Nonlinear Sampled-data Systems: A lifting framework

Yamamoto, Yutaka
Graduate School of Informatics, Kyoto University : Professor Emeritus

Yamamoto, Kaoru
Faculty of Information Science and Electrical Engineering, Kyushu University

KYUSHU UNIVERSITY

Nonlinear Sampled-data Systems—A lifting framework [★]

Yutaka Yamamoto [*] **Kaoru Yamamoto** [**]

[*] *Professor Emeritus, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (e-mail: yy@i.kyoto-u.ac.jp)*
[**] *Faculty of Information Science and Electrical Engineering, Kyushu University, Fukuoka 819-0395, Japan (e-mail: yamamoto@ees.kyushu-u.ac.jp)*

Abstract: This short note gives a new framework for dealing with nonlinear sampled-data systems. We introduce a new idea of lifting, which is well known for linear systems, but not successfully generalized to nonlinear systems. This paper introduces a new lifting technique for nonlinear, time-invariant systems, which are different from the linear counterpart as developed in Bamieh et al. (1991); Yamamoto (1994) etc. The main difficulty is that the direct feedthrough term effective in the linear case cannot be generalized to the nonlinear case. Instead, we will further lift the state trajectory, and obtain an equivalent time-invariant discrete-time system with function-space input and output spaces. The basic framework, as well as the closed-loop equation with discrete-time controller, is given. As an application of this framework, we give a representation for the Koopman operator derived from the given original nonlinear system.

Keywords: Time-invariant nonlinear system, nonlinear lifting, closed-loop equation, the Koopman operator *AMS subject classification*: 46F10, 46J15

1. INTRODUCTION

The lifting technique (Bamieh et al. (1991); Yamamoto (1994)), etc., has played a crucial role in modernizing the treatment of linear sampled-data control systems, e.g., (Yamamoto (1999); Yamamoto and Nagahara (2018)). To see the basic idea, let us take the following linear, time-invariant system:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{1}$$

where x, u, y are the state, input, and output, respectively, and A, B, C are constant matrices of suitable dimensions.

The *lifting* converts this system to a linear, time-invariant, *discrete-time* system as follows.

Fix an arbitrary $T > 0$ (which may later work as a sampling period), and let $\mathscr{L}$ be the *lifting* operator defined by

$$\mathscr{L} : \psi \mapsto \{\psi_k(\cdot)\}, \psi_k(\theta) := \psi(kT + \theta),$$

for $0 \le \theta < T$ and ψ belonging to a suitably defined function space on $[0, \infty)$, e.g., L^2 or $L^2_{loc}[0, \infty)$, etc. The lifting employed in Bamieh et al. (1991) gives

$$x[k+1] = e^{AT}x[k] + \int_0^T e^{A(T-\tau)}Bu[k](\tau)d\tau,$$
$$y[k](\theta) = Ce^{A\theta}x[k] + \int_0^\theta Ce^{A(\theta-\tau)}Bu[k](\tau)d\tau. \tag{2}$$

where $x[k]$ denote the state $x(t)$ at time kT. These formulas now take the form

$$x[k+1] = \mathscr{A}x[k] + \mathscr{B}u[k]$$
$$y[k] = \mathscr{C}x[k] + \mathscr{D}u[k]. \tag{3}$$

Observe that the operators $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ do not depend on the time variable k, and hence the above is a time-invariant discrete-time system, where $u[k]$ and $y[k]$ belong to a suitable function space, say, $L^2[0, T]$.

A crucial point here is the direct feedthrough term $\mathscr{D}$, which does not exist in the original continuous-time system. This term describes the effect of the input to the output, which, in principle, should depend on the state evolution in the mean time. However, the linearity of the original system makes it possible to describe this effect without involving the state transition. When we deal with a nonlinear system, this is clearly impossible, and this is the topic of the present article.

Nonlinear sampled-data systems have been studied in several different contexts. Heemels et al. (2015) studies the $\mathscr{L}_2$ gain analysis for a class of event-triggered switched systems. As stated in Nešić and Teel (2007), there are roughly two ways to design a controller for nonlinear sampled-data systems. One is that of designing a continuous-time controller and then discretizing it. The other is to first discretize the nonlinear plant, and then invoke a discrete-time design procedure. For the second approach, what discretization process should be employed and what descrete-time design method should be used are the issues. Then how a stability or other system properties can be guaranteed become our main concern. Nešić and Teel (2007) discusses the classical Euler one-step discretization, and then gives some convergence results on stability preservation.

Zaccarian et al. (2003) has shown that finite gain $\mathscr{L}_p$ stability can be inherited from a discretized system to the original sampled-data system in consideration.

The development of nonlinear lifting has the potential of enabling direct manipulation of intersample behavior and provide new insights into such nonlinear sampled-data systems.

2. NONLINEAR SYSTEM LIFTING

Suppose we are given the following continuous-time nonlinear system:

$$\frac{d}{dt}x(t) = f(x(t), u(t)),$$
$$y(t) = h(x(t)), \tag{4}$$
$$x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n$ and we assume a suitable condition for f to allow for the existence and uniqueness for the solution to exist, for example, the Lipschitz condition with a global constant not to allow a finite escape time. We also assume a suitable regularity condition for h. The initial condition is $x(0) = x_0$.

Let $\Phi(t, x_0, u)$ denote the solution $x(t)$ at time t of the above differential equation.

Let $T > 0$ be fixed; this will be taken as a sampling period later.

We then define the following nonlinear discrete-time system:

$$x[k+1](\theta) = \Phi(k+\theta, x[k](T), u[k+1](\cdot))$$
$$y[k](\theta) = h(x[k](\theta)). \tag{5}$$

Here, we mean by $u[k+1](\cdot)$ that $x[k+1](\theta)$ depends on $u[k+1](\tau)$, $0 \leq \tau \leq \theta$. We call this system the *lifting* of the system (4). Note that it does not satisfy the strict causality condition in that $u[k+1]$ appears on the right-hand side.

Remark 2.1. To be precise, (5) requires a little more care. When lifted, $x[k](\theta)$ is defined only for $0 \leq \theta < T$, and hence not for T. To circumvent this, we should require that the trajectory $x(t)$ of (4) is continuous, and interpret that $x[k](T)$ as $\lim_{\theta \uparrow T} x[k](\theta)$. Also, the point evaluation $h(x[k](\theta))$ at $t = kT + \theta$ is not necessarily well defined. To take care of this, we should assume that h is defined on a dense $\mathscr{D}(h)$ subspace of the state space, and the mapping

$$h : \mathscr{D}(h) \to \mathbb{C} \tag{6}$$

gives rise to a continuous mapping

$$\tilde{h} : \mathscr{D}(h) \to L^2_{loc}[0, \infty) : x \mapsto h(x(t)). \tag{7}$$

We will write $h(x(t))$ as a shorthand for $\tilde{h}(x)(t)$. By the assumed continuity, this mapping $\tilde{h}$ can be extended continuously to the whole state space $\mathbb{R}^n$. For this extension, see Yamamoto (1981), for details.

Remark 2.2. As is well recognized (e.g., Neşić and Teel (2007); Neşić and Postoyan (2015)), it is generally not possible to obtain an explict solution formula for system (4). For example, even for the rare case where an explicit solution can be derived, the input to state transition still cannot be explicitly captured. Take, for example, the cubic nonlinear equation

$$\dot{x} = -x - x^3 + u. \tag{8}$$

Suppose for the moment $U \equiv 0$. Then this admits an explicit solution Mauroy et al. (2020):

$$x(t) = \frac{te^{-t}}{\sqrt{1 + x^2 - x^2 e^{-2t}}} \tag{9}$$

with initial condition $x(0) = x$. However, if u is nonzero, it is difficult to derive a closed form for $\Phi(t, x, u)$ (8).

We also note that lifting has been utilized to give an $\mathscr{L}_2$-gain analysis for a class of switched sampled-data systems Heemels

et al. (2015). But note that the equation itself studied there is of linear nature, and different from the present case.

To see the difference from the usual linear case, consider the following example.

Example 2.3. Consider the linear system (1). Then (1) takes the following form:

$$x[k+1](\theta) = e^{A\theta} x[k](T) + \int_0^{\theta} e^{A(\theta-\tau)} Bu[k+1](\tau) d\tau$$
$$y[k](\theta) = Cx[k](\theta).$$

Note that this looks very different from (2) in that the state $x(t)$ is also lifted. This is the lifting employed in Yamamoto (1994). In this linear case, one can make it strictly causal by introducing the new state variable $x[k] - \mathscr{B}u[k]$. But this technique cannot be applied to the nonlinear case considered here.

3. FAST-SAMPLING APPROXIMATION

The difference of the new nonlinear lifting (5) from the linear one is that we cannot have a closed-form expression as (3). This is inevitable at the abstract level, but one can instead introduce a fast-sampling approximation formula as in the case with the linear case.

The idea is similar to the linear case. We introduce the subdivision of the sampling interval $[0, T]$ at each sampling period as $[0, T/N], [T/N, 2T/N], \ldots, [T - T/N, T]$. Let us temporarily write $x[T/N], x[2T/N]$, etc. for $x(t)|_{t=T/N}, x(t)|_{t=2T/N}$, and so forth, to indicate their discrete-time values, although T/N, $2T/N$ etc. do not take on integer values. In view of the differential equation (4), we have

$$x[T/N] - x_0 = \int_0^{T/N} f(x(\tau), u[0](\tau)) d\tau.$$

One can invoke a suitable approximation for this equation, e.g., the Runge-Kutta method, or even forward-difference approximation [1]. Repeating this process inductively, we obtain an approximant of $x[T/N], x[2T/N], \ldots, x[T]$ in (5), which gives an approximation of $\Phi(\cdot, x_0, u)$ in (5). It can be used as a substitute of the lifted nonlinear system (5). Note here that we do not require the actual sampling of the trajectory, but it is only an artificial approximation of the lifted system.

4. CLOSED-LOOP EQUATION

Consider the unity feedback sampled-data control system in Fig. 4. The plant P is the nonlinear continuous-time system

$$\dot{x} = f(x, u)$$
$$y(t) = h(x). \tag{10}$$

The controller C is a discrete-time system

$$z[k+1] = \phi(z[k], e[k](0))$$
$$v[k] = \psi(z[k]),$$

where $z[k], e[k], v[k]$ belong to $\mathbb{R}^{n_c}, \mathbb{R}^{m_c}, \mathbb{R}^{p_c}$, respectively.

As before, (10) is lifted as (5):

$$x[k+1](\theta) = \Phi(k+\theta, x[k](T), u[k+1](\cdot))$$
$$y[k](\theta) = h(x[k](\theta)).$$

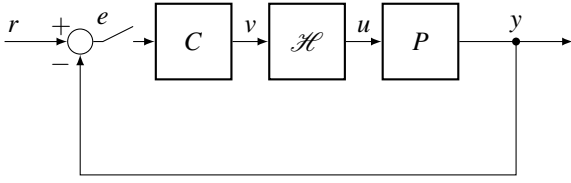[1] For some higher-order formulas, see, for example, Grüne and Kloeden (2001).

Fig. 1. Unity feedback nonlinear sampled-data system

Equating $u[k+1](\cdot) = v[k]$, considering the delay induced by the hold element $\mathscr{H}$, we obtain the closed-loop equation

$$
\begin{aligned}
x[k+1](\theta) &= \Phi(k+\theta, x[k](T), v[k]) \\
z[k+1] &= \phi(z[k], e[k](0)) \\
v[k] &= \psi(z[k]) \\
e[k](\theta) &= r[k](\theta) - y[k](\theta).
\end{aligned}
\tag{11}
$$

In the above equation, the lack of strict causality disappears due to the delay induced by the hold element $\mathscr{H}$.

5. THE KOOPMAN OPERATOR

The Koopman operator (Koopman (1931)) has been actively studied for nonlinear systems; see, for example, Mauroy et al. (2020). The crux of the idea is in introducing the time-transition of the *observables*. That is, considering the adjoint (dual) system of the original system. While the original system may be nonlinear, the transition of the adjoint system turns out to be linear, and this is a great advantage of the Koopman operator.

5.1 Some backgrounds

While the study of the Koopman operator has become popular relatively recently, the very fact that the introduction of a duality for a nonlinear object has been long known. Let us note some of this background.

R. E. Kalman had discussed a lot about dual systems, particularly duality between reachability (controllability) and observability (Kalman (1968); Kalman et al. (1969)), and this idea was further explored in the context of realization theory; see, e.g., Sontag (1979, 1998); Yamamoto (1981). The system derived from the evolution of observables is called the dual (adjoint) system, and this is precisely the Koopman operator idea. Various observability notions can be translated into some system properties (e.g., reachability) by going into duals. For example, in Sontag (1979), a class of systems in an algebraic category is studied; a system is *algebraically observable* if the algebra generated by the observables agrees with the algebra of polynomials. It is algebraically observable if and only if its adjoint system satisfies a strong reachability condition. In the context of continuous-time linear systems, a system is *topologically observable* if and only if its dual system is exactly reachable (Yamamoto (1981)). These recognitions date back to the intuition shown in R. E. Kalman's idea given in Kalman (1968) where a finite-dimensional linear system is reachable if and only if its dual (or adjoint) system is observable, and vice versa.

5.2 The Koopman operator for nonlinear systems

We now proceed to consider the system (4) without input u for simplicity:

$$
\begin{aligned}
\frac{d}{dt}x(t) &= f(x(t)), \\
y(t) &= h(x(t)), \\
x(0) &= x_0.
\end{aligned}
\tag{12}
$$

We will see the correspondence of spectra between the original system and its lifted counterpart. Under this assumption, we denote $\Phi(t, x, 0)$ by $\Phi(t)(x)$. $\Phi(t)$ satisfies the semigroup property: $\Phi(t+s) = \Phi(t)\Phi(s)$, $\Phi(0) = I$, where the product is understood as the composition.

The Koopman operator for (12) is defined by

$$
(U(t)h)(x) := h(\Phi(t)(x)).
\tag{13}
$$

We may also write $\langle U(t)h, x \rangle$ for $(U(t)h)(x)$ in view of the linearity of $U(t)$ (Mauroy et al. (2020)). Then (13) can be written as $\langle U(t)h, x \rangle = h(\Phi(t)(x))$.

The lifted version of (12) is given by

$$
\begin{aligned}
x[k+1](\cdot) &= \Phi(T, x[k]) = \Phi(T)(x[k]) \\
y[k] &= h(x[k]).
\end{aligned}
\tag{14}
$$

Then the discrete-time Koopman operator U_d corresponding to the lifted system (14) is defined as

$$
U_d(h)(x) := h(\Phi(T)(x)).
\tag{15}
$$

It is clear that $U_d(h) = U(T)(h)$.

Example 5.1. Let us consider the simple case of a continuous-time linear system. Consider the linear differential equation given in a Banach space X:

$$
\frac{d}{dt}x(t) = Ax(t), x \in X
\tag{16}
$$

and an observation operator

$$
H : X \to \mathbb{C} : x \mapsto H(x).
$$

We here assume the following:

(1) X is a reflexive Banach space;
(2) A gives rise to a strongly continuous semigroup $S(t)$; that is, $S(t) = e^{At}$.
(3) For simplicity, we assume H to be a continuous linear functional: $X \to \mathbb{C}$.

Since X is reflexive, there exists a dual semigroup $S'(t)$ which satisfies

$$
\langle S'(t)H, x \rangle = \langle H, S(t)x \rangle
$$

For every $H \in X', x \in X$. This is nothing but the definition of the Koopman operator (13), and hence $S'(t) = U(t)$. If we denote the infinitesimal generator (see Subsection 5.3 below) of $S(t)$ by A, then the generator of $U(t)$ is given by A' or A^*, the adjoint operator of A.

Naturally, in this case, the lifted Koopman operator U_d agrees with $S'(T)$.

Remark 5.2. It is possible to weaken the third requirement on H above that it is defined on a dense subspace $D(H) \subset X$, and the mapping

$$
D(H) \to \mathscr{F} : x \mapsto HS(t)x
$$

is extensible to a continuous linear map from X to $\mathscr{F}$, where $\mathscr{F}$ is a suitably defined function space on $[0, \infty)$. In this case, the definition of the Koopman operator needs to be modified a little, but the formality works mutatis mutandis. See, e.g., Yamamoto (1981) for such an extension.

5.3 Spectrum of the Koopman operator

Let us assume that $U(t)$ is strongly continuous. The infinitesimal generator of the semigroup $U(t)$ is then defined by

$$L := \lim_{t \to 0} \frac{U(t) - I}{t}.$$

It is important to realize that $U(t)$, and hence L also is a linear operator despite the fact that f is nonlinear.

Let us calculate Lh, where h is an observable. By the chain rule of differentiation, we have

$$\langle \frac{1}{t}(U(t)h - h), x \rangle = \frac{1}{t}(h(\Phi(t)x - h(x)) \to \nabla h \dot{x} = \nabla h f(x).$$

This should hold for any x, and hence $Lh = \nabla h f$. (We will see below a more explicit expression when the system is linear.)

On the other hand, since L is the infinitesimal generator of the strongly continuous semigroup $U(t)$, we may write $U(t) = e^{Lt}$ for $t \geq 0$. Since $U(h)$ is equal to $(U(t)h)_{t=T}$ as noted above, we have that $U_d(h) = e^{LT}h$.

Now let us compare the eigenvalues of the Koopman operator (13) and its lifted version (15). We have the following proposition:

Proposition 5.3. Let λ be an eigenvalue of L with a corresponding eigenfunction ϕ:

$$L\phi = \lambda\phi.$$

Then

$$U(t)\phi = e^{\lambda t}\phi. \tag{17}$$

In particular, the Koopman operator U_d of the lifted system (5) has an eigenvalue $e^{\lambda T}$.

Proof Since $U(t) = e^{Lt}$, (17) follows from the spectral mapping theorem (Yosida, 1978, Chapter IX). The second claim is obvious because $U_d = U(T)$. □

Example 5.4. Let us return to the linear case Example 5.1. Let us assume that $\dim X = n < \infty$ for simplicity. Then the state transition derived by (16) is simply

$$x(t) = e^{At}x$$

for the initial state x. Then the Koopman operator $U(t)$ must be its dual, and hence is given by

$$U(t) = e^{A^\top t},$$

where $A^\top$ denotes the transpose of A. If λ is an eigenvalue of A, then it is also an eigenvalue of $A^\top$, and hence

$$U(t)\phi = e^{\lambda t}\phi$$

for some ϕ as expected. In particular, this implies $e^{\lambda T}$ is an eigenvalue of the lifted system (5).

This correspondence can be viewed from a more general viewpoint.

Let us view the above correspondence from a different angle. The infinitesimal generator L of $U(t)$ is given by $Lh = \nabla h f$. In the present case, $\nabla h = h$, $f = A$. Noting

$$\langle Lh, x \rangle = \langle h, Ax \rangle = \langle A^\top h, x \rangle,$$

for every $x \in X$, we have $L = A^\top$ again, as expected.

5.4 Eigenfunctions

Eivenfunctions play an important roles in the Koopman mode expansion. Take the linear equation (16), and let λ be an eigenvalue of A with associated eigenfunction ϕ. The associated Koopman operator is $e^{A^\top t}$, along with the same eigenvalue and left eigenfunction $\phi^\top$. That is,

$$\phi^\top A^\top = \lambda \phi^\top.$$

This readily yields

$$\phi^\top U(t) = \phi^\top e^{A^\top t} = e^{\lambda t}\phi^\top.$$

In particular, for the lifted system,

$$\phi^\top U_d = \phi^\top U(T) = \phi^\top e^{A^\top T} = e^{\lambda T}\phi^\top.$$

6. CONCLUDING REMARKS

We have introduced a lifting framework for time-invariant nonlinear systems. This can have impacts on the study of nonlinear sampled-data systems, especially when one needs to discuss intersampling behavior. For some pertinent studies, see, e.g., Nešić and Teel (2007) and references therein. While the introduced lifted system (5) is not strictly causal as a discrete-time system, this drawback can be surpassed by considering its connection with a discrete-time controller (11) as seen in Section 4.

We have also given a straightforward calculation for the Koopman operators for the system and the corresponding lifted system, where the input term is absent. The generalization to the case with the input term is attempted by viewing u as part of the state: see, e.g., Mauroy et al. (2020); Brunton and Kutz (2019).

Different discretization schemes have been addressed by Barbot et al. (1991), e.g., singular perturbation, fast-sampling approximation, etc.; see also Castillo et al. (1997) for a treatment of regulation under the first-order approxiation.

ACKNOWLEDGEMENT

REFERENCES

Bamieh, B., Pearson, J.B., Francis, B.A., and Tannenbaum, A. (1991). A lifting technique for linear periodic systems with applications to sampled-data control. *Syst. Control Lett.*, 17(2), 79–88.

Barbot, J.P., Monaco, S., Normmand-Cyrot, D., and Pantalos, N. (1991). Discretization schemes for nonlinear singularly perturbed systems. In *Proc. 48th IEEE CDC*, 443–448. IEEE.

Brunton, S.L. and Kutz, J.N. (2019). *Data Driven Science and Engineering—Machine Learning, Dynamical Systems, and Control*. Cambridge University Press, Cambridge, UK.

Castillo, B., Di Gennaro, S., Monaco, S., and Normmand-Cyrot, D. (1997). On regulation under sampling. *IEEE Trans. Autom. Control*, 42(6), 864–868.

Grüne, L. and Kloeden, P.E. (2001). Higher order numerical schemes for affinely controlled nonlinear systems. *Numerische Mathematik*, 89, 669–690.

Heemels, W.P.M.H., Dullerud, G.E., and Teel, A.R. (2015). A lifting approach to $\mathscr{L}_2$-gain analysis of periodic event-triggered and switching sampled-data control systems. In *Proc. 54th IEEE CDC*, 4176–4182. IEEE.

Kalman, R.E. (1968). *Lecture on Controllability and Observability*. CIME Summer Course, Roma.

Kalman, R.E., Falb, P.L., and Arbib, M.A. (1969). *Topics in Mathematical System Theory*. McGraw-Hill, New York.

Koopman, B.O. (1931). Hamiltonian systems and transformations in hilbert space. *SIAM J. Aplied Dynamical Systems*, 17–1, 315–318.

Mauroy, A., Mezić, I., and Susuki, Yoshihiko, E. (2020). *The Koopman Operator in Systems and Control*. Springer, New York.

Neşić, D. and Postoyan, R. (2015). Nonlinear sampled-data systems. In *Encyclopedia of Systems and Control, Ed. Baillieul, J., Samad, T.* Springer-Verlag, Berlin.

Neşić, D. and Teel, A.R. (2007). Sampled-data control of nonlinear systems: an overview of recent results. In *Perspectives in Robust Control; Reza Moheimani Ed., Lecture Notes in Control and Information*, volume 268, 221–239. Springer-Verlag, Berlin.

Sontag, E.D. (1979). *Polynomial Response Maps*. Springer-Verlag, Berlin.

Sontag, E.D. (1998). *Mathematical Control Theory*. Springer-Verlag, Berlin.

Yamamoto, Y. (1981). Realization theory of infinite-dimensional linear systems, parts I and II. *Math. Syst. Theory*, 15, 55–77 and 169–190.

Yamamoto, Y. (1994). A function space approach to sampled-data control systems and tracking problems. *IEEE Trans. Autom. Control*, AC-39(4), 703–713.

Yamamoto, Y. (1999). Digital control. In *Wiley Encyclopedia of Electrical and Electronics Engineering*, volume 5, 445–457. John Wiley.

Yamamoto, Y. and Nagahara, M. (2018). Digital control. In *Wiley Encyclopedia of Electrical and Electronics Engineering, J. Webster Ed.*, 1–19. John Wiley. doi:DOI: 10.1002/047134608X.W1010.pub2.

Yosida, K. (1978). *Functional Analysis*. Springer, New York, fifth edition.

Zaccarian, L., Teel, A.R., and Neşić, D. (2003). On finite gain $\mathscr{L}_p$ stability of nonlinear sampled-daa systems. *Systems and Control Letters*, 49, 669–690.