beta^2}}x\right)+\right.$$
$$\left.+\eta^2B(\eta, s)\exp\left(-\sqrt{\eta^2+s^2}\,x\right)\right]\cos(y\eta)\,d\eta, \tag{6.13}$$

$$2\mu\hat{u}_y(x, y, s)=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\eta\sqrt{\eta^2+\frac{s^2}{\beta^2}}A(\eta, s)\exp\left(-\sqrt{\eta^2+\frac{s^2}{\beta^2}}x\right)+\right.$$
$$\left.+\eta\sqrt{\eta^2+s^2}\,B(\eta, s)\exp\left(-\sqrt{\eta^2+s^2}\,x\right)\right]\sin(y\eta)\,d\eta, \tag{6.14}$$

$$\hat{\sigma}_{xx}(x, y, s)=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\left(\eta^2+\frac{s^2}{2}\right)\sqrt{\eta^2+\frac{s^2}{\beta^2}}A(\eta, s)\exp\left(-\sqrt{\eta^2+\frac{s^2}{\beta^2}}x\right)+\right.$$

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* The result with a restriction on the contact surface will be submitted to the Report of Research Institute for Applied Mechanics in near future.

$$+\eta^2\sqrt{\eta^2+s^2}B(\eta,s)\exp\left(-\sqrt{\eta^2+s^2}\,x\right)\Big]\cos(y\eta)d\eta, \tag{6.15}$$

$$\hat{\sigma}_{yy}(x,y,s)=\sqrt{\frac{2}{\pi}}\int_0^\infty\Big[\Big\{-\eta^2+\Big(\frac{1}{2}-\frac{1}{\beta^2}\Big)s^2\Big\}\sqrt{\eta^2+\frac{s^2}{\beta^2}}A(\eta,s)\exp\Big(-$$

$$-\sqrt{\eta^2+\frac{s^2}{\beta^2}}x\Big)-\eta^2\sqrt{\eta^2+s^2}\,B(\eta,s)\exp(-\sqrt{\eta^2+s^2}x)\Big]\cos(y\eta)d\eta, \tag{6.16}$$

$$\hat{\sigma}_{xy}(x,y,s)=\sqrt{\frac{2}{\pi}}\int_0^\infty\Big[\eta(\eta^2+\frac{s^2}{\beta^2}A(\eta,s)\exp(-\sqrt{\eta^2+\frac{s^2}{\beta^2}}x)+$$

$$+\eta\left(\eta^2+\frac{s^2}{2}\right)B(\eta,s)\exp(-\sqrt{\eta^2+s^2}x)\Big]\sin(y\eta)d\eta. \tag{6.17}$$

Putting $x=0$ in (6.17) and considering (6.10), we obtain that

$$A(\eta,s)=-\frac{\eta^2+\dfrac{s^2}{2}}{\eta^2+\dfrac{s^2}{\beta^2}}B(\eta,s). \tag{6.18}$$

Inserting this equation into (6.15) and (6.13) with $x=0$, we find that

$$\hat{\sigma}_{xx}(0,y,s)=-\sqrt{\frac{2}{\pi}}\int_0^\infty\left\{\frac{\left(\eta^2+\dfrac{s^2}{2}\right)^2}{\sqrt{\eta^2+\dfrac{s^2}{\beta^2}}}-\eta^2\sqrt{\eta^2+s^2}\right\}B(\eta,s)\cos(y\eta)d\eta, \tag{6.19}$$

and $$2\mu\hat{u}_x(0,y,s)=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{s^2}{2}B(\eta,s)\cos(y\eta)d\eta. \tag{6.20}$$

Thus, from the remaining boundary conditions (6.11), (6.12) and the equation of motion of the beam (6.8), the equations to determine an indefinite constant $B\ (\eta,s)$ can be obtained as follows:

$$\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{s^2}{2}B(\eta,s)\cos(y\eta)d\eta=\frac{2\mu}{s^2}\left(\frac{v}{c}-\frac{\hat{f}(s)}{mc^2}\right),\ y<a, \tag{6.21}$$

$$-\sqrt{\frac{2}{\pi}}\int_0^\infty\left\{\frac{\left(\eta^2+\dfrac{s^2}{2}\right)^2}{\sqrt{\eta^2+\dfrac{s^2}{\beta^2}}}-\eta^2\sqrt{\eta^2+s^2}\right\}B(\eta,s)\cos(y\eta)d\eta=0,\ a<y. \tag{6.22}$$

These are dual integral equations which frequently, as is known, appear when a solution of mixed boundary value problem is sought through the integral transform technique. Provided we can obtain $B(\eta,s)$ from (6.21) and (6.22), $A(\eta,s)$ will be determined according to (6.18). Then, the displacement and the stress will be in principle found by performing the Laplace inversion of (6.13) through (6.17). In this report, however, we shall avoid solving the dual integral equations directly.[9)]

One of the boundary conditions was, as given in (6.6), that the normal stress vanishes on the free surface when $x=0$ and $|y|>a$. Since we are now interested in the stress distribution on the collision surface between the rigid beam and the semi-infinite elastic plane from the physical point of view, we shall rewrite the boundary condition, instead of (6.6), as follows[9]:

$$\sigma_{xx}(0, y, T)=\begin{cases}-p(y, T), & |y|<a,\\ 0, & a<|y|.\end{cases} \tag{6.23}$$

Through Laplace transformation, it becomes that

$$\hat{\sigma}_{xx}(0, y, s)=\begin{cases}-\hat{p}(y, s), & |y|<a,\\ 0, & a<|y|.\end{cases} \tag{6.24}$$

Denoting the Fourier cosine transform of (6.24) by $-P\ (\eta,s)$, we obtain that

$$-P(\eta, s)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{\sigma}_{xx}(0, y, s)\cos(\eta y)\,dy=-\sqrt{\frac{2}{\pi}}\int_0^a \hat{p}(y, s)\cos(\eta y)\,dy. \tag{6.25}$$

Due to inversion formula, we have that

$$\hat{\sigma}_{xx}(0, y, s)=-\sqrt{\frac{2}{\pi}}\int_0^\infty P(\eta, s)\cos(y\eta)\,d\eta=\begin{cases}-\hat{p}(y, s), & y<a,\\ 0, & a<y.\end{cases} \tag{6.26}$$

Comparing (6.26) with (6.19), we find that $B\ (\eta, s)$ can be expressed in terms of $P\ (\eta, s)$ by

$$B(\eta, s)=\frac{\sqrt{\eta^2+s^2/\beta^2}}{\left(\eta^2+\dfrac{s^2}{2}\right)^2-\eta^2\sqrt{\eta^2+s^2/\beta^2}\,\sqrt{\eta^2+s^2}}P(\eta, s)\,. \tag{6.27}$$

If we use $P\ (\eta, s)$ given by (6.25), a half of the dual integral equations (6.22) is always satisfied according to (6.26) and the remainder (6.21) becomes

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \Omega(\eta, s)P(\eta, s)\cos(y\eta)\,d\eta=\frac{2\mu}{s^2}\left(\frac{v}{c}-\frac{\hat{f}(s)}{mc^2}\right),\ y<a, \tag{6.28}$$

where

$$\Omega(\eta, s)=\frac{s^2}{2}\,\frac{\sqrt{\eta^2+s^2/\beta^2}}{(\eta^2+s^2/2)^2-\eta^2\sqrt{\eta^2+s^2/\beta^2}\,\sqrt{\eta^2+s^2}}. \tag{6.29}$$

Inserting (6.9) and (6,25) into (6.28), we obtain, after a little reduction of using (2.5), that

$$\int_0^\infty k(y, \xi, s)\hat{p}(\xi, s)\,d\xi=\frac{2\mu}{s^2}\,\frac{v}{c},\ y<a, \tag{6.30}$$

in which

$$k(y, \xi, s)=\frac{4\rho}{m}\,\frac{1}{s^2}+\frac{2}{\pi}\int_0^\infty \Omega(\eta, s)\cos(y\eta)\cos(\xi\eta)\,d\eta. \tag{6.31}$$

This is the Fredholm integral equation of the first kind to determine the function $\hat{p}(\xi, s)$ which is the Laplace transform of the stress distribution on the contact surface of the elastic half plane with the impinged rigid beam. The kernel function $k(y, \xi, s)$ is symmetrical about y and ξ, as it is seen from (6.31). It is also understood from (6.29) and the considerations to be made in the next article that $\Omega(\eta, s)$ contributes to the propagtion of longitudinal, transverse, and surface waves. The problem of propagation of stress wave is by itself an interesting subject about which various investigations have already been made. Since it seems to the author that the behaviour of wave propagation of the present problem may essentially be covered by those results thus far obtained, any analyses will not be carried out about it. We shall devote ourselves to the evaluation of the stress distribution on the collision surface in the next article.

7. Stress Distribution on the Collision Surface

As it has been made clear in the preceeding article, the stress distribution on the collision surface of the elastic half plane with the rigid beam is determined: first, by seeking the solution $\hat{p}(\xi, s)$ of the Fredholm integral equation of the first kind of which kernel is symmetrical

$$\int_0^\infty k(y, \xi, s)\hat{p}(\xi, s)d\xi = \frac{2\mu}{s^2}\frac{v}{c}, \quad y<a, \tag{7.1}$$

where
$$k(y, \xi, s) = \frac{4\rho}{m}\frac{1}{s^2} + \frac{2}{\pi}\int_0^\infty \Omega(\eta, s)\cos(y\eta)\cos(\xi\eta)d\eta, \tag{7.2}$$

and second, by means of Laplace inverse transformation.

When we attack the boundary value problem through the integral transform technique, a difficulty usually arises in the course of inversion. As for Laplace transformation, it has such a great advantage that the inverse transform can simply be determined by means of the theory of residue. But this very merit often turns out to be a demerit that makes the inversion impossible unless the behaviours of the function to be inverted are prescribed over the entire complex plane. It is not so easy to obtain the analytic solution of (7.1) that numerical method must be consulted in order to solve it. It is, however, practically much difficult to evaluate the numerical solution of (7.1) over the entire complex s plane without losing any sight of the essence of the problem at hand. Thus, we have to give up, as a matter of fact, to carry out Laplace inverse transformation by the theorem of residue.

In the meanwhile, Bellman et al.[10] have already developed an instructive method of Laplace inversion on the basis of the values of the image function on a real axis, even if the behaviours of the function are not prescribed over the entire complex region. It will be briefly mentioned below. Now let us take a certain function $h(T)$ and assume its Laplace transform $\hat{h}(s)$. If the values of $\hat{h}(s)$ are known over the entire s plane, it is indeed possible to get a Laplace inversion transform through the theorem of residue. We shall proceed to consider the case in which $\hat{h}(s)$ is known on real s axis only. By definition, it is obvious that

$$\int_0^\infty h(T)e^{-ST}dT=\hat{h}(s). \qquad (7.3)$$

Changing an integral variable

$$x=e^{-T}, \qquad (7.4)$$

we have that

$$\int_0^1 x^{S-1}g(x)\,dx=\hat{h}(s), \qquad (7.5)$$

in which

$$g(x)=h(-\ln x). \qquad (7.6)$$

According to Gauss'es quadrature, (7.5) can be appoximated by

$$\sum_{i=1}^{N} w_i\, x_i^{S-1}\, g(x_i)=\hat{h}(s), \qquad (7.7)$$

where x_i represents one of N zeros of the shifted Legendre polynomial $P_N^*(x)$ and w_i is a weight. Values of x_i and w_i are already known for various values of N. Putting $s=1,2,\ \dots\ , N$ in (7.7)

$$\left.\begin{aligned}&\sum_{i=1}^{N} w_i x_i^0 g(x_i)=\hat{h}(1),\\ &\sum_{i=1}^{N} w_i x_i^1 g(x^i)=\hat{h}(2),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \text{and}\quad &\sum_{i=1}^{N} w_i x_1^{N-1} g(x_i)=\hat{h}(N),\end{aligned}\right\} \qquad (7.8)$$

we find that $g(x_i)=h(-\ln x_i)$ $i=1,2.,$ N can be determined by solving the linear algebraic equation (7.8), provided $\hat{h}(1)$, $\hat{h}(2),\dots,\hat{h}.(N)$ are known. The inverse matrix of the coefficients $w_i x_i^{j-1}$ in (7.8) are also obtained for $N=3$ through 15 by Bellman et al. and are conviniently utilized to seek $g(x_i)$, $i=1,\ 2,\ \dots\dots, N$ directly,

$$\begin{bmatrix} g(x_1)\\ g(x_2)\\ \cdot\\ g(x_N)\end{bmatrix}=\begin{bmatrix} B_{11} & B_{12} & \cdot & B_{1N}\\ B_{21} & B_{22} & \cdot & B_{2N}\\ \cdot & \cdot & \cdot & \cdot\\ B_{N1} & B_{N2} & \cdot & B_{NN}\end{bmatrix}\begin{bmatrix}\hat{h}_{(1)}\\ \hat{h}_{(2)}\\ \cdot\\ \hat{h}_{(N)}\end{bmatrix}. \qquad (7.9)$$

In order to make an accurate approximation of integral (7.5), it is in principle desired to choose N as large as possible in (7.7). On the contrary, the coefficients matrix $(w_i\ x_i^{j-1})$ in (7.8) becomes to exhibit an ill-conditioned character, as N takes large value. Thus, we have in practice to select N through several trials.

Now let us proceed to determone $\hat{p}\ (\xi, s)$ for certain values of s. Introducing non-dimensional quantities as follows:

$$\left.\begin{aligned} \eta &= sx, \\ y &= aY, \\ \xi &= aX, \\ s &= \sigma/a, \end{aligned}\right\} \qquad \left.\begin{aligned} x &= \eta/s, \\ Y &= y/a, \\ X &= \xi/a, \\ \sigma &= as, \end{aligned}\right\} \tag{7.10}$$

and

$$\hat{p}(\xi,s)=2\mu\frac{v}{c}a\frac{\hat{\tau}(X,\sigma)}{\sigma^2}, \tag{7.11}$$

we find that (7.1) becomes

$$\int_0^1 K(Y,X,\sigma)\hat{\tau}(X,\sigma)dX=1, \tag{7.12}$$

where

$$K(Y,X,\sigma)=k(y,\xi,s)= \\ =\frac{4\rho a^2}{m}\frac{1}{\sigma^2}+\frac{2}{\pi}\int_0^\infty \Upsilon(x)\cos(\sigma Yx)\cos(\sigma Xx)dx, \tag{7.13}$$

and

$$\Upsilon(x)=s\Omega(\eta,s)=\frac{1}{2}\frac{\sqrt{x^2+1/\beta^2}}{(x^2+1/2)^2-x^2\sqrt{x^2+1/\beta^2}\sqrt{x^2+1}}. \tag{7.14}$$

Direct numerical calculus of the kernel function $K(Y,X,\sigma)$ in the present form given by (7.13) will be ineffective because of slow convergency with respect to the integral calculus to be due to the cosine terms in an integrand and the upper integration limit being infinity. Hence, we shall evaluate the kernel function in the following way.

Denoting the three roots of Rayleigh's equation which gives the ratio of propagation velocity of surface wave to that of transverse one

$$c'^6-8c'^4+16\left(\frac{3}{2}-\frac{1}{\beta^2}\right)c'^2-16\left(1-\frac{1}{\beta^2}\right)=0, \tag{7.15}$$

by k_A^2, k_B^2, and k_s^2, respectively, we find that

$$\left.\begin{aligned} &k_A^2+k_B^2+k_s^2=8, \\ &k_B^2k_s^2+k_s^2k_A^2+k_A^2k_B^2=16\left(\frac{3}{2}-\frac{1}{\beta^2}\right), \\ &k_A^2k_B^2k_s^2=16\left(1-\frac{1}{\beta^2}\right). \end{aligned}\right\} \tag{7.16}$$

Since (7.15) is a cubic equation with respect to c'^2, three roots can easily be obtained for any real Poisson's ratio. Because coefficients are real, (7.15) has at least one real root. The remaining two roots are also real as long as Poisson's ratio is small; whereas they become complex number and are conjugate each other with the increase of Poisson's ratio, as it is seen from Table 1. The one that gives the propagation velocity of surface wave is a real root smaller than 1, which is expressed by k_s in the table. The another two roots have no physical significance from a stand-point of wave propagation. Rationalizing the denom-

Table 1. Roots of (7.15)

ν	$k_A{}^2$	$k_B{}^2$	$k_S{}^2$	
0.	5.2361	2.0000	0.7639	three real roots
.05	5.1080	2.1111	.7809	
.10	4.9519	2.2504	.7976	
.15	4.7537	2.4322	.8140	
.20	4.4812	2.6889	.8299	
.25	4.0000	3.1547	.8457	
.30	0.7369	3.5700	.8601	one real root and two comlpex roots
.35	1.1764	3.5629	.8742	
.40	1.5406	3.5561	.8877	
.45	1.8846	3.5497	.9005	
.50	2.2303	3.5437	.9126	
ν	$\dfrac{k_A{}^2+k_B{}^2}{2}$	$\dfrac{k_A{}^2-k_B{}^2}{2\sqrt{-1}}$	$k_S{}^2$	

inator of $\Upsilon(x)$ given by (7.14) and utilizing the relations (7.16), we can reduce that

$$\Upsilon(x)=(1-\nu)\frac{\sqrt{x^2+1/\beta^2}(x^2+1/2)^2+\sqrt{x^2+1}\,x^2(x^2+1/\beta^2)}{(x^2+1/k_A^2)(x^2+1/k_B^2)(x^2+1/k_s^2)}. \tag{7.17}$$

Then we are ready to develop $\Upsilon(x)$ into partial fraction in either case of existing three real roots or not. When three distinct real roots appear, we have that

$$\begin{aligned}\frac{\Upsilon(x)}{1-\nu}=&\frac{1}{\sqrt{x^2+1/\beta^2}}+\frac{1}{\sqrt{x^2+1}}+\\ &+A_A\left(\sqrt{\frac{1}{\beta^2}-\frac{1}{k_A^2}}\cdot\frac{1}{\sqrt{x^2+1/\beta^2}}-\sqrt{1-\frac{1}{k_A^2}}\cdot\frac{1}{\sqrt{x^2+1}}\right)\frac{1}{x^2+1/k_A^2}-\\ &-A_B\left(\sqrt{\frac{1}{\beta^2}-\frac{1}{k_B^2}}\cdot\frac{1}{\sqrt{x^2+1/\beta^2}}-\sqrt{1-\frac{1}{k_s^2}}\cdot\frac{1}{\sqrt{x^2+1}}\right)\frac{1}{x^2+1/k_B^2}-\\ &-A_s\left(\sqrt{\frac{1}{k_2^s}-\frac{1}{\beta^2}}\cdot\frac{1}{\sqrt{x^2+1/\beta^2}}+\sqrt{\frac{1}{k_s^2}-1}\right)\frac{1}{\sqrt{x^2+1}}\Bigg)\frac{1}{x^2+1/k_s^2},\end{aligned} \tag{7.18}$$

in which

$$\left.\begin{aligned}A_A&=\frac{1}{D}\left(\frac{1}{k_s^2}-\frac{1}{k_B^2}\right)\left(\frac{1}{\beta^2}-\frac{1}{k_A^2}\right)\frac{1}{k_A^2}\sqrt{1-\frac{1}{k_B^2}},\\ A_B&=\frac{1}{D}\left(\frac{1}{k_s^2}-\frac{1}{k_A^2}\right)\left(\frac{1}{\beta^2}-\frac{1}{k_B^2}\right)\frac{1}{k_B^2}\sqrt{1-\frac{1}{k_B^2}},\\ A_s&=\frac{1}{D}\left(\frac{1}{k_B^2}-\frac{1}{k_A^2}\right)\left(\frac{1}{k_s^2}-\frac{1}{\beta^2}\right)\frac{1}{k_s^2}\sqrt{\frac{1}{k_s^2}-1},\end{aligned}\right\} \tag{7.19}$$

and $$\dot{D}=\left(\frac{1}{k_s^2}-\frac{1}{k_B^2}\right)\left(\frac{1}{k_s^2}-\frac{1}{k_A^2}\right)\left(\frac{1}{k_B^2}-\frac{1}{k_A^2}\right), \tag{7.20}$$

A_A, A_B, A_s, and D being all positive.

Now the following formulae[11] concerning to the modified Bessel function of the second kind of order n, $K_n(Z)$, are known for positive a, b, and Z:

$$K_0(aZ)=\int_0^\infty \frac{\cos(xZ)}{\sqrt{x^2+a^2}}dx, \tag{7.21}$$

$$\frac{1}{(2n-1)!!}\frac{(bz)^n}{b^{2n}}K_n(bZ)=\int_0^\infty \frac{\cos(xZ)}{(x^2+b^2)^{n+1/2}}dz, \tag{7.22}$$

in which $$\left.\begin{aligned}&(2n-1)!!=(2n-1)(2n-3)\ldots\ldots 3\cdot 1,\\&(-1)!!=1.\end{aligned}\right\} \tag{7.23}$$

Let us calculate the integral

$$\int_0^\infty \frac{\cos(xZ)}{\sqrt{x^2+a^2}\,(x^2+b^2)}dx. \tag{7.24}$$

Provided $a^2<b^2$, using the formula

$$\frac{1}{\sqrt{1-Z}}=\sum_{n=0}^\infty \frac{(2n-1)!!}{(2n)!!}Z^n, \quad |Z|<1, \tag{7.25}$$

in which

$$\left.\begin{aligned}&(2n)!!=2n(2n-2)\ldots\ldots 4\cdot 2,\\&0!!=1,\end{aligned}\right\} \tag{7.26}$$

we obtain that

$$\frac{1}{\sqrt{x^2+a^2}\,(x^2+b^2)}=\sum_{n=0}^\infty (b^2-a^2)^n\frac{(2n-1)!!}{(2n)!!}\frac{1}{(x^2+b^2)^{n+3/2}}. \tag{7.27}$$

If $a^2>b^2$ conversly, making use of the relation

$$\frac{1}{1-Z}=\sum_{n=0}^\infty Z^n, \quad |Z|<1, \tag{7.28}$$

we find that

$$\frac{1}{\sqrt{x^2+a^2}\,(x^2+b^2)}=\sum_{n=0}^\infty (a^2-b^2)^n\frac{1}{(x^2+a^2)^{n+3/2}}. \tag{7.29}$$

According to (7.22), it is reduced that

$$\int_0^\infty \frac{\cos(xZ)}{\sqrt{x^2+a^2}\,(x^2+b^2)}dx=$$

$$\left\{\frac{1}{b^2}\sum_{n=0}^\infty\left(\frac{b^2-a^2}{b^2}\right)^n\frac{1}{2n+1}\frac{(bZ)^{n+1}}{(2n)!!}K_{n+1}(bZ), \quad a^2<b^2,\right.$$

$$= \begin{cases} \dfrac{1}{a^2}\sum\limits_{n=0}^{\infty}\left(\dfrac{a^2-b^2}{a^2}\right)^n \dfrac{(2n)!!}{(2n+1)!!}\dfrac{(aZ)^{n+1}}{(2n)!!}K_{n+1}(aZ), \quad a^2>b^2, \\ \dfrac{1}{b^2}\, bZ\, K_1(bZ), \quad a^2=b^2. \end{cases} \tag{7.30}$$

Putting
$$L_{n+1}(Z) = \frac{Z^{n+1}}{(2n)!!}K_{n+1}(Z), \tag{7.31}$$

we can rewrite the recurrence formula[11)]

$$K_{n+1}(Z) = \frac{2n}{Z}K_n(Z) + K_{n-1}(Z), \tag{7.32}$$

in the form that

$$L_{n+1}(Z) = L_n(Z) + \frac{Z^2}{2^2(n-1)n}L_{n-1}(Z). \tag{7.33}$$

Moreover it is known that[11)]

$$\lim_{n\to\infty} L_n(Z) = 1. \tag{7.34}$$

Consequently it is found that (7.30) is indeed a quick convergent series.

Thus, the kernel function given by (7.13) can be simply evaluated by summ ing up, according to (7.18), several terms of those which are given in the form of either (7.21) or (7.30) where Z should be replaced by $\sigma\,|X-Y|$ or $\sigma\,(X+Y)$; a by $1/\beta$ or 1 ; and b by one of $1/k_A$, $1/k_B$, or $1/k_s$. The kernel function is singular of logarithmic order at $X=Y$, for the modified Bessel function of second kind of order zero with argument $|X-Y| \times$ *constant* is contained within the kernel. Since a logarithmci function can, however, be integrated,

$$\int_0^1 \ln x dx = \Big[x\ln x - x\Big]_0^1 = -1, \tag{7.35}$$

the integral equation (7.12) is of Fredholm type with a singular symmetric kernel. Special attention must be paid in an actual numerical caluculation.

The closed sections $[0,1]$ of X and Y are divided n equal segments of which width are given by

$$H = \frac{1}{n}\cdot \tag{7.36}$$

The middle point of each divided section with respect to X is denoted by X_j:

$$X_j = \left(j-\frac{1}{2}\right)H, \quad j=1,\ 2, \dots,\ n. \tag{7.37}$$

As for Y, the point except the middle is chosen

$$Y_i = (i-e)H, \quad e \neq \frac{1}{2}, \quad i=1,\ 2, \dots,\ n, \tag{7.38}$$

so as to avoid $X=Y$.

Using X_j and Y_i, we can approximate (7.12) through MacLaurin quadrature in the form

$$\sum_{j=1}^{n} K(Y_i, X_j, \sigma)\hat{\tau}(X_j, \sigma)H=1, \quad i=1, 2, \ldots, n. \tag{7.39}$$

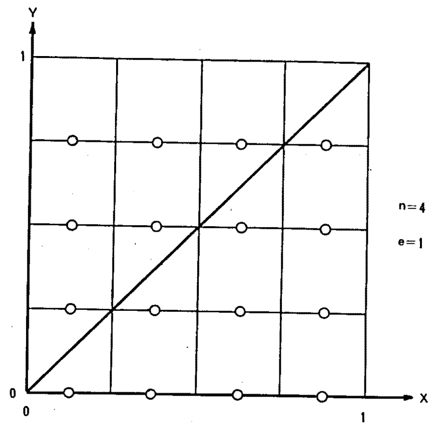

Fig. 2 Illustration of Selecting X_j and Y_i

As was stated at the beginning of this article, (7.39) is first solved with respect to $\hat{\tau}$ (X_j, σ) for $\sigma=1, 2, \ldots, N$, and then the stress distribution on the collision surface is determined by means of the method of Bellman et al.

8. Numerical Results and Discussions

Typical results are shown in Fig. 3, 5, and 6. The force interacting between the rigid beam and the elastic half plane is presented in Fig. 3, which were obtained by putting $N=3$ through 7. The results for $N\geqq 8$ were not shown because of appreciable discrepancy with the broken line which seems to give a true solution. This tendency is caused by the unboundness of the Laplace inverse transform, as it was already pointed out by Bellman et al.. In order to deduce a true solution, it has been recommended that several trials should be made for various values on N. It is found from the figure that the interacting force decays monotonously with time. This may also be expected from the following simple impact problem in which an exact solution can be obtained.

The rigid body of mass m with initial velocity v is assumed to impinge upon the infinitely long elastic bar whose Young's modulus is denoted by E, cross sectional area by A, and mass density by ρ. As to the bar, if the radial motion is neglected, the equation of motion and the Hooke's law are given respectively by

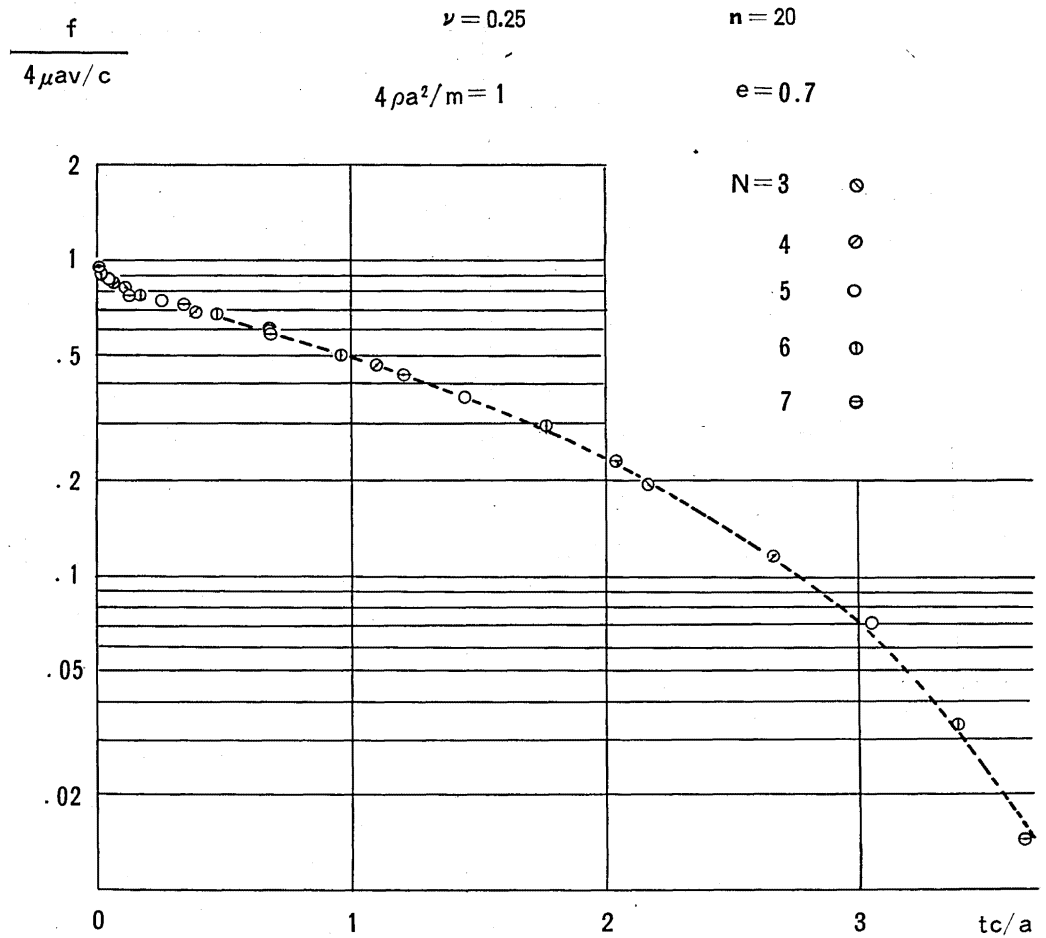

Fig. 3 Interacting Force

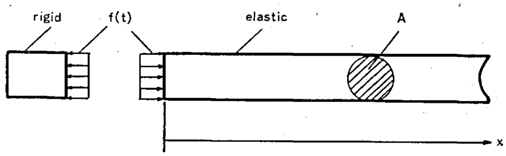

Fig. 4 One-Dimensional Impact Problem

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial \sigma(x,\ t)}{\partial x}, \tag{8.1}$$

and
$$\sigma(x,\ t) = E\frac{\partial u(x,\ t)}{\partial x}, \tag{8.2}$$

in which x represents coordinate, t time, u displacement, and σ stress. With respect to the rigid mass if the displacement is denoted by w,

$$w(t) = u(0,\ t), \tag{8.3}$$

the equation of motion is written as

$$m\frac{d^2 w(t)}{dt^2} = A\sigma(0,\ t), \tag{8.4}$$

with the initial conditions

$$w(0) = 0, \tag{8.5}$$

and
$$\left.\frac{dw(t)}{dt}\right|_{t=0} = v. \tag{8.6}$$

Now, (8.1), (8.2), and (8.4) can be analytically solved with respect to u, σ, and w as follows:

$$u(x,\ t) = \begin{cases} \frac{m}{\rho A}\frac{v}{c}\left[1-\exp\left\{-\frac{\rho A}{m}(ct-x)\right\}\right], & x<ct, \\ 0, & ct<x, \end{cases} \tag{8.7}$$

$$\sigma(x,\ t) = \begin{cases} -E\frac{v}{c}\exp\left\{-\frac{\rho A}{m}(ct-x)\right\}, & x<ct, \\ 0, & ct<x, \end{cases} \tag{8.8}$$

and
$$w(t) = \frac{m}{\rho A}\frac{v}{c}\left\{1-\exp\left(-\frac{\rho A}{m}ct\right)\right\}, \tag{8.9}$$

where
$$c = \sqrt{E/\rho}. \tag{8.10}$$

Putting $x=0$ in (8.8), we find that the interacting force $f(t)$ between the mass and the bar decays exponentially:

$$f(t) = -A\sigma(0,t) = AE\frac{v}{c}\exp\left(-\frac{\rho A}{m}ct\right). \tag{8.11}$$

The stress distribution on the collision surface is presented in Fig. 5. It is found by examining the figure that the curve of $tc/a=0.048$ has the reflection point and the slightly decreasing portion, which may not be allowed from an exact physical point of view and are conjectured to be caused by the errors arising in the numerical caluculation. However, the figure will be effective to illustrate the general transitional behaviour of stress distribution, Through the method of Bellman et al. only, the distribution at $T=0$ can not be obtained. Applying the formula

$$\lim_{s\to\infty} s\hat{h}(s)=h(0), \tag{8.12}$$

to the present problem with the aid of (7.11) in the following way:

$$p(x,\,0)=\lim_{s\to\infty} s\hat{p}(x,\,s)=2\,\mu\frac{v}{c}\lim_{\sigma\to\infty}\frac{\hat{\tau}\,(x,\;\sigma)}{\sigma}\;, \tag{8.13}$$

we can estimate the initial stress distribution. According to (7.39), $\hat{\tau}\,(X_j,\,\sigma)$ and conseguently $\hat{\tau}(X_j,\sigma)/\sigma$ can be calculated for, say, $\sigma=15$, $16,\ldots$, etc.. Extrapolating these results, we can evaluate the values which are presented by a broken line in the figure. Thus, it is obviously concluded that: at the instant of the collision, stress is uniformly distributed over the entire contact surface; while, as time passes, the corner parts of the contact surface are loaded much higher than the middle region. The conclusions just mentioned seem to reflect the following physical sequences of phenomena. Since no stress wave is transmitted into the body at the moment of impact, the entire region of elastic contact surface is equally affected by the impinged beam. As stress waves propagate, the local parts of elastic plane are deformed and consequently the corner region of the contact surface is stressed more severely than the middle.

In Fig. 6 the stress distributions are again given in the non-dimensional form $2ap/f$ on the basis of Fig. 3 and 5. The broken line represents the Sadowsky's result[12]

$$p=\frac{f}{2a}\frac{2}{\pi}\frac{1}{\sqrt{1-(y/a)^2}}, \tag{8.14}$$

which has been valid for the two dimensional static indentation problem. It is found from the figure that the Sadowsky's result is situated below the curve of $tc/a=0.693$ at almost every point of y/a, zero through one. Accordingly it is shown that the relation (6.2) does not exactly hold but approximately for the curve of $tc/a=0.693$, while it completely does for the Sadowsky's. This is again supposed to reflect the error in the course of the numerical integration by means of MacLaurin quadrature. The form of stress distribution can roughly be classified in three stages of: uniform distribution at the initial impact, approaching the distribution given by Sadowsky, and deviating from the static one.

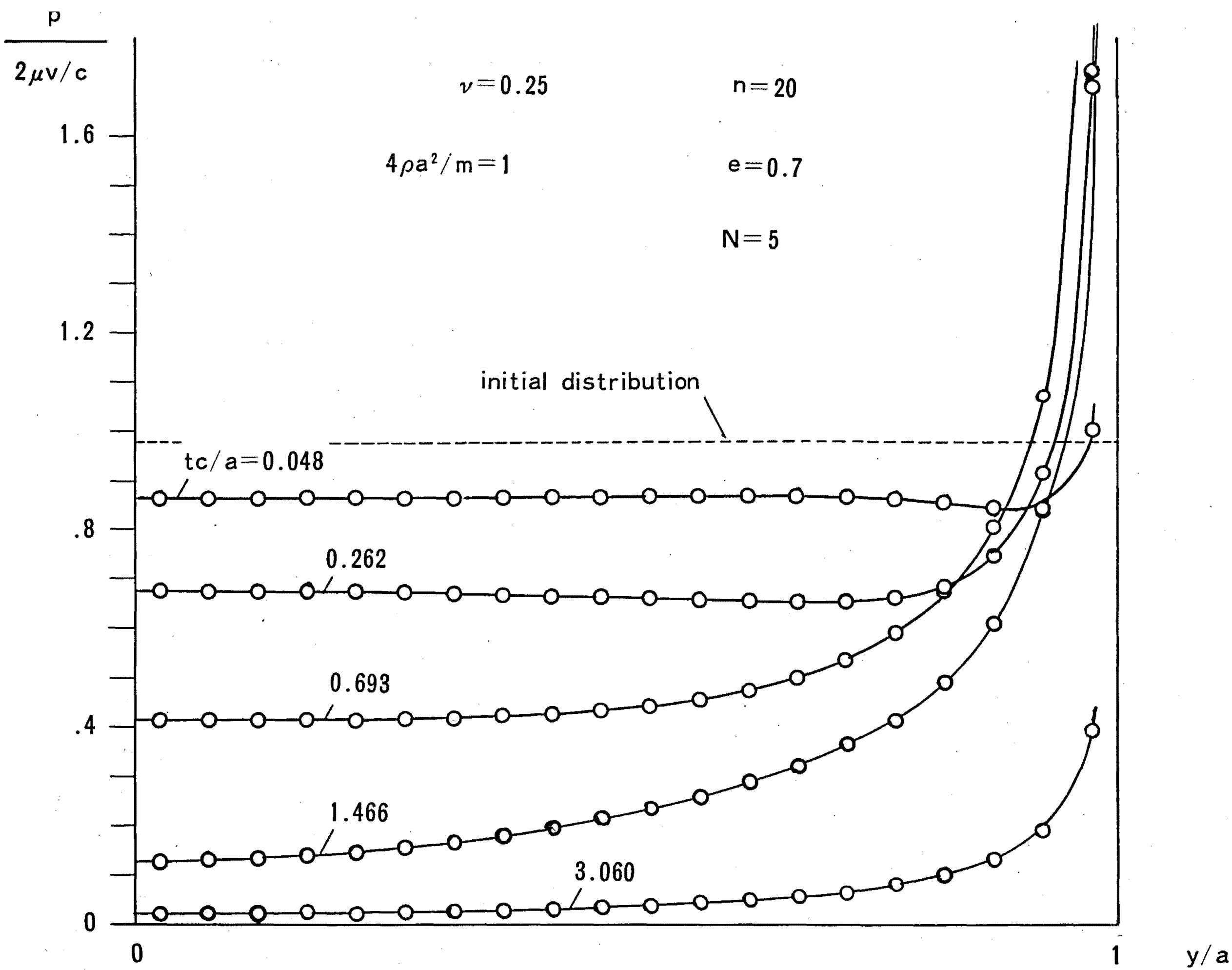

Fig. 5 Stress Distribution

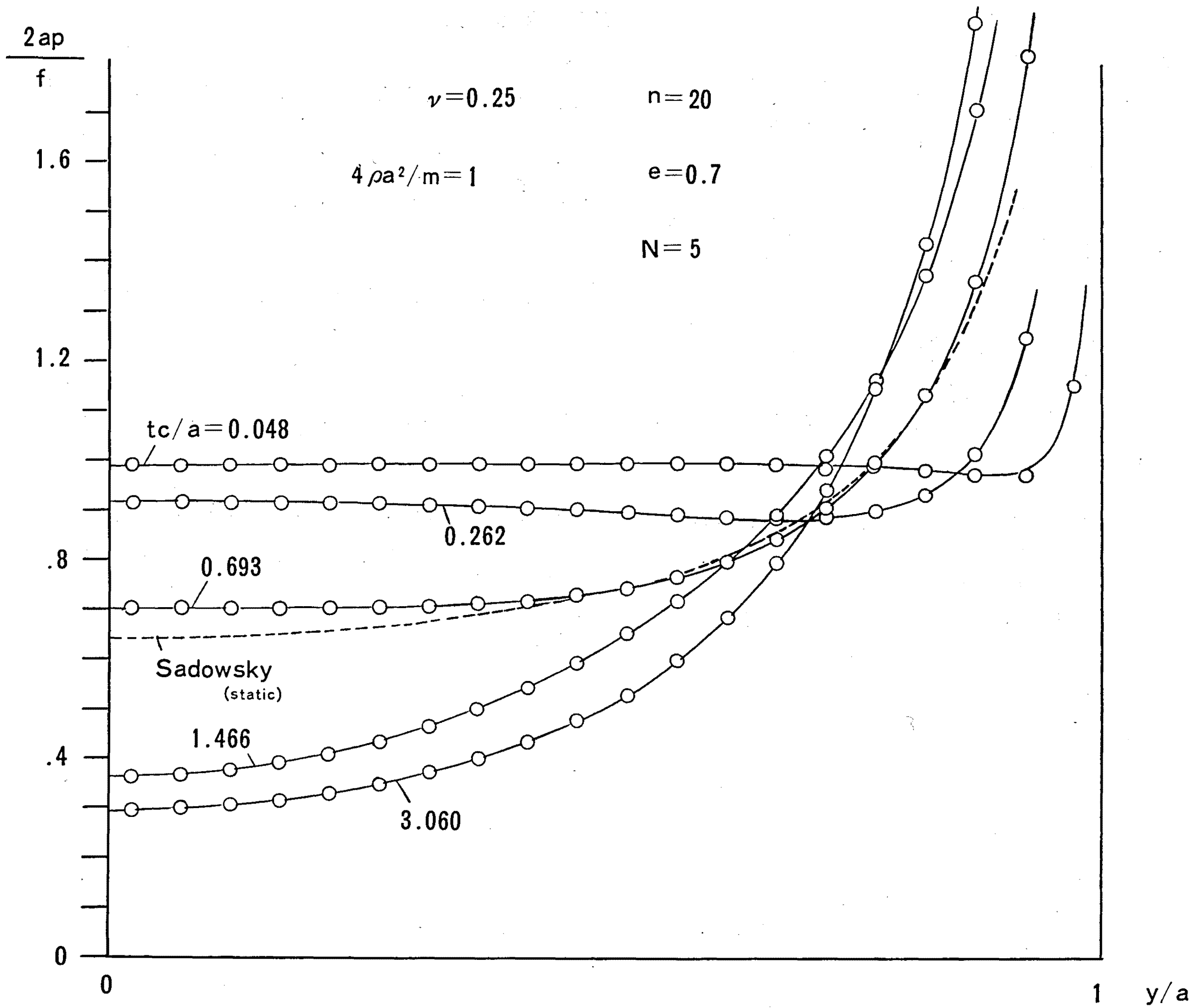

Fig. 6 Normalized Stress Distribution

Acknowledgements

The author wishes to express his sincere gratitude to Professor M. Higuchi, Member of Research Institute for Applied Mechanics of Kyushu University, who has given him valuable advices and warmly encouraged him in promoting the present work.

He is also indebted to Lecturer M. Takematsu, Member of our Institute, for the courtesy of calling his attention to the work of Bellman, Kalaba, and Lockett.

Numerical calculations were performed with the aid of FACOM 230-60 of Computor Center of Kyushu University, FACOM 230-60 of Kyoto University, and HITAC 5020 of Tokyo University. To the staffs of each center, he offers grateful thanks.

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(Received September 26, 1969)