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STABILITY OF A LAMINAR FLOW ALONG A FLEXIBLE BOUNDARY (II)

By Masaki TAKEMATSU*

A laminar flow along a flexible boundary is investigated for stability on the basis of the Orr-Sommerfeld equation. The primary flow considered is of the boundary layer type. A conventional eigenvalue problem is formulated and solved asymptotically for large values of αR.

Special emphasis is placed on the *inviscid* type of instability, which has so far been treated exclusively by means of the Kelvin-Helmholtz theory where the primary flow is assumed to be inviscid and uniform initially. It is first confirmed that for this type of instability, in particular, the eigenvalues in the complete viscous problem are expressed in terms of the corresponding inviscid eigenvalues. A method is then given for evaluating those inviscid eigenvalues for small values of α, thereby the corresponding viscous ones being determined. A short accout is also given of the K-H theory itself.

Numerial calculations are carried out only for non-dispersive flexible boundaries. In the calculations the primary flow is taken to be $U(y)=\tanh(2y)$ as a representative of the boundary layer type flows. Results are presented in a stability diagram, illustrating the inviscid type of instability and the related modification of the Tollmien-Schlichting instability.

1. Introduction

The stability problem for a laminar boundary layer along a flexible surface was previously treated by Benjamin[1], Landahal[2] and others. They first recognized that when the boundary is made flexible three distinct modes of instability may be possible, and then revealed several important features of the respective mode of instability. However it has not been discussed exhaustively as yet how those different waves are described in a stability diagram. It is with this aspect of the problem that the present paper is primarily concerned. In all of the existing analyses except a previous one of the author[3], the so-called Kelvin-Helmholtz instability, among others, has been treated so far only by the Kelvin-Helmholtz theory where the primary flow is assumed to be inviscid and uniform initially, so that the resulting eigenvalues as such cannot be represented in an ordinary stability diagram. Regarding the other *viscous* modes of instability (designated by Benjamin as 'Class A' and 'Class B') there remain also a few respects for which further study is needed.

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As a preliminary to the present study, the author[3)] previously treated the subject by using a broken linear velocity profile for which exact solutions of the Orr-Sommerfeld equation are available and clarified those aspects of the stability characteristics which are effectively independent of the profile curvature of the primary flow. In the present paper is considered a more realistic velocity profile of the boundary layer type. For completeness, the problem formulation is made without any restriction on the values of the parameters involved. When the boundary is flexible, it cannot necessarily be expected that instability occurs only for large values of αR as in the case of a rigid boundary. For analytical convenience, however, solution of the problem is restricted to the case $\alpha R \gg 1$. Special interest is given to the K–H instability, which may be regarded as one of the *inviscid* modes of instability as exemplified by 'Inflection-point' instabilities.

2. Basic equations and boundary conditions

We consider the stability of a plane parallel flow along a flexible boundary (located at rest at $y=0$) with respect to infinitesimal wave-like disturbances. The primary flow is in the x-direction and has the velocity profile $U(y)$ of the boundary layer type. In the following analysis, we work mainly in terms of dimensionless quantities by referring all velocities to the free stream velocity U_∞ and all lengths to a thickness δ of the shear layer, thereby the Reynolds number being defined by $R=U_\infty\delta/\nu$.

The behaviour of infinitesimal disturbances in the primary flow is governed by the familiar Orr-Sommerfeld equation

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = (i\alpha R)^{-1}(\phi^{iv}-2\alpha^2\phi''+\alpha^4\phi), \tag{1}$$

where ϕ, c, α are defined by noting that the stream function associated with the disturbance velocity field is given by

$$\Psi(x,\ y,\ t) = \phi(y)\exp[i\alpha(x-ct)].$$

The boundary conditions to be satisfied at infinity are the usual ones; i.e.

$$\phi'(\infty) = \alpha\phi(\infty) = 0. \tag{2}$$

We now take the formula for the lateral displacement of the flexible boundary* accompanying the fluid motion to be

$$y = \eta(x,\ t) = \hat{\eta}\exp[i\alpha(x-ct)], \tag{3}$$

with the infinitesimal amplitude $\hat{\eta}$. Then, in the linearized approach, the kinematical and non-slipping conditions at the boundary can be written respectively as

$$\phi(0)-c\hat{\eta} = 0, \tag{4}$$

$$\phi'(0)+U'(0)\hat{\eta} = 0. \tag{5}$$

* The boundary is assumed to be not free to move in the x-direction.

Eliminating $\hat{\eta}$ from (4) and (5), we have

$$c\phi'(0)+U'(0)\phi(0)=0. \tag{6}$$

The dynamical condition relating the sinusoidal transverse force F exerted by the fluid and the corresponding surface displacement may generally be expressed as

$$Z\eta=F, \tag{7}$$

by means of a complex stiffness Z which is determined by the mechanical property of the boundary. A variety of flexible boundaries could be characterized by an effective mass (m) per unit area, a damping coefficient (κ) and a free wave velocity ($\bar{a}$). For such a system, the stiffness Z takes the following form

$$Z=M\alpha^2(a^2-c^2)-ic\alpha K, \tag{8}$$

where M, a, K are the dimensionless parameters defined by

$$M=m/\rho\delta, \quad a=\bar{a}/U_\infty \text{ and } K=\kappa/\rho U_\infty.$$

Here we note that for the case of a non-dispersive boundary the free wave velocity a takes a constant value independent of α. On the other hand, the disturbance force F is evaluated as the sum of the pressure, normal stress and the normal component of the tangential stress at the boundary, its linearized form being

$$F=-[\{c\phi'(0)+U'(0)\phi(0)\}+(i\alpha R)^{-1}\{\phi'''(0)-3\alpha^2\phi'(0)+ +U'(0)\alpha^2\hat{\eta}\}]\exp[i\alpha(x-ct)]. \tag{9}$$

Inserting (9) in (7), eliminating $\hat{\eta}$ by the use of (5) and rearranging, we have

$$(\alpha R)^{-1}\phi'''(0)-\left[\frac{iZ}{U'(0)}+4\alpha^2(\alpha R)^{-1}\right]\phi'(0)=0, \tag{10}$$

where use is also made of the relation (6).

The stability problem concerned has now been reduced to a conventional eigenvalue problem of the basic equation (1) with the homogeneous boundary conditions (2), (6), (10); the usual secular determinant leads to a relation of the form $\Delta(c, \alpha, \alpha R, Z)=0$. If the mechanical property of the boundary is specified, then the (complex) eigenvalues $c=c_r+ic_i$ are determined as functions of the real parameter α and αR for a prescribed profile $U(y)$. We now note that thus far no restriction has been made on the values of the parameters involved: Hence the problem formulation made above should cover all possible modes of (linear) instability for any specified boundary. In the case of a flexible boundary, it can not necessarily be expected that instability ($c_i>0$) occures only for large values of αR as in the case of a rigid wall. For analytical convenience, however, the following discussion is restricted to the case $\alpha R\gg 1$.

3. Eigenvalue problem for large values of αR

The analytical property of the solutions of the Orr-Sommerfeld equation (1) for large values of αR is now well known[4]. Its four particular solutions are usually classified into two inviscid solutions $\{\phi_1, \phi_2\}$ and two viscous solutions $\{\phi_3, \phi_4\}$. In each pair of solutions one becomes infinite as $y \to \infty$, while the other tends to zero in that limit. For definitness let ϕ_1 and ϕ_3 be the solutions which vanish at infinity. In view of the vanishing conditions (2) at infinity ϕ_2 and ϕ_4 must be rejected, so that the appropriate general solution of (1) is expressed by a linear combination of ϕ_1 and ϕ_3 alone:

$$\phi = B_1\phi_1 + B_3\phi_3, \tag{11}$$

where B's are arbitrary constants. Imposing the remaining boundary conditions (6), (10) on (11), we have the secular equation

$$\Delta = \begin{vmatrix} \omega_{11} & \omega_{13} \\ \omega_{21} & \omega_{23} \end{vmatrix} = 0\,, \tag{12}$$

with the abbreviations

$$\omega_{1j} = c\phi_j'(0) + U'(0)\phi_j(0),$$

$$\omega_{2j} = (\alpha R)^{-1}\phi_j'''(0) - \left[\frac{iZ}{U'(0)} + 4\alpha^2(\alpha R)^{-1}\right]\phi_j'(0).$$

$$(j = 1,\ 3)$$

The required solutions ϕ_1 and ϕ_3 are obtained asymptotically in the form

$$\phi_1 \sim \Phi^{(0)}(y) + (\alpha R)^{-1}\Phi^{(1)}(y) + \cdots\cdots, \tag{13$_1$}$$

$$\phi_3 \sim \exp[-\sqrt{\alpha R}\,Q(y)]\cdot\{\varphi^{(0)}(y) + (\alpha R)^{-1/2}\varphi^{(1)}(y) + \cdots\cdots\}, \tag{13$_2$}$$

with

$$\varphi^{(0)} = (U-c)^{-5/4},$$

$$Q(y) = \int_{y_c}^{y} \sqrt{i(U-c)}\,dy, \quad U(y_c) = c.$$

Here the first approximation $\Phi^{(0)}$ to the inviscid solution satisfies the Rayleigh equation (obtained from (1) by letting $\alpha R \to \infty$)

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0. \tag{14}$$

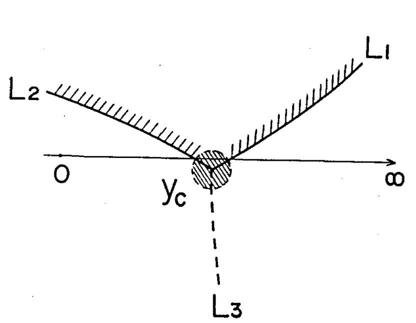

Fig. 1 Domain of validity of the asymptotic solutions (13).

The domain of validity* of these asymptotic solutions in the complex y plane is represented by the unshaded portion in Fig. 1: L_1, L_2, L_3 are the three anti-Stokes lines (defined by the conditions $R_e\{Q(y)\} = 0$) which branch off from the critical point y_c with arguments $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}\right\} - \frac{1}{3}\arg(U_c')$ respectively, the small circle centred at the critical point having a radius of order $(\alpha R)^{-1/3}$. Here we note that the domain of validity thus defined specifies at once the proper branch of the multi-valued expressions

* For detailed discussions, see, e.g., 3), 4)

in (13).

The relevant solutions (the rejected ones as well) can be obtained in an alternative form. That is

$$\phi_j = X_j^{(0)}(z) + \varepsilon X_j^{(1)}(z) + \cdots \quad (j = 1, 3), \tag{15}$$

where

$$\varepsilon = (U_c'\alpha R)^{-1/3}, \quad z = (y - y_c)/\varepsilon.$$

These solutions are all regular, and are also applicable in the circular region including the critical point. However they are usually not so convenient for numerical purposes as the asymptotic solutions given by (13).

The two alternative representations of the solutions are now to be used properly according to the nature of the eigenvalues concerned. In the stability problem for the boundary layer flow along a rigid wall, as is well known, the required solutions ϕ_1 and ϕ_3 are conveniently approximated by $\Phi^{(0)}(y)$ and $X_3^{(0)}(z)$ respectively; the unstable waves which can occur in the presence of a rigid wall are usually designated as Tollmien-Schlichting waves. Then it may well be expected that also in the case of a flexible boundary the same approximation procedure can be used successfully, at least, to examine the modification of the Tollmien-Schlichting wave with wall flexibility. Approximating $\{\phi_1, \phi_3\}$ by $\{\Phi^{(0)}, X_3^{(0)}\}$ in (12), expanding the determinant and neglecting the terms of higher order than the errors already involved in the solutions, we have a simplification of the determinantal equation (12): That is

$$u + iv + \frac{U'(0)c}{Z} = \mathscr{F}(z_c) \tag{16}$$

where

$$u + iv = \left\{1 + \frac{U'(0)}{c} \cdot \frac{\Phi^{(0)}(0)}{\Phi^{(0)\prime}(0)}\right\}^{-1},$$

and $\mathscr{F}(z_c) = \mathscr{F}_r + i\mathscr{F}_i$ is the modified Tietjens function with $z_c = y_c\{U_c'\alpha R\}^{1/3}$. This was first derived by Benjamin[1]. Now we note that the equation (16) is closely related to the Lin's formula* for the case of a rigid boundary (i.e. for the T-S wave). By means of (16) Benjamin[1] gave a lucid account of the modification of the T-S wave with wall flexibility. The numerical examples based on (16) were presented by Landahl[2]. In Benjamin's analysis, however, it was mentioned that the eq. (16) is inappropriate to account directly for the inviscid mode of instability as predicted by the Kelvin-Helmholtz theory; in fact the stability diagrams computed by Landahl did not contain any branch corresponding to the inviscid mode of instability. In the subsequent sections then we deal with this type of instability by using a more appropriate approximation procedure.

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* The eq. (16) reduces to the Lin's formula $u+iv = \mathscr{F}(z_c)$ if the complex stiffness Z is made infinite.

4. Inviscid mode of instability

4-1 Eigenvalue equation for large αR

We now consider the inviscid type of instability which survives even in the inviscid limit $\alpha R \to \infty$. For this type of instability, the condition that both $c(\alpha R)^{1/3}$ and $(1-c)(\alpha R)^{1/3}$ are sufficiently large compared with unity may generally be satisfied for large values of αR: This means that the end points where the boundary conditions are to be imposed are always included in the domain of validity of the asymptotic solutions (13) (cf. Fig. 1). Then we see that in this case it is sufficient to use the asymptotic solutions given by (13) so far as the eigenvalue problem is concerned.

By the use of (13), the following formal asymptotic expansion* of the secular equation (12) is obtained:

$$\Delta \sim \Delta_0 + (\alpha R)^{-1/2}\left[\Delta_0 \cdot J_1(c, \alpha) - \frac{Z \cdot G(c, \alpha)}{c^2 Q'(0)}\right] + \cdots\cdots = 0, \tag{17}$$

with

$$\Delta_0 = Z - G(c, \alpha). \tag{18}$$

In these expressions, $G(c, \alpha)$ represents the (inviscid) function which depends only on the solution of the inviscid equation (14) in the form

$$G(c, \alpha) = -c\left[U'(0) + c \cdot \frac{\Phi^{(0)\prime}(0)}{\Phi^{(0)}(0)}\right], \tag{19}$$

and $J_1(c, \alpha)$ does a bounded function of c and α involving the second approximation $\varphi^{(1)}(0)$ to the viscous solution.

Letting $\alpha R \to \infty$ in (17), we have the inviscid eigenvalue equation

$$\Delta_0 = Z - G(c, \alpha) = 0, \tag{20}$$

which is also obtainable by neglecting the viscous effect initially. In what follows, let c_0 denote the inviscid eigenvalues given by the roots of (20). The corresponding viscous eigenvalues can be evaluated from (17) by iteration: That is

$$Z - G(c, \alpha) = (\alpha R)^{-1/2}\left[\frac{Z \cdot G(c, \alpha)}{c^2 Q'(0)}\right]_{c=c_0}, \tag{21}$$

to the first approximation. To see the relationship between these viscous and inviscid eigenvalues, however, it is more illuminating to solve (17) for eigenvalues in the form

$$c = c_0 + (\alpha R)^{-1/2} c_1 + (\alpha R)^{-1} c_2 + \cdots\cdots. \tag{22}$$

Expanding (17) again by the use of (22) and equating the coefficients of like powers of $(\alpha R)^{-1/2}$ to zero, then we have the set of equations

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* The irrelevant multiplication factors which do not affect the eigenvalue problem are omitted.

$$\left.\begin{aligned} &\Delta_0 = Z - G(c_0, \alpha) = 0, \\ &c_1\left(\frac{\partial \Delta_0}{\partial c}\right)_{c=c_0} - \left[\frac{Z\cdot G(c, \alpha)}{c^2 Q'(0)}\right]_{c=c_0} = 0, \\ &\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots \quad . \end{aligned}\right\} \quad (23)$$

Thus c_0, $c_1, \cdots$ can be determined successively; the initial term c_0 is, of course, the inviscid eigenvalue and the coefficients of $(\alpha R)^{-1/2}$ is found to be

$$c_1 = \left[\frac{Z\cdot G(c, \alpha)}{c^2 Q'(0)}\right]_{c=c_0} \bigg/ \left(\frac{\partial \Delta_0}{\partial c}\right)_{c=c_0}, \qquad (24)$$

provided c_0 is a simple root of $\Delta_0 = 0$, i.e $\left(\frac{\partial \Delta_0}{\partial c}\right)_{c=c_0} \neq 0$. When c_0 is a multiple root of $\Delta_0 = 0$, two cases may be considered according as $[Z\cdot G(\mathrm{c}, \alpha)]_{c=c0} \neq 0$ or $[Z\cdot G(\mathrm{c}, \alpha)]_{c=c_0} = 0$: The above analysis suggests that in the first case such an inviscid eigenvalue is unrelated to the complete viscous problem, while in the second case the dependence on $(\alpha R)^{-1}$ of the corresponding viscous eigenvalue becomes of higher order than one-half. By continuing similar process the coefficients c_2,... can also be found, but the resulting expressions become too complicated to be handled with a reasonable amount of numerical work.

Now that the eigenvalues in the complete viscous problem have been expressed asymptotically in terms of the inviscid eigenvalues, we are only to deal with the inviscid problem; once the inviscid eigenvalues are found, the corresponding viscous eigenvalues can readily be computed by either (21) or (22).

4-2 The inviscid problem

We may now investigate the asymptotic behaviours of the inviscid mode of instability at $\alpha R \to \infty$ by means of (20). For this purpose, it is necessary to evaluate the function $G(c, \alpha)$ for a prescribed velocity profile.

Evaluation of $G(c, \alpha)$: Now it is convenient to introduce the new independent variable

$$g(y) = (U-c)\left[U'\Phi^{(0)} - (U-c)\,\Phi^{(0)\prime}\right]/\Phi^{(0)}, \qquad (25)$$

in terms of which the required function is given simplly as

$$G(c, \alpha) = g(0). \qquad (26)$$

Applying the transformation, the Rayleigh equation (14) is reduced to the first-order equation

$$g' - (U-c)^{-2} g^2 + \alpha^2 (U-c)^2 = 0. \qquad (27)$$

On the other hand the boundary condition on $g(y)$ is found as follows: At $y \gg 1$, where the primary flow is substantially uniform, the eq. (14) reduces to $\phi'' - \alpha^2\phi = 0$. This, together with the vanishing condition at infinity, reads $\Phi^{(0)} \sim e^{-\alpha y}$ as $y \to \infty$. With this property of $\Phi^{(0)}$, the required condition is readily found from (25) as

$$g(\infty) = \alpha(1-c)^2. \qquad (28)$$

The solution of (27) subject to (28) may be obtained in the form

$$g(y) = \alpha(1-c)^2[1+\alpha g_1(y,c)+\cdots]. \tag{29}$$

This form is particularly convenient for small values of α. Inserting (29) in (27) and equating the coefficients of like power of α to zero, we have

$$\begin{aligned} g_1' &= (1-c)^2(U-c)^{-2}-(1-c)^{-2}(U-c)^2, \\ g_2' &= -2g_1(1-c)^2(U-c)^{-2}, \\ &etc. \end{aligned} \tag{30}$$

Integrating these equations with $g_1(\infty)=g_2(\infty)=\cdots=0$, we obtain

$$\begin{aligned} g_1(y,\ c) &= \int_y^\infty \left\{\frac{(1-c)^2}{(U-c)^2}-\frac{(U-c)^2}{(1-c)^2}\right\} dy, \\ g_2(y,\ c) &= -2(1-c)^2\int_y^\infty \frac{g_1(y,\ c)}{(U-c)^2}\, dy, \\ &etc. \end{aligned} \tag{31}$$

Refering to the relation (26), the function $G(c,\ \alpha)$ is given as

$$G(c,\ \alpha) = \alpha(1-c)^2[1+\alpha g_1(o,\ c)+\alpha^2 g_2(o,\ c)+\cdots]. \tag{32}$$

Thus we see that the actual elvaluation of $G(c,\ \alpha)$ is reduced to the calculation of the integrals given by (31). The function G is also related to the Lin's function $u+iv$ by the relation (cf. (17))

$$G(c,\ \alpha) = U'(0)c\{u+iv-1\}^{-1}. \tag{33}$$

In the neutral case (i.e. for real c) this relation may be conveniently used for numerical purposes since for real c, in particular, u and v can be calculated with sufficient accuracy by means of the approximation procedure suggested by Lin[5].

If use is made of the expressions (8) and (32) for Z and G respectively, the inviscid eigenvalue equation is written finally as

$$\Delta_0 = \alpha[M\alpha(a^2-c^2)-icK-(1-c)^2\{1+\alpha g_1(0,\ c)+\cdots\}] = 0. \tag{34}$$

All that remains is now to find the roots of (34) with (31) for specific values of M, a^2 and K; the inviscid eigenvalues thus found are now known to be related to the complete viscous problem by the arguments made in the previous section. As a preliminary to the task, here we give a short account of the Kelvin-Helmholtz theory.

The Kelvin-Helmholtz theory: This simple theory is essentially a long wave approximation and assumes (inviscid) uniform flow initially. In all the previous analyses, as mentioned in the introduction, the inviscid mode of instability has been treated exclusively on the basis of the theory or of its modified form. However it must be recognized that the K-H theory as such is unrelated to the complete viscous problem. In this special case of uniform flow, as is well known[7], the driving force F exerted by the fluid is expressed simply as

$$F = (1-c)^2\alpha\eta. \tag{35}$$

Using this expression in (7) and eliminating η, we have the simple eigenvalue equation

$$\alpha[M\alpha(a^2-c^2)-icK-(1-c)^2]=0. \tag{36}$$

Eq. (36) can readily be solved in terms of c. The result is

$$c_0^{(0)}=\frac{1-i\left(\frac{K}{2}\right)\pm\sqrt{a^2s(s-s_0)-iK-\left(\frac{K}{2}\right)^2}}{1+s}, \tag{37}$$

where

$$s_0=(1-a^2)/a^2,$$

and $s=M\alpha$ ($=m\bar{\alpha}/\rho$ with actual wave number $\bar{\alpha}$) represents the mass ratio refferred to actual wave length. As a matter of course, all the parameters involved in (37) are independent of the reference length δ. Comparing (36) with (34), we see that the use of the K-H theory is tantamount to approximating the function $G(c, \alpha)$ only by the first term of (32); the succeeding terms of (32) are then to be interpreted as the effect of finite thickness of the shear layer. For the special case of a non-dissipative boundary with $K=0$ eq. (37) predicts instability if $s<s_0$ and hence if $c_r<a^2$, while it gives no such definite criterion for instability for the case of a dissipative boundary with $K\neq 0$. This feature is clearly seen in Fig. 2, where eigenvalues evaluated from (37) for $a^2=0.5$ and $K=0$, 0.05 are shown by chain-lines. With these results of the K-H theory in mind, we return to the more general treatment represented by (34).

Omitting the trivial factor α from (34), we have the net eigenvalue equation in the form

$$s(a^2-c^2)-icK-(1-c)^2\{1+\alpha g_1(0, c)+\cdots\}=0, \tag{34$'$}$$

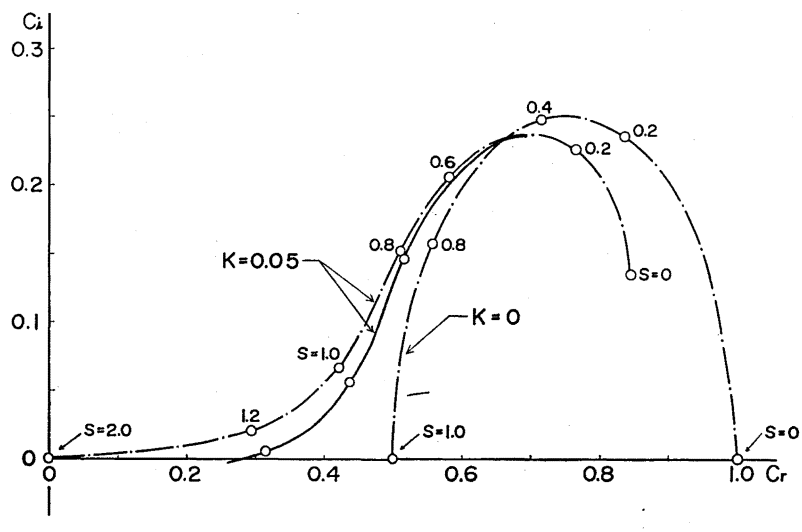

Fig. 2 Inviscid eigenvalues for $a^2=0.5$.
(—·—, from (37); ——, from (38) with $M=30$)

with $s = M\alpha = m\overline{\alpha}/\rho$.

We now find the roots of this equation approximately for small but finite values of α. In doing so it is particularly convenient to regard s as a parameter independent of α, since s is not necessarily be small depending on the values of M: Formally this is to interpret the variation of $\alpha(=\overline{\alpha}\delta)$ as that of δ. According to this interpretation, eq. (34)′ reduces in the limit $\alpha \to 0$ to the K-H equation (36). [Alternatively, if we interpret the variation of α as that of $\overline{\alpha}$, (34)′ reduces in the same limit to the other equation $(1-c)^2+ick=0$, which is, however, included in (36) as a special case $s=0$.] As the zeroth approximation in the more realistic problem concerned, then we can take the K-H equation, whose roots $c_0^{(0)}$ are given by (37). With this choice of the zeroth approximation, the first approximation is obtained in the form

$$c_0 = c_0^{(0)} + \alpha c_0^{(1)} + 0(\alpha^2), \tag{38}$$

where $c_0^{(1)}$ is given in terms of $c_0^{(0)}$ as

$$c_0^{(1)} = \left[\frac{(1-c)^2 g_1(0,\ c)}{2(1-c-sc)-iK}\right]_{c=c^{(0)}}. \tag{39}$$

If use is made of $c_0^{(0)}$ and $c_0^{(1)}$ so determined, the desired inviscid eigenvalues may be evaluated from (38) with acceptable accuracy regardless of the values of s, so long as α is sufficiently small. It is clear from the way eq. (38) has been derived that its second term represents the effect of the non-uniformity of the velocity profile.

In all the numerical examples which follow, we take $U(y)=\tanh(2y)$ as a representative of the velocity profiles of the boundary layer type. In this case, the function $g_1(0,\ c)$ is expressed in the form

$$g_1(0,\ c) = \frac{1}{2(1+c)} - \frac{1}{2(1-c)^2} + \left[\frac{c}{(1+c)^2} - \frac{c}{(1-c)^2}\right]\log_e 2$$
$$- \frac{c}{(1+c)^2}\left[\log_e \frac{(1-c)}{c} + i\pi\right]. \tag{40}$$

Using the values of $c_0^{(0)}$ for $a^2=0.5$, $K=0.05$, which have been shown in Fig. 2 by the chain-line, inviscid eigenvalues are calculated according to the foregoing procedure for several values of M. The numerical results for $M=30$ are shown partly $(c_i \geq 0)$ in Fig. 2 by the full line with s as a parameter, illustrating the existence of a definite stability criterion. The instability condition $(c_i > 0)$ for this specific boundary is found to be $s < 1.27$, i.e. $\alpha < 0.0423$, and the phase velocity of the disturbance which first becomes unstable is $c_r = 0.27$. This is in contrast to the simplified treatment of the K-H theory, according to which no definite criterion is predicted except for the special case $K=0$. Thus the present investigation suggests that the K-H theory should be used reservedly, in particular, for the predition of stability limits.

In particular for the determination of neutral roots $(c_i = 0)$ which provide at once stability limits, a graphical solution of (20) is available, where the functional values of $G(c,\ \alpha)$ can be calculated more accurately from (33) by means of the Lin's function for real c. The numerical values for a neutral root so determined are employed to make an accuracy check of the results obtained above from (38).

We remark that in the exceptional case $K=0$ the inviscid eigenvalue equation (20) has no neutral solution, if account is taken of the non-uniformity of the velocity profile or more precisely of the effect of the profile curvature ($U''(y)$) in the evaluation of $G(c, \alpha)$. This is also the point in which the simplified treatment of the K-H theory or of the previous paper[3] differs from the more general treatment concerned.

5. Stability diagram and discussions

We now calculate the viscous eigenvalues for large αR by means of (16) and (21) or (22) and present the results in a stability diagram. As mentioned in section 3, the neutral stability condition for the modified Tollmien-Schlichting wave may be determined by assuming c to be real in eq. (16). In doing so, we use a usual graphical method. The complex function $\mathscr{F}$ on the right-hand side of (16) which has already been calculated by Lin[6] for real values of z_c (i.e. for real c) can be represented in a Argand diagram. On the same diagram we plot the left-hand side of the equation as a function of c_r at constant values α (Fig. 3). The intersections of the two curves then determine the required neutral eigenvalues c, α and the corresponding values of z_c, and hence αR. The neutral curve thus found for the same flexible boundary as treated in the preceding section is shown in Fig. 4 with the label (A) attached. For comparison the corresponding one for the case of a rigid boundary (i.e. $Z\to\infty$) is also included in the figure. On the other hand, the stability characteristics of the inviscid mode may be calculated according to the procedure described in the preceding section: Using the known inviscid eigenvalues (plotted in Fig. 2) in

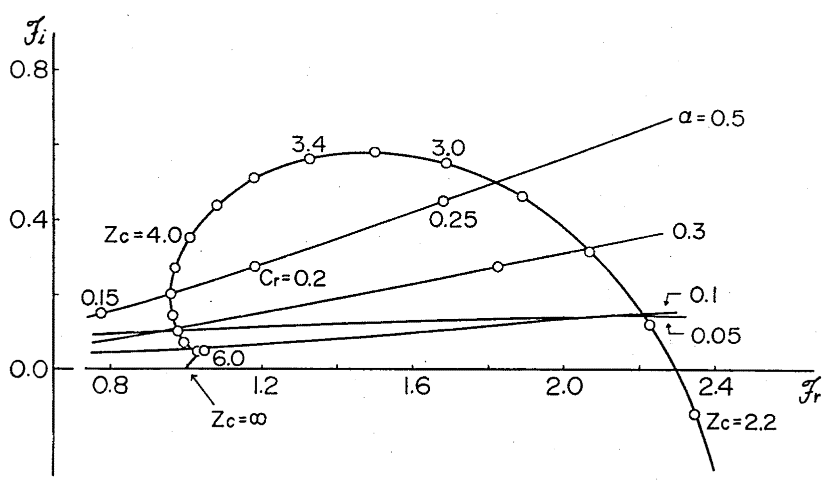

Fig. 3. Argand diagram for the solution of (16).

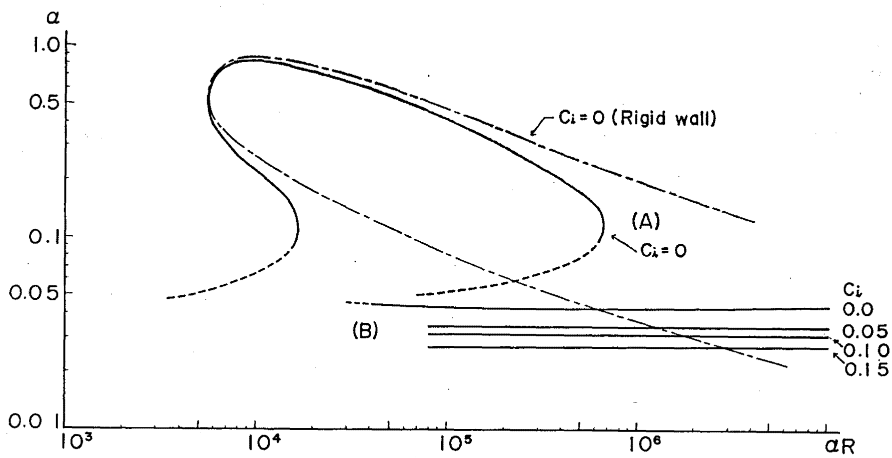

Fig. 4. Example of stability diagram for $U = \tanh 2y$.
($M = 30$, $a^2 = 0.5$, $K = 0.05$)

(21), the corresponding viscous eigenvalues are determined graphically at constant (large) values of αR. The $c_i =$ constant curves so obtained are shown in the same figure.

Similar calculations were also carried out for other various combinations of boundary parameters. The example presented in Fig. 4 is intended to illustrate some of the interesting features characteristic to the stability diagrams so obtained. The branches (B) of the inviscid type always occur so long as the boundary is flexible. As is seen from the figure, they are largely independent of Reynolds numbers, and hence the unstable region of this type may be defined approximately by the line $\alpha = \alpha_0$ for wide range of αR where α_0 is the wave number for the neutral disturbance in the inviscid limit. It must be also noted, however, that those branches are not restricted only in the region of large αR, but that they are likely to extend to fairly small αR. These characteristics of the branches (B) are similar to those observed in the stability problem of the free boundary layer between two parallel streams[8]. On the other hand, the loop (A) represents the modification of T-S waves which are typical ones observed in the presence of a rigid boundary. The loop shows a sharp contrast to the corresponding one for a rigid boundary, in particular, as α approaches α_0 at which the inviscid type of instability may occur: It bends left-ward to small αR without penetrating into the unstable region of the inviscid type. The unstable region enclosed by the loop (A) is thus displaced by the other unstable region, so that the former region appears only when the latter region is restricted to relatively small α, i.e. when the stiffness of the boundary is relatively high.

The present analysis seems to indicate that when the boundary is flexible, we cannot determine the critical Reynolds number so successfully as in the case of a rigid boundary by the use of the asymptotic theory for large αR alone.

Further, according to Landhal[2], another type of instability may occur at quite low Reynolds numbers. In order to cover all these branches at moderate or small values of αR, it is necessary to extend the calculations beyond the limit of applicability of the ordinary asymptotic theory. This may be done most effectively by numerical solution of the complete problem formulated in section 2.

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