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https://doi.org/10.5109/7170226

出版情報：Reports of Research Institute for Applied Mechanics. 17 (56), pp.1-24, 1969. 九州大学応用力学研究所

Reports of Research Institute for Applied Mechanics
Vol. XVII, No. 56, 1969

BOTTOM FRICTION PROBLEM IN COASTAL HYDRAULICS

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In the present status of mathematical analysis of problems encountered in coastal hydraulics, the velocity of water is replaced conventionally by its horizontal component averaged throughout the depth of water, and the frictional stress acting upon the bottom makes its appearance in various expressions as an unknown quantity. It is the mathematical basis of the usual approximation procedure to assume the bottom friction in a functional form linear or quadratic with regard to this mean velocity in order to complete the system of equations. In this paper, degree of accuracy attained by this approximation is studied in detail. From a viewpoint of turbulent flow theory in three-dimensions, if we employ moments of an infinite number of equations of motion to represent the degree of freedom in the direction of the depth of water, it is readily proved that the present averaging technique is nothing but the approximation achieved by adopting only the first member of moment equations. It is the subject of this paper to elucidate the behavior of the approximate solution by showing, in a few of fundamental problems of typical character, that it approaches the exact solution steadily when the number of the moment equations to be taken into consideration is increased. At the same time, our objects in view are, in the first place, to prove that the average method generally employed has tolerably satisfactory accuracy for practical purposes under the restrictions that the change of the environmental condition is gradual and that the depth of water is quite small, and, in the second place, to suggest a procedure of improving the approximation, when the circumstance varies with a steep gradient in space and/or in time on deep water.

1. Introduction

It is not always possible in coastal hydraulics to work out in great detail the motion of water as a function of four variables, representing time and three-dimensional stretch of space, through an analytical method nor by means of a numerical procedure with the aid of a digital computer. Generally speaking, it is a customary practice, therefore, to simplify the whole phenomenon by constructing differential equations valid only 'in the mean' with regard to the depth

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of water.[1] Owing to this technique, we are enabled to define, as functions of time t over a two-dimensional region (x, y) of water, known quantities such as the atmospheric pressure gradient, the surface traction due to wind, as well as unknown quantities such as the velocity of water averaged through its depth, and the surface elevation. In doing so, we are confronted with a difficulty: the frictional force acting upon the bottom of water makes its appearance in an indeterminate form mathematically. Thus we have to correlate the bottom friction somehow with those functions defined already and/or x, y, t, the coordinates. This is the subject to be studied in the present paper.

Let us begin with the discussions of a most general problem. In hydrodynamics of three-dimensional flows, it is a usual convention to give the bottom friction, $\boldsymbol{T}^{(b)}$, a non-linear form

$$\boldsymbol{T}^{(b)} = \rho|\boldsymbol{v}^*|\boldsymbol{v}^*, \tag{1.1}$$

ρ and $\boldsymbol{v}^*$ being the density of water and the friction velocity, respectively. By assuming some relationship between $\boldsymbol{v}^*$ and the velocity of the stream at this point on the bottom, we are led to an expression of $\boldsymbol{T}^{(b)}$ as a function of the flow velocity $\boldsymbol{v}$.[2] (Usually we may approximate the velocity of the stream to a satisfactory accuracy by its horizontal component. In the followings the latter will be denoted as $\boldsymbol{v}$ for simplicity.) However, the fact that the friction formula derived in this way as well as inertia terms in NAVIER-STOKES equations take non-linear forms without fail constitutes the central part of difficulty encountered in flow problems. And in coastal hydraulics in which rough treatments are permissible to a great extent, we have not always to preserve the non-linearity. Namely, like in a number of cases in which inertia terms may be neglected safely, we should employ a linear form for the expression of the bottom friction. By virtue of this linearization, coastal hydraulics, which is essentially the science of flows for studying the synthetic effect resulting invariably from a great number of factors, reduces itself into superposition of independent problems as many, and is found so simplified as to allow mathematical analysis.

Looking from this standpoint, we might conclude that we should desirably substitute an appropriate constant for $|\boldsymbol{v}^*|$ in our formulation of the bottom friction. Writing κ^* for it, we have accordingly

$$\boldsymbol{T}^{(b)} = \rho\kappa^*\boldsymbol{v}^* \tag{1.2}$$

in place of (1.1). Furthermore, neglecting the boundary layer thickness on the bottom compared with the depth of the mainstream, and assuming at the same time a linear relationship between $\boldsymbol{v}^*$ and the velocity at the outer edge of the bottom boundary layer, $\boldsymbol{v}(0)$, we may write down the equation in an alternative form

$$\boldsymbol{T}^{(b)} = \rho\kappa\boldsymbol{v}(0), \tag{1.3}$$

which is regarded as the approximate formula for the law of bottom friction; κ is a constant (or a constant speed, more precisely) to be determined in reference to the prevailing stream velocity, being dependent also upon environmental circumstances.

Now, as long as neglect of inertia terms is permissible, the equations governing $\boldsymbol{v}$, the horizontal component of the velocity, and ζ, the surface elevation, may be written approximately

$$\Gamma \equiv \frac{\partial \boldsymbol{v}}{\partial t} + g\frac{\partial \zeta}{\partial \boldsymbol{r}} + g\frac{\partial P}{\partial \boldsymbol{r}} - \frac{\partial}{\partial z}\left(\nu \frac{\partial \boldsymbol{v}}{\partial z}\right) = 0, \tag{1.4}$$

and

$$\varDelta \equiv \frac{\partial \zeta}{\partial t} + \operatorname{div}\int_0^h \boldsymbol{v}\, dz = 0, \tag{1.5}$$

where $\partial/\partial \boldsymbol{r}$ and div denote respectively the gradient and the divergence on the x-y plane. $z=0$ and $z=h(x, y)$ correspond to the bottom, and the surface, of water respectively, P being the atmospheric pressure in terms of water head. ν, the eddy viscosity, is regarded as a function of z only. The boundary conditions may be summarized as

$$\nu \frac{\partial \boldsymbol{v}}{\partial z} = \frac{\boldsymbol{T}^{(b)}}{\rho} = \kappa \boldsymbol{v}(0) \tag{1.6}$$

at $z=0$, and

$$\nu \frac{\partial \boldsymbol{v}}{\partial z} = \frac{\boldsymbol{T}^{(s)}}{\rho} \tag{1.7}$$

at $z=h$, $\boldsymbol{T}^{(s)}$ being the traction due to wind acting on the surface of water. With the aid of the measured wind velocity at a standard height, $\boldsymbol{V}$ say, we may put approximately, as is usual,

$$\boldsymbol{T}^{(s)} = k\rho |\boldsymbol{V}| \boldsymbol{V}; \tag{1.8}$$

although measurements of k, a pure constant, reveal sensible scatterings, validity of the relation (1.8) might be preserved nevertheless provided we could substitute for k a value appropriately chosen.

For the convenience of the following analysis, let us introduce a new variable z' in place of z as defined by $z=hz'$. Then (1.4)—(1.7) can be rewritten as follows.

$$\Gamma \equiv \frac{\partial \boldsymbol{v}}{\partial t} + g\frac{\partial \zeta}{\partial \boldsymbol{r}} + g\frac{\partial P}{\partial \boldsymbol{r}} - \frac{1}{h^2}\frac{\partial}{\partial z'}\left(\nu \frac{\partial \boldsymbol{v}}{\partial z'}\right) = 0, \tag{1.9}$$

$$\varDelta \equiv \frac{\partial \zeta}{\partial t} + \operatorname{div}\left(h\int_0^1 \boldsymbol{v}\, dz'\right) = 0, \tag{1.10}$$

$$\left(\frac{\nu}{h}\frac{\partial \boldsymbol{v}}{\partial z'}\right)_{z'=0} = \frac{\boldsymbol{T}^{(b)}}{\rho} = \kappa \boldsymbol{v}(0), \tag{1.11}$$

and

$$\left(\frac{\nu}{h}\frac{\partial \boldsymbol{v}}{\partial z'}\right)_{z'=1} = \frac{\boldsymbol{T}^{(s)}}{\rho}. \tag{1.12}$$

Taking the average of (1.9) with regard to z' over the range from 0 through

1, we are led to the result

$$\frac{\partial \bar{\boldsymbol{v}}}{\partial t}+g\frac{\partial \zeta}{\partial \boldsymbol{r}}+g\frac{\partial P}{\partial \boldsymbol{r}}-\frac{\boldsymbol{T}^{(s)}}{\rho h}+\frac{\boldsymbol{T}^{(b)}}{\rho h}=0, \tag{1.13}$$

with the aid of (1.11) and (1.12), where $\bar{\boldsymbol{v}} \equiv \int_0^1 \boldsymbol{v}\, dz'$ denotes the horizontal flow velocity averaged through the depth of water. Thus is achieved the so-called 'simplification-to-plane-problem' frequently employed up to the present, when this equation and (1.10), or more briefly,

$$\frac{\partial \zeta}{\partial t}+\mathrm{div}(h\bar{\boldsymbol{v}})=0, \tag{1.14}$$

are combined together. It is well-known that, if an appropriate estimation is applied for $\boldsymbol{T}^{(b)}=\rho\kappa\boldsymbol{v}(0)$, quite a reasonable result can be yielded through this method.[3]

2. Exact procedure for plane-problem-method and practice of its approximation

Let us study in the following a method of improving the accuracy of the averaging technique above-mentioned. For this purpose, we must call to mind the 'theorem of moments', which reads that if all the moments vanish of a function defined in a finite interval, then the function itself should be zero identically.[4] From this viewpoint, equation (1.9) is found equivalent to the condition

$$\Gamma_n \equiv \int_0^1 \Gamma z'^n\, dz' = 0 \quad (n=0,\ 1,\ 2,\ \dots,\ \infty), \tag{2.1}$$

and we may replace the equations (1.9) and (1.14) with the simultaneous system of equations composed of both (2.1) of a number of enumerable infinity and (1.14) as before. Since these equations do not contain the variable z' any longer, we have succeeded in bringing the problem into two-dimensional formulation, which we have been aiming at. Rewriting (2.1) in detail by taking into account of (1.11) and (1.12), we have

$$\begin{aligned}
&\frac{\partial \boldsymbol{v}_0}{\partial t}+g\frac{\partial \zeta}{\partial \boldsymbol{r}}+g\frac{\partial P}{\partial \boldsymbol{r}}-\frac{\boldsymbol{T}^{(s)}}{\rho h}+\frac{\kappa\boldsymbol{v}(0)}{h}=0 \qquad (n=0),\\
&\frac{\partial \boldsymbol{v}_1}{\partial t}+\frac{1}{2}\left(g\frac{\partial \zeta}{\partial \boldsymbol{r}}+g\frac{\partial P}{\partial \boldsymbol{r}}\right)-\frac{\boldsymbol{T}^{(s)}}{\rho h}+\frac{\tau_0}{h^2}=0 \qquad (n=1),\\
&\frac{\partial \boldsymbol{v}_2}{\partial t}+\frac{1}{3}\left(g\frac{\partial \zeta}{\partial \boldsymbol{r}}+g\frac{\partial P}{\partial \boldsymbol{r}}\right)-\frac{\boldsymbol{T}^{(s)}}{\rho h}+\frac{2\tau_1}{h^2}=0 \qquad (n=2),\\
&\dots,
\end{aligned} \tag{2.2}$$

where the following abbreviations were employed for convenience:

$$\boldsymbol{\nu}_n \equiv \int_0^1 \boldsymbol{\nu} z'^n \, \mathrm{d}z',$$

and

$$\boldsymbol{\tau}_n \equiv \int_0^1 \left(\nu \frac{\partial \boldsymbol{\nu}}{\partial z'}\right) z'^n \, \mathrm{d}z' \qquad (n = 0, 1, 2, \ldots). \tag{2.3}$$

The first equation of (2.2), corresponding to $n = 0$, is found nothing but (1.13), and it is obvious that the accuracy attained by the averaging technique would be improved by the present method provided we could simultaneously put to use the remaining equations for $n \geqq 1$ also. It should be noted that our computation ceases to be exact only when we put an end to the procedure at some finite value of n.

So far the water region was supposed as extending most generally in two dimensions. However, problems of the bottom friction related with possibility of approximation as well as with examination of its accuracy would not lose their generality, even if we should confine ourselves to the case when both of $P(\boldsymbol{r}, t)$ and $\boldsymbol{T}^{(s)}(\boldsymbol{r}, t)$ are uniform in one direction, along the y-axis, say. In the followings, therefore, let us simplify the calculation by assuming that

$$P = P(x, t), \quad \boldsymbol{\nu} = \boldsymbol{i}u,$$
$$\boldsymbol{T}^{(s)} = \boldsymbol{i}T^{(s)}(x, t), \text{ and } \boldsymbol{T}^{(b)} = \boldsymbol{i}T^{(b)}(x, t), \tag{2.4}$$

where $\boldsymbol{i}$ denotes the unit vector along the x-axis, and further that the associated boundary conditions would not violate the one-dimensional property of the flow. Then equations (2.2), (2.3), and (1.10) become respectively

$$\frac{\partial u_0}{\partial t} + g\frac{\partial \zeta}{\partial x} + g\frac{\partial P}{\partial x} - \frac{T^{(s)}}{\rho h} + \frac{\kappa u(0)}{h} = 0 \qquad (n = 0),$$

$$\frac{\partial u_1}{\partial t} + \frac{1}{2}\left(g\frac{\partial \zeta}{\partial x} + g\frac{\partial P}{\partial x}\right) - \frac{T^{(s)}}{\rho h} + \frac{\tau_0}{h^2} = 0 \qquad (n = 1), \tag{2.5}$$

$$\frac{\partial u_2}{\partial t} + \frac{1}{3}\left(g\frac{\partial \zeta}{\partial x} + g\frac{\partial P}{\partial x}\right) - \frac{T^{(s)}}{\rho h} + \frac{2\tau_1}{h^2} = 0 \qquad (n = 2),$$

$$\ldots,$$

where

$$u_n \equiv \int_0^1 u z'^n \, \mathrm{d}z', \text{ and } \tau_n \equiv \int_0^1 \left(\nu \frac{\partial u}{\partial z'}\right) z'^n \, \mathrm{d}z' \quad (n = 0, 1, 2, \ldots), \tag{2.6}$$

and

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (hu_0)}{\partial x} = 0. \tag{2.7}$$

On the other hand, boundary conditions (1.11) and (1.12) reduce respectively into

$$\left(\frac{\nu}{h}\frac{\partial u}{\partial z'}\right)_{z'=0} = \kappa u(0),$$

and

$$\left(\frac{\nu}{h}\frac{\partial u}{\partial z'}\right)_{z'=1} = \frac{T^{(s)}}{\rho}. \tag{2.8}$$

The velocity of the stream, u, is a continuous function of z' in the interval $0 < z' < 1$, and can be approximated therefore to an accuracy as much as we please by means of a polynomial of z'. However, unless we are interested in particularly exact configuration of a velocity distribution, satisfactory approximation may be attained by the use of a polynomial of a lower degree. Now, putting u in the form

$$u(x, z', t) = a_0(x, t) + a_1(x, t)z' + a_2(x, t)z'^2 + \dots, \tag{2.9}$$

(2.8) gives two relationships among a's:

$$\frac{\nu(0)}{h} a_1 = \kappa a_0,$$

$$\text{and} \qquad \frac{\nu(1)}{h}(a_1 + 2a_2 + \dots) = \frac{T^{(s)}}{\rho}, \tag{2.10}$$

through which a_0 and a_1, for example, can be expressed in terms of a_i's ($i \neq 0, 1$) and $T^{(s)}$. Next, substitution of (2.9) for u in (2.6) yields

$$u_n = \frac{a_0}{n+1} + \frac{a_1}{n+2} + \frac{a_2}{n+3} + \dots, \tag{2.11}$$

and

$$\tau_n = \nu_n a_1 + 2\nu_{n+1} a_2 + 3\nu_{n+2} a_3 + \dots, \tag{2.12}$$

where

$$\nu_m \equiv \int_0^1 \nu(z') z'^m \, dz', \tag{2.13}$$

($m, n = 0, 1, 2, \dots$). At the same time, we obtain from (2.9)

$$u(0) \equiv u(x, 0, t) = a_0(x, t). \tag{2.14}$$

By virtue of these equalities, (2.5) and (2.7) are found to constitute an infinite system of linear partial differential equations of the first order with regard to $a_n(x, t)$ ($n = 0, 1, 2, \dots$) and $\zeta(x, t)$. If we should succeed in finding the system of solutions compatible with the initial conditions $\zeta(x, 0)$ and $a_n(x, 0)$, then $u(x, z', t)$ can be easily derived from (2.9) by substituting a_n's thus obtained for those appearing on its right-hand side.

In what follows, however, after expressing a_n's in terms of u_n's through equations (2.10) and (2.11), we proceed to rewrite τ_n's and $u(0)$ as functions of u_n's using equations (2.12) and (2.14) respectively. Then equations (2.5) and (2.7) become the system of linear equations of u_n's and ζ, and our problem reduces to finding the solutions of that system. Since, in particular, $T^{(b)}$, the bottom friction in which we are now interested primarily, may be given in a non-dimensional form

$$\frac{T^{(b)}}{\rho\kappa^2} = \frac{u(0)}{\kappa} = \frac{a_0(x, t)}{\kappa}, \tag{2.15}$$

by substituting for a_0 on the right-hand side its expression in terms of u_n's, we can readily evaluate the frictional force in question. If we assume further that ν is a constant independent of z' for the purpose of simplification, then (2.12) produces the result

$$\tau_n = \nu\left(\frac{a_1}{n+1}+\frac{2a_2}{n+2}+\frac{3a_3}{n+3}+\dots\right) \quad (n=0,\ 1,\ 2,\dots). \tag{2.16}$$

Besides, we shall treat in the followings only the case when $h(x, t)$ may be regarded as invariable.

In the next place, let us turn to an approximate computation. In view of existence of two boundary conditions stated in (2.10), the simplest form permissible for u of our approximate polynomial expressions appearing on the right-hand side of (2.9) should be composed of the first three terms. Accordingly, we are to adopt the first equation of (2.5) together with (2.7) as their determining equations. Now, since

$$u_0 = a_0+\frac{a_1}{2}+\frac{a_2}{3}, \tag{2.17}$$

we obtain by solving (2.10) and (2.17)

$$a_0=\frac{1}{R(1/3+1/R)}\left(u_0-\kappa\frac{R}{6}\frac{T^{(s)}}{\rho\kappa^2}\right),$$

$$a_1=\frac{1}{(1/3+1/R)}\left(u_0-\kappa\frac{R}{6}\frac{T^{(s)}}{\rho\kappa^2}\right), \tag{2.18}$$

and

$$a_2=\frac{1}{2(1/3+1/R)}\left\{-u_0+\kappa R\left(\frac{1}{2}+\frac{1}{R}\right)\frac{T^{(s)}}{\rho\kappa^2}\right\},$$

where $h\kappa/\nu$, the non-dimensional parameter, is denoted as R. Introducing the value of a_0 in (2.15), we are led to an approximate formula for the bottom friction:

$$\frac{T_1{}^{(b)}}{\rho\kappa^2}=\frac{1}{R(1/3+1/R)}\left(\frac{u_0}{\kappa}-\frac{R}{6}\frac{T^{(s)}}{\rho\kappa^2}\right), \tag{2.19}$$

where subscript 1 in $T_1{}^{(b)}$ signifies that only the first equation of (2.5) has been taken into consideration. On the other hand, by applying (2.18) to the first equation of (2.5) and combining the result with (2.7), the equations of motion of water are found to be

$$\frac{\partial u_0}{\partial t}+g\frac{\partial \zeta}{\partial x}+\frac{\kappa}{h}\frac{u_0}{R(1/3+1/R)}+g\frac{\partial P}{\partial x}-\frac{1/2+1/R}{1/3+1/R}\frac{T^{(s)}}{\rho h}=0,$$

and

$$\frac{\partial \zeta}{\partial t}+h\frac{\partial u_0}{\partial x}=0; \tag{2.20}$$

these are nothing but the well-known equations in coastal hydraulics. By substituting the computed value of u_0 into the right-hand side of (2.19), we are enabled to evaluate the bottom friction.[1)]

Now, let us go a step farther in our approximations: by taking the form up to the term of a_3 in (2.9) and adopting therefore the equations

$$a_1 = Ra_0,$$

and
$$a_1+2a_2+3a_3 = \kappa R(T^{(s)}/\rho\kappa^2),$$
as (2.10), and
$$u_0 = a_0+\frac{a_1}{2}+\frac{a_2}{3}+\frac{a_3}{4},$$
and
$$u_1 = \frac{a_0}{2}+\frac{a_1}{3}+\frac{a_2}{4}+\frac{a_3}{5},$$
as (2.11) respectively, we obtain
$$a_0 = \frac{1}{R}\left(\frac{7}{2}u_0-5u_1+\frac{\kappa R}{24}\frac{T^{(s)}}{\rho\kappa^2}\right)\left(\frac{1}{8}+\frac{1}{R}\right)^{-1},$$
$$a_1 = \left(\frac{7}{2}u_0-5u_1+\frac{\kappa R}{24}\frac{T^{(s)}}{\rho\kappa^2}\right)\left(\frac{1}{8}+\frac{1}{R}\right)^{-1},$$
$$a_2 = \left\{-15\left(\frac{8}{15}+\frac{1}{R}\right)u_0+30\left(\frac{5}{12}+\frac{1}{R}\right)u_1-\frac{3}{4}\kappa R\left(\frac{2}{9}+\frac{1}{R}\right)\frac{T^{(s)}}{\rho\kappa^2}\right\}\left(\frac{1}{8}+\frac{1}{R}\right)^{-1}, \tag{2.21}$$
and
$$a_3 = \left\{10\left(\frac{5}{12}+\frac{1}{R}\right)u_0-20\left(\frac{1}{3}+\frac{1}{R}\right)u_1+\frac{5}{6}\kappa R\left(\frac{1}{6}+\frac{1}{R}\right)\frac{T^{(s)}}{\rho\kappa^2}\right\}\left(\frac{1}{8}+\frac{1}{R}\right)^{-1},$$
as the solution of these simultaneous equations. The unknown variables appearing in our calculation are u_0, u_1, and ζ, the determining equations of which are the first and the second equations of (2.5) together with (2.7). With the aid of the relation
$$\tau_0 = \nu(a_1+a_2+a_3) = \nu\{u(1)-u(0)\}, \tag{2.22}$$
they are written respectively
$$\frac{\partial u_0}{\partial t}+g\frac{\partial\zeta}{\partial x}+\frac{\kappa}{h}\frac{7/2}{R(1/8+1/R)}u_0-\frac{\kappa}{h}\frac{5}{R(1/8+1/R)}u_1 + g\frac{\partial P}{\partial x}-\frac{1/12+1/R}{1/8+1/R}\frac{T^{(s)}}{\rho h}=0,$$
$$\frac{\partial u_1}{\partial t}+\frac{g}{2}\frac{\partial\zeta}{\partial x}-\frac{\kappa}{h}\frac{1/3+5/R}{R(1/8+1/R)}u_0+\frac{\kappa}{h}\frac{10(1/12+1/R)}{R(1/8+1/R)}u_1 + \frac{g}{2}\frac{\partial P}{\partial x}-\frac{1/9+11/12R}{1/8+1/R}\frac{T^{(s)}}{\rho h}=0, \tag{2.23}$$
and
$$\frac{\partial\zeta}{\partial t}+h\frac{\partial u_0}{\partial x}=0.$$

Finally, the expression for the bottom friction, which according to our convention should be denoted as $T_2^{(b)}$ because two moment equations have been taken into consideration, becomes

$$\frac{T_2^{(b)}}{\rho\kappa^2}=\frac{a_0}{\kappa}=R^{-1}\left(\frac{1}{8}+\frac{1}{R}\right)^{-1}\left(\frac{7}{2}\frac{u_0}{\kappa}-5\frac{u_1}{\kappa}+\frac{R}{24}\frac{T^{(s)}}{\rho\kappa^2}\right). \tag{2.24}$$

3. Forced tidal waves

With a view to examining the accuracy attained by the approximate expressions of the bottom friction worked out in the preceding sections, we shall take up some typical problems. Since our method is based upon the linearized equations, the general solution of the problem should be composed of the solution corresponding to the forced motion (or, in other words, a particular solution of the equations containing the terms of P and $T^{(s)}$; let us call it the 'forced motion solution' for short) and that representing the free motion (or a solution of the equations derived by putting $P=T^{(s)}=0$, to be mentioned as the 'free motion solution' in what follows). The former is divided again into the terms containing P only and those related to $T^{(s)}$ only. If, therefore, we should succeed in evaluating the exact value of the bottom friction, which implies that an infinite number of moments of the equations has been taken into consideration and therefore should be written as $T_\infty^{(b)}$ according to our abbreviation, out of these three kinds of the solutions mentioned above, it is quite certain that by comparing them with the corresponding $T_1^{(b)}$ and $T_2^{(b)}$, we shall be able to gain a satisfactory understanding of accuracy of our approximate procedures.

Now, let us consider the forced motion solution in the first place. If the impressed forces $g\partial P/\partial x$ and $T^{(s)}/\rho h$ are supposed to be subject to FOURIER analysis with regard to both of x and t, then the whole problem becomes analyzed accordingly, and what we have to do reduces into finding the solutions corresponding to the constituent impressed forces

$$g\frac{\partial P}{\partial x}=F(\alpha,\ \theta)e^{i(\alpha x+\theta t)},$$

and

$$\frac{T^{(s)}}{\rho}=G(\alpha,\ \theta)e^{i(\alpha x+\theta t)}. \tag{3.1}$$

Thus the exact solution of equations (1.9) and (1.10) may be constructed in the following manner. Namely, writing the relevant components of the velocity and the surface elevation as

$$u=U(z';\ \alpha,\ \theta)e^{i(\alpha x+\theta t)},$$

and

$$\zeta=Z(\alpha,\ \theta)e^{i(\alpha x+\theta t)}, \tag{3.2}$$

substitute these expressions into (1.9) and (1.10). Then we shall readily arrive at the results

$$\frac{\nu}{h^2}\frac{d^2U}{dz'^2}-i\theta U=i\alpha gZ+F,$$

and

$$i\theta Z+i\alpha h\int_0^1 U\,dz'=0; \tag{3.3}$$

also the boundary conditions (1.11), (1.12) will be transformed into

$$\frac{\nu}{h}\left(\frac{\mathrm{d}U}{\mathrm{d}z'}\right)_{z'=0} = \kappa U(0),$$

and

$$\frac{\nu}{h}\left(\frac{\mathrm{d}U}{\mathrm{d}z'}\right)_{z'=1} = G, \tag{3.4}$$

respectively.

The general solution of the first equation of (3.3) is found to be

$$U(z') = A\cosh\sqrt{\mathrm{i}\lambda}\,z' + B\sinh\sqrt{\mathrm{i}\lambda}\,z' - (\mathrm{i}\alpha gZ + F)/\mathrm{i}\theta, \tag{3.5}$$

where $\lambda \equiv h^2\theta/\nu$ is the non-dimensional parameter second to R. By employing (3.4) in order to determine A and B, the integration constants, we obtain

$$A = \frac{G\,\mathrm{sech}\sqrt{\mathrm{i}\lambda} - \kappa U(0)}{\mathrm{i}h\theta\tanh\sqrt{\mathrm{i}\lambda}/\sqrt{\mathrm{i}\lambda}},$$

and

$$B = \frac{\kappa\sqrt{\mathrm{i}\lambda}\,U(0)}{\mathrm{i}h\theta}. \tag{3.6}$$

Furthermore, from (3.5) we know that

$$U(0) = A - (\mathrm{i}\alpha gZ + F)/\mathrm{i}\theta. \tag{3.7}$$

On the other hand, integrating the first equation of (3.3) with respect to z' from 0 to 1, and taking account of the second equation as well as (3.4), we have

$$Z = \frac{\mathrm{i}\alpha h}{\alpha^2 gh - \theta^2}\left\{F - \frac{G}{h} + \frac{\kappa}{h}U(0)\right\}. \tag{3.8}$$

When A and Z formulated in (3.6) and (3.8) respectively are introduced into (3.7), $U(0)$ is evaluated quite readily, and by (1.3) the exact value of the bottom friction, $T_\infty^{(b)}$, may be given finally in the form

$$\frac{T_\infty^{(b)}}{\rho\kappa^2} = \frac{U(0)}{\kappa}\mathrm{e}^{\mathrm{i}(\alpha x+\theta t)}$$

$$= \left[-m^2\frac{hF}{\kappa^2} + \left\{1-(1-m^2)\frac{\sqrt{\mathrm{i}\lambda}}{\sinh\sqrt{\mathrm{i}\lambda}}\right\}\frac{G}{\kappa^2}\right]\frac{\mathrm{e}^{\mathrm{i}(\alpha x+\theta t)}}{\varDelta_\infty}, \tag{3.9}$$

where

$$\varDelta_\infty \equiv 1-(1-m^2)\left(\frac{\sqrt{\mathrm{i}\lambda}}{\tanh\sqrt{\mathrm{i}\lambda}} + \mathrm{i}\frac{\lambda}{R}\right), \tag{3.10}$$

$m \equiv \theta/\alpha\sqrt{gh}$ being the third non-dimensional parameter.

In contrast with the exact solution above-mentioned, the bottom friction predicted by the approximate method explained in the previous sections may be found without difficulty. Namely, (2.19) and (2.24) yield respectively

$$\frac{T_1^{(b)}}{\rho\kappa^2} = \left[-m^2\frac{hF}{\kappa^2} + \left\{1-(1-m^2)\left(1-\mathrm{i}\frac{\lambda}{6}\right)\right\}\frac{G}{\kappa^2}\right]\frac{\mathrm{e}^{\mathrm{i}(\alpha x+\theta t)}}{\varDelta_1}, \tag{3.11}$$

where
$$\varDelta_1 \equiv 1-(1-m^2)\left(1+\mathrm{i}\frac{\lambda}{3}+\mathrm{i}\frac{\lambda}{R}\right), \tag{3.12}$$

and
$$\frac{T_2^{(b)}}{\rho\kappa^2} = \left[-m^2\frac{hF}{\kappa^2} + \left\{1-(1-m^2)\frac{1-13\lambda^2/1200-\mathrm{i}(\lambda/6-\lambda^3/2400)}{1+\lambda^2/100}\right\} \times\frac{G}{\kappa^2}\right]\frac{\mathrm{e}^{\mathrm{i}(\alpha x+\theta t)}}{\varDelta_2}, \tag{3.13}$$

where
$$\varDelta_2 \equiv 1-(1-m^2)\left\{\frac{1+37\lambda^2/1200+\mathrm{i}(\lambda/3+\lambda^3/800)}{1+\lambda^2/100}+\mathrm{i}\frac{\lambda}{R}\right\}; \tag{3.14}$$

numerical discussions concerning accuracy of these results will be found in the last section of this paper.

Since we have, by expanding the expression for $T_\infty^{(b)}$ when λ is quite small,

$$\frac{T_\infty^{(b)}}{\rho\kappa^2} = \left[-m^2\frac{hF}{\kappa^2} + \left\{1-(1-m^2)\frac{1-\lambda^2/120+\ldots-\mathrm{i}(\lambda/6-\lambda^3/5040+\ldots)}{1+\lambda^2/90+\ldots}\right\} \times\frac{G}{\kappa^2}\right]\mathrm{e}^{\mathrm{i}(\alpha x+\theta t)} \div\left[1-(1-m^2)\left\{\frac{1+\lambda^2/30+\ldots+\mathrm{i}(\lambda/3+\lambda^3/360+\ldots)}{1+\lambda^2/90+\ldots}+\mathrm{i}\frac{\lambda}{R}\right\}\right], \tag{3.15}$$

we readily know that T_∞, T_1, and T_2 all coincide to each other up to the term of λ.

It would be pertinent to add here, incidentally, that when $m=1$ we have

$$\frac{T_\infty^{(b)}}{\rho\kappa^2} \equiv \frac{T_1^{(b)}}{\rho\kappa^2} \equiv \frac{T_2^{(b)}}{\rho\kappa^2} = \left(-\frac{hF}{\kappa^2}+\frac{G}{\kappa^2}\right)\mathrm{e}^{\mathrm{i}(\alpha x+\theta t)}. \tag{3.16}$$

4. Free tidal waves

When the impressed forces $\partial P/\partial x$ and $T^{(s)}$ are both omitted, equations (1.9) and (1.10) become respectively

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}-\frac{\nu}{h^2}\frac{\partial^2 u}{\partial z'^2}=0,$$

and
$$\frac{\partial \zeta}{\partial t}+h\frac{\partial}{\partial x}\int_0^1 u\,\mathrm{d}z'=0, \tag{4.1}$$

while, on the other hand, (1.11) and (1.12), the boundary conditions imposed upon the bottom and the surface of water, are written

$$\frac{\nu}{h}\left(\frac{\partial u}{\partial z'}\right)_{z'=0} = \kappa u(0), \tag{4.2}$$

and
$$\frac{\nu}{h}\left(\frac{\partial u}{\partial z'}\right)_{z'=1} = 0.$$

Besides, supposing the end of the water region be given by $x = 0$, let us formulate the boundary conditions at this point in the forms

$$h\int_0^1 u\,\mathrm{d}z' = q(t), \tag{4.3}$$
and
$$\zeta = s(t).$$

In solving equations (4.1) and (4.2), however, the boundary conditions at $x = 0$ may be specified not in terms of $q(t)$ nor $s(t)$ but of the constituent elements of their FOURIER integrals, $Q(\theta)\exp(\mathrm{i}\theta t)$ and $S(\theta)\exp(\mathrm{i}\theta t)$ respectively.

Now, assuming the representations
$$u = U^*(x,\ z';\ \theta)\mathrm{e}^{\mathrm{i}\theta t}, \tag{4.4}$$
and
$$\zeta = Z^*(x;\ \theta)\mathrm{e}^{\mathrm{i}\theta t}$$
in (4.1), we have

$$\frac{\nu}{h^2}\frac{\partial^2 U^*}{\partial z'^2} - \mathrm{i}\theta U^* = g\frac{\mathrm{d}Z^*}{\mathrm{d}x}, \tag{4.5}$$
and
$$\mathrm{i}\theta Z^* + h\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 U^*\,\mathrm{d}z' = 0.$$

If we solve the first of these equations with regard to U^* and determine the integration constants in such a way that the two conditions mentioned in (4.2) would be fulfilled, we are led to

$$U^* = \frac{RU^*(0)}{\sqrt{\mathrm{i}\lambda}}\sinh\sqrt{\mathrm{i}\lambda}\,z' - \frac{RU^*(0)}{\sqrt{\mathrm{i}\lambda}\tanh\sqrt{\mathrm{i}\lambda}}\cosh\sqrt{\mathrm{i}\lambda}\,z' + \frac{\mathrm{i}g}{\theta}\frac{\mathrm{d}Z^*}{\mathrm{d}x}, \tag{4.6}$$

from which putting $z' = 0$,

$$U^*(0) = \frac{\mathrm{i}g}{\theta}\left(1 + \frac{R}{\sqrt{\mathrm{i}\lambda}\tanh\sqrt{\mathrm{i}\lambda}}\right)^{-1}\frac{\mathrm{d}Z^*}{\mathrm{d}x}. \tag{4.7}$$

Integrating U^* with respect to z' on the other hand,

$$\int_0^1 U^*\,\mathrm{d}z' = -\frac{RU^*(0)}{\mathrm{i}\lambda} + \frac{\mathrm{i}g}{\theta}\frac{\mathrm{d}Z^*}{\mathrm{d}x}. \tag{4.8}$$

Combining (4.7) with (4.8),

$$\int_0^1 U^*\,\mathrm{d}z' = \mathrm{i}\frac{g}{\theta p_\infty^2}\frac{\mathrm{d}Z^*}{\mathrm{d}x}, \tag{4.9}$$
where
$$p_\infty = \left\{1 - \left(\frac{\sqrt{\mathrm{i}\lambda}}{\tanh\sqrt{\mathrm{i}\lambda}} + \mathrm{i}\frac{\lambda}{R}\right)^{-1}\right\}^{-1/2}.$$

When (4.9) is substituted for the integral of U^* on the right-hand side of the second member of (4.5), left untouched so far, this is transformed into

$$\frac{d^2Z^*}{dx^2}+\frac{\theta^2 p_\infty^2}{c^2}Z^*=0,$$

so that
$$Z^*=A^*\sin\frac{\theta p_\infty}{c}x+B^*\cos\frac{\theta p_\infty}{c}x,$$

where $c\ (\equiv\sqrt{gh})$ represents the propagation speed of the tidal wave. In order to determine A^* and B^*, the integration constants, in accordance with the conditions (4.3) which are rewritten as

$$h\int_0^1 U^*(0, z'; \theta)\,dz'=Q(\theta),$$
and
$$Z^*(0;\ \theta)=S(\theta), \tag{4.10}$$
use must be made of the relation (4.9). The final result is found to be

$$Z^*=-i\frac{Qp_\infty}{c}\sin\frac{\theta p_\infty}{c}x+S\cos\frac{\theta p_\infty}{c}x. \tag{4.11}$$

In the last place, (4.11) is to be substituted for Z^* on the right-hand side of (4.7). Then, however, if we note that a convenient transformation,

$$\left(1+\frac{R}{\sqrt{i\lambda}\tanh\sqrt{i\lambda}}\right)^{-1}=i\frac{\lambda}{R}\left(1-\frac{1}{p_\infty^2}\right),$$

is possible by virtue of the definition of p_∞, cf. (4.9), we may write in conclusion

$$\frac{U^*(0)}{\kappa}=i\frac{\lambda}{R}(p_\infty^2-1)\left(\frac{Q}{h\kappa}\cos\frac{\theta p_\infty}{c}x-i\frac{cS}{h\kappa p_\infty}\sin\frac{\theta p_\infty}{c}x\right), \tag{4.12}$$

in terms of which the bottom friction for the free motion solution is given by

$$\frac{T_\infty^{(b)}}{\rho\kappa^2}=\frac{U^*(0)}{\kappa}e^{i\theta t}. \tag{4.13}$$

Let us examine the approximate solutions explained in **2** with a view to comparing them with this exact solution just obtained. $T_1^{(b)}$, the first approximation, may be derived from (2.19) and (2.20) by putting $P=T^{(s)}\equiv 0$. Namely, assuming the forms

$$u_n=U_n^*(x;\ \theta)e^{i\theta t}\qquad (n=0,\ 1,\ 2,\ldots),$$
and
$$\zeta=Z^*(x;\ \theta)e^{i\theta t} \tag{4.14}$$
we have from (2.20)

$$i\theta p_1^2U_0^*+g\frac{dZ^*}{dx}=0,$$
and
$$i\theta Z^*+h\frac{dU_0^*}{dx}=0, \tag{4.15}$$

where
$$p_1\equiv\left\{1-\left(1+i\frac{\lambda}{3}+i\frac{\lambda}{R}\right)^{-1}\right\}^{-1/2}. \tag{4.16}$$

Constructing the solution compatible with the boundary conditions (4.10), it is found from (2.19) that

$$\frac{T_1^{(b)}}{\rho\kappa^2}=\frac{1}{R(1/3+1/R)}\frac{U_0^*}{\kappa}\mathrm{e}^{\mathrm{i}\theta t}=\mathrm{i}\frac{\lambda}{R}(p_1^2-1)\left\{\frac{Q}{h\kappa}\cos\frac{\theta p_1}{c}x\right.$$

$$\left.-\mathrm{i}\frac{cS}{h\kappa p_1}\sin\frac{\theta p_1}{c}x\right\}\mathrm{e}^{\mathrm{i}\theta t}. \tag{4.17}$$

It should be remarked that this is nothing but the solution stated as (4.12), except only that p_∞ has been replaced by p_1.

Concerning $T_2^{(b)}$, the second approximation, (2.23) and (2.24) are to be treated in the same manner. Out of the first and the second equations of (2.23), in which u_0 and ζ have been transformed into U_0^* and Z^* respectively according to the definitions stated in (4.14), namely

$$\begin{aligned}&\mathrm{i}\theta U_0^*+g\frac{\mathrm{d}Z^*}{\mathrm{d}x}+\frac{7\kappa/2h}{R(1/8+1/R)}U_0^*-\frac{5\kappa/h}{R(1/8+1/R)}U_1^*=0,\\ \text{and}\quad&\mathrm{i}\theta U_1^*+\frac{g}{2}\frac{\mathrm{d}Z^*}{\mathrm{d}x}-\frac{(1/3+5/R)\kappa/h}{R(1/8+1/R)}U_0^*+\frac{10(1/12+1/R)\kappa/h}{R(1/8+1/R)}U_1^*=0,\end{aligned} \tag{4.18}$$

if we eliminate U_1^* first and then combine the result with the third equation transformed in the same way, we shall arrive at

$$\begin{aligned}&\mathrm{i}\theta p_2^2U_0^*+g\frac{\mathrm{d}Z^*}{\mathrm{d}x}=0,\\ &\mathrm{i}\theta Z^*+h\frac{\mathrm{d}U_0^*}{\mathrm{d}x}=0,\end{aligned} \tag{4.19}$$

where
$$p_2\equiv\left[1-\left\{\frac{1+37\lambda^2/1200+\mathrm{i}(\lambda/3+\lambda^3/800)}{1+\lambda^2/100}+\mathrm{i}\frac{\lambda}{R}\right\}^{-1}\right]^{-1/2}. \tag{4.20}$$

Comparing these equations with (4.15), we find readily that the solution should have just the same form as (4.17), except that p_1 has to be replaced by p_2. Thus

$$\frac{U_0^*}{\kappa}=\frac{Q}{h\kappa}\cos\frac{\theta p_2}{c}x-\mathrm{i}\frac{cS}{h\kappa p_2}\sin\frac{\theta p_2}{c}x. \tag{4.21}$$

Since the relationship between U_0^* and U_1^*, on the other hand, is found to be

$$5\frac{U_1^*}{\kappa}=\left\{\frac{7}{2}+\mathrm{i}\lambda\left(\frac{1}{8}+\frac{1}{R}\right)(1-p_2^2)\right\}\frac{U_0^*}{\kappa}, \tag{4.22}$$

by eliminating $\mathrm{d}Z^*/\mathrm{d}x$ between both of the first equations of (4.18) and (4.19), we find after introducing (4.22) into (2.24) that

$$\frac{T_2^{(b)}}{\rho\kappa^2}=\mathrm{i}\frac{\lambda}{R}(p_2^2-1)\frac{U_0^*}{\kappa}\mathrm{e}^{\mathrm{i}\theta t},$$

or, with the aid of (4.21), we may write further

$$\frac{T_2^{(b)}}{\rho\kappa^2}=\mathrm{i}\frac{\lambda}{R}(p_2^2-1)\left\{\frac{Q}{h\kappa}\cos\frac{\theta p_2}{c}x-\mathrm{i}\frac{cS}{h\kappa p_2}\sin\frac{\theta p_2}{c}x\right\}e^{\mathrm{i}\theta t}, \tag{4.23}$$

which is nothing but the exact solution (4.12)−(4.13), provided we ignore that p_2 appears in place of p_∞.

5. Numerical discussions

With a view to gathering a general idea of the accuracy of the proposed approximations, values of various constants appearing in the computations are chosen in such a way that would be plausible for a problem of a typhoon passing across a bay or a harbor:

$$h=500\sim3000 \text{ cm},\ \kappa=0.15 \text{ cm/sec},\ \nu=2h\kappa/3=0.1h \text{ cm}^2\text{/sec},$$

$$\theta=0.00014\sim0.0035 \text{ sec}^{-1} \text{ (the corresponding period being } 0.5\sim12 \text{ hr)},$$

and

$$\theta/\alpha=500\sim1500 \text{ cm/sec}.$$

Out of these data it is found that

$$R=h\kappa/\nu=1.5,\ \lambda=h^2\theta/\nu=0.7\sim105,$$

and

$$m=\theta/\alpha c=0.3\sim2.0.$$

Various formulas to be put to test are shown in tabular forms (see Tables 1 through 11 at the end of the paper) as functions of λ, containing m (= 0.3, 0.8, 1.2, and 2.0) as a parameter. Expressions chosen for examination are the followings: (i) in the case of forced tidal waves, coefficients of the terms

$$-\frac{hF}{\kappa^2}e^{\mathrm{i}(\alpha x+\theta t)},\quad \text{and}\quad \frac{G}{\kappa^2}e^{\mathrm{i}(\alpha x+\theta t)},$$

constituting $T^{(b)}/\rho\kappa^2$, cf. (3.9)−(3.14), and (ii) for free tidal waves, coefficients of the terms

$$\frac{Q}{h\kappa}\cos\frac{\theta p}{c}x\,e^{\mathrm{i}\theta t},\quad \text{and}\quad -\mathrm{i}\frac{cS}{h\kappa}\sin\frac{\theta p}{c}x\,e^{\mathrm{i}\theta t},$$

which make up again $T^{(b)}/\rho\kappa^2$, together with p standing for p_1, p_2, and p_∞, cf. (4.12), (4.13), (4.17), and (4.23). Since the coefficients for the case (ii) do not contain the parameter m, labor involved in the calculation will be lightened more or less. This time, however, we have to compare values of p_1, p_2, and p_∞ with each other as functions of λ, since they are found in the arguments of sines and cosines.

In all of our tables, complex quantities are expressed in the forms $r_n \exp(\mathrm{i}\phi_n)$ ($n=1$, 2, and ∞), r and ϕ being respectively the absolute value and the argument. The subscripts 1, 2, and ∞ correspond respectively to T_1, T_2, and T_∞. Furthermore, in order to show the degree of approximation, also are given in the tables the relative errors of r and ϕ defined as

$$\varepsilon_{rn}\equiv\frac{r_n-r_\infty}{r_\infty}, \tag{5.1}$$

and
$$\varepsilon_{\phi n} \equiv \frac{\phi_n - \phi_\infty}{\phi_\infty}, \tag{5.2}$$

where n takes the values 1 and 2.

Out of these numerical tables we may conclude as follows. Accuracy with which approximate solution are enabled to predict the bottom friction is observed to differ largely with orders of those approximations. When the pressure gradient does exist (a non-vanishing F, in other words; see (3.1)), or when we are dealing with free tidal waves, we may notice that the error exceeding 20 % is produced for the range $\lambda > 30 \sim 40$ out of the *first* approximation (in which only the zeroth order moment is taken into account) and in no case the error greater than 10 % can be yielded by the *second* approximation (in which both of the zeroth and the first order moments are considered). On the contrary, when the surface traction plays an important part (or $G \doteqdot 0$; see (3.1)), the error of the *first* approximation is over 20 % for $\lambda > 5 \sim 9$, and the error in excess of 20 % is resulted in the range $\lambda > 13 \sim 30$ even from the *second* approximation. Besides, it is well-known that the wind traction is a leading part in a typhoon. As long as, therefore, the water depth remains small and at the same time the time variation of meteorological elements is gradual, the *first* approximation suffices assuredly to compute the whole situation. However, in cases when those elements change abruptly with time over a deep sea, as experienced, for example, when the core of a typhoon is passing, values of $\lambda \gtrsim 30$ are attained unseldom, and the prevailing calculation in which only the first approximation formula is taken into consideration is liable, as is naturally expected, to yield a result accompanied by sensible error. The situation would be very favorable, provided that the high water originated from this central region is small compared with the surface elevation brought about by the skirt of a typhoon in which only gradual change is given rise to (this is in fact quite probable). But under circumstances in which this presumption should be invalidated, or in a computation aiming at an estimation of that component of high water itself, it is certain that a considerable improvement might be achieved by the *second* approximation. Furthermore, the error distribution of the *second* approximation seems to suggest a need of employing even the *third* approximation for a computation of the problem when things vary impulsively in space and/or in time.

In case we have to use the *second* approximation, it would mean only waste of labor and time for us to make use of it, over the whole domain of our computation. As a matter of fact, the advanced approximation should be taken into account exclusively in the region where terms showing steep change take place when we analyze (not necessarily in terms of FOURIER analysis) applied meteorological stresses into constituent simple elements. Confining ourselves to the problem of meteorological high waters, we might express the procedure, composed of separating the steeply changing part from the mild environment and of applying the *second* approximation for that restricted region only, in another way more tangible by saying that we should pick out the strong traction of wind and sharp gradient of pressure, both existing just in the neighborhood of the core of a typhoon, from the flat distribution of the stress extending over the remaining whole region. In other words, by analyzing an actual typhoon into a soft disturbance diffused over a wide area and a sharp one concentrated into a small region, we might next proceed to compute two kinds of high water levels independently through applying the *first* approximation to the former and the *second* to the latter. Finally, the actual water level might be worked out by summing up both of these constituent high waters. We hope to discuss this practical method of computation in near future.

Acknowledgments

The whole of numerical works involved in this study was performed with the aid of OKITAC 5090H. The present writers take this chance to express their gratitude to the staff of the Computation Center of Kyushu University for their cooperation. They are grateful also to Miss Mieko FURUMOTO for her assistances in preparing the manuscript.

The project was supported by the Grant-in-Aid for the Scientific Research from Ministry of Education.

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(Received April 24, 1969)

Appendix——Tables

(1) Complex quantities are expressed in the polar form $r_n \exp(i\phi_n)$, where n indicates the number of moments which have been taken into consideration; $n=\infty$ corresponds accordingly to the exact solution.

(2) ε_{rn} and $\varepsilon_{\phi n}$ $(n=1, 2)$ are respectively the relative errors of r and ϕ defined as follows.

$$\varepsilon_{rn} \equiv \frac{r_n - r_\infty}{r_\infty}, \qquad \text{and} \qquad \varepsilon_{\phi n} \equiv \frac{\phi_n - \phi_\infty}{\phi_\infty}.$$

Numerical values of errors are given in parentheses beneath the corresponding quantities in permillage.

(3) The following abbreviation is employed throughout the tables: e.g. for .1234 **—1** read 0.1234×10^{-1}.

Table 1. Coefficient of the term $-\frac{hF}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m=0.3$

λ \ n	r_n (ε_{rn})			ϕ_n $(\varepsilon_{\phi n})$		
	1	2	∞	1	2	∞
0.7	0.1399 0 (−3)	0.1403 0 (−0)	0.1403 0	0.1430 +1 (−10)	0.1445 +1 (−1)	0.1446 +1
1.0	.9482 −1 (−4)	.9880 −1 (−0)	.9882 −1	.1472 +1 (−15)	.1493 +1 (−1)	.1494 +1
2.0	.4939 −1 (−9)	.4985 −1 (−0)	.4986 −1	.1521 +1 (−27)	.1561 +1 (−2)	.1564 +1
3.0	.3295 −1 (−18)	.3353 −1 (−0)	.3354 −1	.1538 +1 (−39)	.1596 +1 (−3)	.1600 +1
4.0	.2472 −1 (−28)	.2543 −1 (−0)	.2543 −1	.1546 +1 (−48)	.1619 +1 (−3)	.1625 +1
5	0.1978 −1 (−40)	0.2059 −1 (−0)	0.2060 −1	0.1551 +1 (−57)	0.1637 +1 (−4)	0.1644 +1
10	.9890 −2 (−100)	.1098 −1 (−0)	.1098 −1	.1561 +1 (−77)	.1676 +1 (−9)	.1690 +1
15	.6593 −2 (−142)	.7663 −2 (−2)	.7681 −2	.1564 +1 (−79)	.1675 +1 (−14)	.1698 +1
20	.4945 −2 (−168)	.5908 −2 (−6)	.5945 −2	.1566 +1 (−77)	.1665 +1 (−18)	.1696 +1
30	.3297 −2 (−199)	.4047 −2 (−17)	.4116 −2	.1567 +2 (−71)	.1644 +1 (−26)	.1687 +1
40	0.2473 −2 (−217)	0.3070 −2 (−28)	0.3158 −2	0.1568 +1 (−66)	0.1629 +1 (−30)	0.1679 +1
50	.1978 −2 (−229)	.2471 −2 (−37)	.2567 −2	.1569 +1 (−62)	.1618 +1 (−32)	.1672 +1
60	.1648 −2 (−239)	.2066 −2 (−46)	.2165 −2	.1569 +1 (−59)	.1611 +1 (−33)	.1667 +1
80	.1236 −2 (−251)	.1555 −2 (−59)	.1651 −2	.1570 +1 (−53)	.1601 +1 (−34)	.1658 +1
100	.9890 −3 (−260)	.1246 −2 (−68)	.1337 −2	.1570 +1 (−50)	.1596 +1 (−34)	.1652 +1

Table 2. Coefficient of the term $\frac{G}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m=0.3$

λ \ n	r_n (ε_{rn})			ϕ_n $(\varepsilon_{\phi n})$		
	1	2	∞	1	2	∞
0.7	0.2163 0 (−39)	0.2258 0 (+3)	0.2252 0	0.2298 +1 (+15)	0.2261 +1 (−2)	0.2265 +1
1.0	.1929 0 (−47)	.2032 0 (+4)	.2025 0	.2507 +1 (+27)	.2435 +1 (−3)	.2442 +1
2.0	.1736 0 (−48)	.1836 0 (+7)	.1824 0	.2804 +1 (+64)	.2618 +1 (−6)	.2635 +1
3.0	.1698 0 (−34)	.1776 0 (+11)	.1758 0	.2913 +1 (+100)	.2623 +1 (−10)	.2649 +1
4.0	.1684 0 (−12)	.1732 0 (+16)	.1705 0	.2970 +1 (+136)	.2579 +1 (−14)	.2615 +1
5	0.1678 0 (+17)	0.1687 0 (+22)	0.1651 0	0.3004 +1 (+172)	0.2518 +1 (−18)	0.2563 +1
10	.1670 0 (+241)	.1444 0 (+73)	.1345 0	.3072 +1 (+351)	.2177 +1 (−43)	.2275 +1
15	.1668 0 (+571)	.1222 0 (+151)	.1062 0	.3095 +1 (+512)	.1890 +1 (−77)	.2047 +1
20	.1667 0 (+985)	.1055 0 (+256)	.8400 −1	.3107 +1 (+648)	.1665 +1 (−117)	.1885 +1
30	.1667 0 (+2020)	.8472 −1 (+535)	.5520 −1	.3119 +1 (+839)	.1337 +1 (−212)	.1696 +1
40	0.1667 0 (+3263)	0.7364 −1 (+883)	0.3910 −1	0.3124 +1 (+932)	0.1110 +1 (−314)	0.1617 +1
50	.1667 0 (+4597)	.6727 −1 (+1259)	.2978 −1	.3128 +1 (+956)	.9434 0 (−410)	.1599 +1
60	.1667 0 (+5909)	.6333 −1 (+1625)	.2412 −1	.3130 +1 (+947)	.8172 0 (−492)	.1608 +1
80	.1667 0 (+8280)	.5900 −1 (+2285)	.1796 −1	.3133 +1 (+913)	.6406 0 (−609)	.1638 +1
100	.1667 0 (+10397)	.5683 −1 (+2886)	.1462 −1	.3135 +1 (+898)	.5244 0 (−683)	.1652 +1

Table 3. Coefficient of the term $-\frac{hF}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m=0.8$

λ \ n	r_n (ε_{rn}) 1	r_n 2	r_n ∞	ϕ_n ($\varepsilon_{\phi n}$) 1	ϕ_n 2	ϕ_n ∞
0.7	0.9305 0 (−5)	0.9352 0 (−0)	0.9355 0	0.3751 0 (−5)	0.3767 0 (−0)	0.3768 0
1.0	.8716 0 (−10)	.8798 0 (−1)	.8803 0	.5124 0 (−9)	.5165 0 (−1)	.5168 0
2.0	.6644 0 (−26)	.6808 0 (−1)	.6818 0	.8442 0 (−24)	.8630 0 (−2)	.8646 0
3.0	.5098 0 (−39)	.5295 0 (−2)	.5306	.1036 +1 (−37)	.1072 +1 (−3)	.1075 +1
4.0	.4061 0 (−51)	.4272 0 (−2)	.4282 0	.1153 +1 (−47)	.1204 +1 (−4)	.1209 +1
5	0.3350 0 (−64)	0.3569 0 (−2)	0.3578 0	0.1229 +1 (−55)	0.1294 +1 (−5)	0.1300 +1
10	.1750 0 (−118)	.1978 0 (−3)	.1984 0	.1395 +1 (−72)	.1489 +1 (−10)	.1503 +1
15	.1177 0 (−155)	.1384 0 (−5)	.1392 0	.1453 +1 (−73)	.1545 +1 (−15)	.1568 +1
20	.8854 −1 (−178)	.1067 0 (−9)	.1077 0	.1482 +1 (−71)	.1564 +1 (−19)	.1595 +1
30	.5916 −1 (−205)	.7293 −1 (−20)	.7440 −1	.1512 +1 (−65)	.1575 +1 (−26)	.1617 +1
40	0.4440 −1 (−221)	0.5528 −1 (−30)	0.5701 −1	0.1526 +1 (−61)	0.1576 +1 (−30)	0.1625 +1
50	.3553 −1 (−233)	.4446 −1 (−40)	.4630 −1	.1535 +1 (−57)	.1576 +1 (−32)	.1629 +1
60	.2962 −1 (−241)	.3716 −1 (−48)	.3902 −1	.1541 +1 (−55)	.1576 +1 (−33)	.1630 +1
80	.2222 −1 (−253)	.2796 −1 (−60)	.2975 −1	.1549 +1 (−50)	.1575 +1 (−34)	.1630 +1
100	.1777 −1 (−261)	.2240 −1 (−69)	.2407 −1	.1553 +1 (−47)	.1574 +1 (−33)	.1629 +1

Table 4. Coefficient of the term $\frac{G}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m=0.8$

λ \ n	r_n (ε_{rn}) 1	r_n 2	r_n ∞	ϕ_n ($\varepsilon_{\phi n}$) 1	ϕ_n 2	ϕ_n ∞
0.7	0.9325 0 (−11)	0.9425 0 (+0)	0.9425 0	0.4406 0 (−2)	0.4415 0 (−0)	0.4416 0
1.0	.8754 0 (−20)	.8937 0 (+0)	.8935 0	.6059 0 (−4)	.6078 0 (−1)	.6082 0
2.0	.6759 0 (−62)	.7218 0 (+1)	.7208 0	.1030 +1 (−5)	.1032 +1 (−2)	.1034 +1
3.0	.5296 0 (−108)	.5959 0 (+4)	.5935 0	.1310 +1 (+5)	.1298 +1 (−4)	.1303 +1
4.0	.4338 0 (−149)	.5137 0 (+8)	.5098 0	.1511 +1 (+25)	.1466 +1 (−6)	.1474 +1
5	0.3700 0 (−183)	0.4581 0 (+12)	0.4527 0	0.1668 +1 (+53)	0.1571 +1 (−8)	0.1584 +1
10	.2399 0 (−224)	.3213 0 (+39)	.3093 0	.2148 +1 (+230)	.1707 +1 (−23)	.1747 +1
15	.2031 0 (−136)	.2521 0 (+73)	.2349 0	.2405 +1 (+395)	.1649 +1 (−44)	.1724 +1
20	.1881 0 (+14)	.2067 0 (+114)	.1855 0	.2563 +1 (+525)	.1564 +1 (−69)	.1681 +1
30	.1766 0 (+404)	.1520 0 (+209)	..1258 0	.2741 +1 (+691)	.1405 +1 (−133)	.1621 +1
40	0.1723 0 (+851)	0.1219 0 (+310)	0.9307 −1	0.2837 +1 (+774)	0.1272 +1 (−205)	0.1599 +1
50	.1703 0 (+1315)	.1035 0 (+407)	.7856 −1	.2896 +1 (+811)	.1158 +1 (−276)	.1599 +1
60	.1692 0 (+1773)	.9148 −1 (+499)	.6102 −1	.2936 +1 (+827)	.1059 +1 (−341)	.1607 +1
80	.1681 0 (+2647)	.7725 −1 (+676)	.4609 −1	.2987 +1 (+841)	.8982 0 (−446)	.1622 +1
100	.1676 0 (+3484)	.6950 −1 (+859)	.3738 −1	.3018 +1 (+852)	.7736 0 (−525)	.1629 +1

Table 5. Coefficient of the term $-\frac{hF}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m=1.2$

$\lambda \backslash n$	r_n (ε_{rn}) 1		2		∞		ϕ_n ($\varepsilon_{\phi n}$) 1		2		∞	
0.7	0.9779 (+3)	0	0.9750 (+0)	0	0.9748	0	$\bar{0}$.2107 (+4)	0	$\bar{0}$.2099 (+0)	0	$\bar{0}$.2098	0
1.0	.9564 (+6)	0	.9510 (+0)	0	.9507	0	-.2965 (+8)	0	-.2942 (+0)	0	-.2941	0
2.0	.8533 (+17)	0	.8401 (+1)	0	.8391	0	-.5485 (+28)	0	-.5343 (+1)	0	-.5335	0
3.0	.7372 (+23)	0	.7218 (+2)	0	.7204	0	-.7419 (+51)	0	-.7079 (+3)	0	-.7060	0
4.0	.6332 (+22)	0	.6211 (+3)	0	.6195	0	-.8851 (+72)	0	-.8290 (+4)	0	-.8258	0
5	0.5477 (+15)	0	0.5412 (+3)	0	0.5396	0	$\bar{0}$.9912 (+89)	0	$\bar{0}$.9146 (+5)	0	$\bar{0}$.9100	0
10	.3110 (−51)	0	.3291 (+4)	0	.3278	0	-.1255 (+131)	+1	-.1122 (+12)	+1	-.1109	+1
15	.2132 (−106)	0	.2393 (+3)	0	.2386		-.1356 (+133)	+1	-.1219 (+18)	+1	-.1197	+1
20	.1615 (−142)	0	.1883 (−0)	0	.1883	0	-.1409 (+125)	+1	-.1283 (+24)	+1	-.1252	+1
30	.1084 (−183)	0	.1314 (−11)	0	.1328	0	-.1462 (+109)	+1	-.1363 (+33)	+1	-.1319	+1
40	0.8155 (−206)	−1	0.1005 (−22)	0	0.1027	0	$\bar{0}$.1489 (+97)	+1	$\bar{0}$.1409 (+38)	+1	$\bar{0}$.1357	+1
50	.6531 (−221)	−1	.8115 (−33)	−1	.8388	−1	-.1505 (+88)	+1	-.1440 (+41)	+1	-.1383	+1
60	.5446 (−232)	−1	.6800 (−42)	−1	.7094	−1	-.1516 (+82)	+1	-.1460 (+42)	+1	-.1402	+1
80	.4087 (−247)	−1	.5129 (−55)	−1	.5430	−1	-.1530 (+72)	+1	-.1487 (+42)	+1	-.1428	+1
100	.3271 (−257)	−1	.4114 (−66)	−1	.4403	−1	-.1538 (+65)	+1	-.1504 (+41)	+1	-.1445	+1

Table 6. Coefficient of the term $\frac{G}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m = 1.2$

$\lambda \backslash n$	r_n (ε_{rn}) 1		2		∞		ϕ_n ($\varepsilon_{\phi n}$) 1		2		∞	
0.7	0.9785 (+6)	0	0.9726 (−0)	0	0.9726	0	$\bar{0}$.2463 (+4)	0	$\bar{0}$.2454 (+0)	0	$\bar{0}$.2454	0
1.0	.9576 (+12)	0	.9462 (−0)	0	.9463	0	-.3474 (+8)	0	-.3448 (+0)	0	-.3447	0
2.0	.8577 (+41)	0	.8235 (−1)	0	.8240	0	-.6501 (+28)	0	-.6334 (+1)	0	-.6325	0
3.0	.7457 (+77)	0	.6910 (−2)	0	.6924	0	-.8936 (+52)	0	-.8515 (+3)	0	-.8494	0
4.0	.6462 (+116)	0	.5761 (−5)	0	.5789	0	-.1086 (+77)	+1	-.1012 (+4)	+1	-.1008	+1
5	0.5651 (+159)	0	0.4835 (−9)	0	0.4878	0	$\bar{0}$.1241 (+103)	+1	$\bar{0}$.1130 (+5)	+1	$\bar{0}$.1125	+1
10	.3490 (+420)	0	.2330 (−52)	0	.2458	0	-.1726 (+246)	+1	-.1395 (+7)	+1	-.1385	+1
15	.2682 (+721)	0	.1361 (−127)	0	.1559	0	-.2008 (+417)	+1	-.1401 (−11)	+1	-.1417	+1
20	.2305 (+987)	0	.9239 (−203)	−1	.1160	0	-.2203 (+593)	+1	-.1283 (−72)	+1	-.1383	+1
30	.1980 (+1387)	0	.6146 (−259)	−1	.8297	−1	-.2453 (+852)	+1	-.9422 (−289)	0	-.1325	+1
40	0.1850 (+1741)	0	0.5409 (−199)	−1	0.6750	−1	$\bar{0}$.2604 (+968)	+1	$\bar{0}$.6785 (−487)	0	$\bar{0}$.1323	+1
50	.1787 (+2128)	0	.5226 (−85)	−1	.5712	−1	-.2702 (+1008)	+1	-.5152 (−617)	0	-.1346	+1
60	.1751 (+2555)	0	.5185 (+53)	−1	.4926	−1	-.2771 (+1017)	+1	-.4124 (−700)	0	-.1374	+1
80	.1715 (+3501)	0	.5188 (+362)	−1	.3810	−1	-.2860 (+1017)	+1	-.2944 (−792)	0	-.1418	+1
100	.1698 (+4511)	0	.5206 (+690)	−1	.3081	−1	-.2915 (+1018)	+1	-.2294 (−841)	0	-.1445	+1

Table 7. Coefficient of the term $-\frac{hF}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m=2.0$

λ \ n	r_n (ε_{rn}) 1	2	∞	ϕ_n ($\varepsilon_{\phi n}$) 1	2	∞
0.7	0.8854 0 (+6)	0.8803 0 (+0)	0.8800 0	$\bar{0}$.4834 0 (+8)	$\bar{0}$.4799 0 (+0)	$\bar{0}$.4797 0
1.0	.8000 0 (+10)	.7927 0 (+1)	.7922 0	-.6435 0 (+14)	-.6352 0 (+1)	-.6347 0
2.0	.5547 0 (+15)	.5473 0 (+1)	.5467 0	-.9828 0 (+35)	-.9518 0 (+2)	-.9499 0
3.0	.4061 0 (+10)	.4028 0 (+1)	.4022 0	-.1153 +1 (+52)	-.1099 +1 (+3)	-.1095 +1
4.0	.3162 0 (−0)	.3167 0 (+2)	.3162 0	-.1249 +1 (+67)	-.1176 +1 (+4)	-.1171 +1
5	0.2577 0 (−13)	0.2614 0 (+2)	0.2609 0	$\bar{0}$.1310 +1 (+79)	$\bar{0}$.1221 +1 (+5)	$\bar{0}$.1214 +1
10	.1322 0 (−80)	.1439 0 (+2)	.1436 0	-.1438 +1 (+109)	-.1312 +1 (+11)	-.1297 +1
15	.8854 −1 (−128)	.1015 0 (+0)	.1015 0	-.1482 +1 (+110)	-.1358 +1 (+17)	-.1335 +1
20	.6652 −1 (−158)	.7873 −1 (−4)	.7901 −1	-.1504 +1 (+105)	-.1393 +1 (+23)	-.1361 +1
30	.4440 −1 (−193)	.5423 −1 (−14)	.5501 −1	-.1526 +1 (+93)	-.1440 +1 (+31)	-.1396 +1
40	0.3331 −1 (−213)	0.4125 −1 (−25)	0.4232 −1	$\bar{0}$.1537 +1 (+84)	$\bar{0}$.1469 +1 (+36)	$\bar{0}$.1418 +1
50	.2666 −1 (−226)	.3323 −1 (−35)	.3445 −1	-.1544 +1 (+78)	-.1488 +1 (+38)	-.1433 +1
60	.2222 −1 (−236)	.2780 −1 (−44)	.2908 −1	-.1549 +1 (+73)	-.1501 +1 (+39)	-.1444 +1
80	.1666 −1 (−250)	.2094 −1 (−57)	.2221 −1	-.1554 +1 (+65)	-.1518 +1 (+40)	-.1460 +1
100	.1333 −1 (−259)	.1678 −1 (−67)	.1799 −1	-.1557 +1 (+59)	-.1528 +1 (+39)	-.1471 +1

Table 8. Coefficient of the term $\frac{G}{\kappa^2}\exp i(\alpha x+\theta t)$ for forced tidal waves expressed in a polar form $r_n \exp(i\phi_n)$, $m=2.0$

λ \ n	r_n (ε_{rn}) 1	2	∞	ϕ_n ($\varepsilon_{\phi n}$) 1	2	∞
0.7	0.8888 0 (+13)	0.8770 0 (−0)	0.8771 0	$\bar{0}$.5707 0 (+6)	$\bar{0}$.5673 0 (+0)	$\bar{0}$.5671 0
1.0	.8062 0 (+25)	.7866 0 (−0)	.7869 0	-.7679 0 (+11)	-.7599 0 (+1)	-.7593 0
2.0	.5718 0 (+74)	.5307 0 (−3)	.5323 0	-.1228 +1 (+26)	-.1200 +1 (+2)	-.1197 +1
3.0	.4338 0 (+144)	.3760 0 (−8)	.3791 0	-.1511 +1 (+35)	-.1467 +1 (+4)	-.1461 +1
4.0	.3536 0 (+239)	.2809 0 (−16)	.2855 0	-.1713 +1 (+38)	-.1661 +1 (+7)	-.1649 +1
5	0.3038 0 (+360)	0.2175 0 (−26)	0.2233 0	$\bar{0}$.1869 +1 (+39)	$\bar{0}$.1819 +1 (+11)	$\bar{0}$.1799 +1
10	.2116 0 (+1486)	.7442 −1 (−126)	.8510 −1	-.2334 +1 (+13)	-.2446 +1 (+61)	-.2305 +1
15	.1881 0 (+4051)	.2693 −1 (−277)	.3725 −1	-.2563 +1 (−14)	-.3242 +1 (+247)	-.2600 +1
20	.1791 0 (+10225)	.1968 −1 (+234)	.1596 −1	-.2695 +1 (−5)	-.4534 +1 (+674)	-.2709 +1
30	.1723 0 (+38204)	.3364 −1 (+6653)	.4395 −2	-.2837 +1 (+877)	-.5438 +1 (+2599)	-.1511 +1
40	0.1699 0 (+23023)	0.4125 −1 (+4833)	0.7071 −2	$\bar{0}$.2911 +1 (+1709)	$\bar{0}$.5691 +1 (+4296)	$\bar{0}$.1075 +1
50	.1687 0 (+20784)	.4514 −1 (+4828)	.7745 −2	-.2956 +1 (+1572)	-.5821 +1 (+4064)	-.1150 +1
60	.1681 0 (+21784)	.4735 −1 (+5418)	.7378 −2	-.2987 +1 (+1387)	-.5903 +1 (+3718)	-.1251 +1
80	.1675 0 (+27115)	.4962 −1 (+7329)	.5957 −2	-.3025 +1 (+1163)	-.6001 +1 (+3290)	-.1399 +1
100	.1672 0 (+34367)	.5069 −1 (+9723)	.4727 −2	-.3048 +1 (+1073)	-.6058 +1 (+3119)	-.1471 +1

Table 9. Coefficient of the term $\frac{Q}{h\kappa}\cos\frac{\theta p_n}{c}x\cdot\exp(i\theta t)$ for free tidal waves expressed in a polar form $r_n\exp(i\phi_n)$

λ \ n	r_n (ε_{rn})			ϕ_n ($\varepsilon_{\phi n}$)		
	1	2	∞	1	2	∞
0.7	0.6667 0 (−1)	0.6673 0 (−0)	0.6673 0	0.0000 0 (−1000)	0.1453 −1 (−63)	0.1550 −1
1.0	.6667 0 (−2)	.6679 0 (−0)	.6679 0	.0000 0 (−1000)	.2067 −1 (−63)	.2206 −1
2.0	.6667 0 (−7)	.6715 0 (−0)	.6715 0	.0000 0 (−1000)	.4037 −1 (−64)	.4314 −1
3.0	.6667 0 (−16)	.6772 0 (−0)	.6772 0	.0000 0 (−1000)	.5828 −1 (−67)	.6244 −1
4.0	.6667 0 (−26)	.6845 0 (−0)	.6845 0	.0000 0 (−1000)	.7383 −1 (−70)	.7940 −1
5	0.6667 0 (−38)	0.6930 0 (−0)	0.6930 0	0.0000 0 (−1000)	0.8674 −1 (−75)	0.9374 −1
10	.6667 0 (−98)	.7392 0 (−0)	.7394 0	.0000 0 (−1000)	.1158 0 (−114)	.1306 0
15	.6667 0 (−141)	.7742 0 (−2)	.7759 0	.0000 0 (−1000)	.1119 0 (−173)	.1352 0
20	.6667 0 (−168)	.7960 0 (−6)	.8008 0	.0000 0 (−1000)	.9967 −1 (−239)	.1310 0
30	.6667 0 (−199)	.8181 0 (−17)	.8319 0	.0000 0 (−1000)	.7677 −1 (−360)	.1199 0
40	0.6667 0 (−217)	0.8277 0 (−28)	0.8511 0	0.0000 0 (−1000)	0.6090 −1 (−452)	0.1110 0
50	.6667 0 (−229)	.8326 0 (−37)	.8649 0	.0000 0 (−1000)	.5006 −1 (−518)	.1040 0
60	.6667 0 (−238)	.8354 0 (−46)	.8753 0	.0000 0 (−1000)	.4235 −1 (−568)	.9813 −1
80	.6667 0 (−251)	.8383 0 (−59)	.8904 0	.0000 0 (−1000)	.3225 −1 (−638)	.8902 −1
100	.6667 0 (−260)	.8396 0 (−68)	.9011 0	.0000 0 (−1000)	.2598 −1 (−684)	.8215 −1

Table 10. Coefficient of the term $-i\frac{cS}{h\kappa}\sin\frac{\theta p_n}{c}x\cdot\exp(i\theta t)$ for free tidal waves expressed in a polar form $r_n\exp(i\phi_n)$

λ \ n	r_n (ε_{rn})			ϕ_n ($\varepsilon_{\phi n}$)		
	1	2	∞	1	2	∞
0.7	0.3823 0 (+7)	0.3798 0 (+0)	0.3797 0	0.4800 0 (−21)	0.4899 0 (−1)	0.4906 0
1.0	.4714 0 (+10)	.4670 0 (+1)	.4667 0	.3927 0 (−41)	.4087 0 (−3)	.4097 0
2.0	.5963 0 (+11)	.5902 0 (+1)	.5896 0	.2318 0 (−148)	.2695 0 (−9)	.2719 0
3.0	.6325 0 (+5)	.6304 0 (+1)	.6296 0	.1609 0 (−276)	.2183 0 (−18)	.2222 0
4.0	.6468 0 (−6)	.6516 0 (+1)	.6507 0	.1225 0 (−394)	.1969 0 (−26)	.2023 0
5	0.6537 0 (−19)	0.6671 0 (+1)	0.6662 0	0.9870 −1 (−491)	0.1871 0 (−35)	0.1939 0
10	.6634 0 (−85)	.7256 0 (+1)	.7246 0	.4983 −1 (−730)	.1699 0 (−79)	.1846 0
15	.6652 0 (−131)	.7653 0 (−0)	.7656 0	.3328 −1 (−808)	.1500 0 (−134)	.1732 0
20	.6658 0 (−161)	.7900 0 (−4)	.7932 0	.2498 −1 (−844)	.1292 0 (−195)	.1605 0
30	.6663 0 (−194)	.8149 0 (−15)	.8270 0	.1666 −1 (−881)	.9709 −1 (−309)	.1405 0
40	0.6665 0 (−214)	0.8258 0 (−26)	0.8477 0	0.1250 −1 (−901)	0.7636 −1 (−398)	0.1268 0
50	.6665 0 (−227)	.8313 0 (−36)	.8623 0	.9999 −2 (−914)	.6251 −1 (−465)	.1168 0
60	.6666 0 (−237)	.8345 0 (−44)	.8732 0	.8333 −2 (−924)	.5277 −1 (−516)	.1090 0
80	.6666 0 (−250)	.8377 0 (−58)	.8890 0	.6250 −2 (−936)	.4010 −1 (−588)	.9732 −1
100	.6666 0 (−259)	.8393 0 (−67)	.9000 0	.5000 −2 (−944)	.3227 −1 (−637)	.8888 −1

Table 11. Parameter p for free tidal waves expressed in a polar form $r_n \exp(i\phi_n)$

λ \ n	r_n (ε_{rn}) 1	2	∞	ϕ_n $(\varepsilon_{\phi n})$ 1	2	∞
0.7	0.1744 +1 (−8)	0.1757 +1 (−0)	0.1758 +1	$\bar{0}$.4800 0 (+10)	$\bar{0}$.4754 0 (+1)	$\bar{0}$.4751 0
1.0	.1414 +1 (−12)	.1430 +1 (−1)	.1431 +1	-.3927 0 (+13)	-.3880 0 (+1)	-.3876 0
2.0	.1118 +1 (−18)	.1138 +1 (−1)	.1139 +1	-.2318 0 (+13)	-.2291 0 (+1)	-.2288 0
3.0	.1054 +1 (−20)	.1074 +1 (−1)	.1076 +1	-.1609 0 (+7)	-.1600 0 (+2)	-.1598 0
4.0	.1031 +1 (−20)	.1051 +1 (−1)	.1052 +1	-.1225 0 (−3)	-.1231 0 (+2)	-.1229 0
5	0.1020 +1 (−20)	0.1039 +1 (−1)	0.1040 +1	$\bar{0}$.9870 −1 (−15)	$\bar{0}$.1004 0 (+2)	$\bar{0}$.1002 0
10	.1005 +1 (−15)	.1019 +1 (−2)	.1020 +1	-.4983 −1 (−77)	-.5416 −1 (+3)	-.5399 −1
15	.1002 +1 (−11)	.1012 +1 (−2)	.1013 +1	-.3328 −1 (−123)	-.3806 −1 (+2)	-.3797 −1
20	.1001 +1 (−8)	.1008 +1 (−2)	.1010 +1	-.2498 −1 (−153)	-.2949 −1 (−1)	-.2951 −1
30	.1001 +1 (−5)	.1004 +1 (−2)	.1006 +1	-.1666 −1 (−189)	-.2032 −1 (−11)	-.2053 −1
40	0.1000 +1 (−4)	0.1002 +1 (−2)	0.1004 +1	$\bar{0}$.1250 −1 (−209)	$\bar{0}$.1546 −1 (−22)	$\bar{0}$.1580 −1
50	.1000 +1 (−3)	.1002 +1 (−1)	.1003 +1	-.9999 −2 (−223)	-.1246 −1 (−32)	-.1287 −1
60	.1000 +1 (−2)	.1001 +1 (−1)	.1002 +1	-.8333 −2 (−233)	-.1042 −1 (−41)	-.1086 −1
80	.1000 +1 (−2)	.1001 +1 (−1)	.1002 +1	-.6250 −2 (−247)	-.7850 −2 (−54)	-.8302 −2
100	.1000 +1 (−1)	.1000 +1 (−1)	.1001 +1	-.5000 −2 (−257)	-.6293 −2 (−65)	-.6727 −2