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ON THE DYNAMICAL STRESS DISTRIBUTION OF STIFFENED PLATE DURING VIBRATION

By

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Analysing the causes of the crack damage of stiffened bulkhead of ship structure, the dynamical stress distributions of the stiffened plate at vibration are investigated theoretically and experimentally. The bending stresses at the intersection of the plate and stiffener are pretty high, and eigen-values are nearly equal to the ones of the plate fixed at this edge. The effective breadth of the vibrating stiffened plate is generally equal to the Schade's formulas of the case subjected to the sinusoidal load statically.

1. Introduction

We have had some reports concerning with the crack damage on the stiffened bulkhead in the neighbourhood of the joint stiffeners. One of the causes of these damages is the fatigue failure due to the vibration of the plate accompanied by the torsional and bending vibration of the stiffeners. To make these causes clear, it is necessary to investigate the fatigue limit of the material of the part welded the stiffener, the natural frequencies and dynamical stress distribution of stiffened plate at vibration.

We have many data concerning with the fatigue limit of the structural materials, but few investigations into the fatigue failure of the stiffened plate caused by vibration. R. Weck [1] made an experiment using the plate stiffened by the two inverted angle to determine the effect on fatigue endurance of different types of joints between plating and stiffeners, and the effect on fatigue endurance of different types joints in stiffeners. For the case with continuous fillet welds attaching the stiffeners to the plating, he showed the stress range was 12 kg/mm^2, number of cycles to failure was 9×10^6, and failed in consequence of the transverse bending of the plate by the crack running along the edge of the continuous fillet weld. But in order to investigate the fatigue failure of stiffened plate, it is more necessary to consider the stress concentration at the welded part, the change of quality of material caused by welding and the effect of corrosion.

Meanwhile, a number of calculation methods of the eigen-values of stiffened plate have been investigated by various investigators [2)~13)], and some of them are in agreement with the experimental results. The eigen-values of coupled bending and torsional vibrations of stiffener have also been studied

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by H. R. Lembecke [14], J. M. Gere [15], Y. K. Lin [16], Wai Keung Tso [17] and K. H. Schrader [18]. R. F. Beresford and D. J. Mead [19] have analysed the natural frequencies of an angle section attached to the rigid wall involving cross-sectional distortion. However, we have little studies on the dynamical stress distribution of the stiffened plate at vibration, only C. A. Mercer and C. Seavey [20] have studied the bending moment distributions of the vibrating stiffened plate using the transfer matrix method. The effective breadth of the vibrating stiffened plate has been approximately calculated by F. H. Schroeder [21].

It is usual in calculating the bending stiffness of the stiffeners in the above-mentioned various calculation methods of natural frequencies of stiffened plate to assume some effective breadth of plate, but effective breadth at vibration is not yet elucidated, because it is difficult to analyse exactly that the stress distributions of vibrating stiffened plate are complicated.

The aim of this paper is to make clear the distributions of the dynamical bending stress and plane stress of vibrating stiffened plate, and to analyse the causes of the crack damage of stiffened bulkhead.

For this purpose, the basic equation for vibration of stiffened plate is introduced as a stiffness matrix form by combining the plane stress theory and the equation for the lateral vibration of the plate.

Besides, a consideration on effective breadth of vibrating stiffened plate is described.

2 Assumptions

i) The plate is stiffened in one direction, and the two edges which are perpendicular to the stiffeners are supported freely.

ii) Though both the ends of the stiffeners are restrained for torsional rotation, they are free to warp.

iii) Since the natural frequencies of the longitudinal vibration of stiffeners are extremely higher than those of the lateral vibration, the effect of the longitudinal vibration of stiffeners is neglected.

iv) The cross-sectional distortion of stiffeners is not involved.

v) The tagential component of inertia forces of the plate is so small compared with the normal ones, that it is neglected. The plane stresses of the plate can be assumed as the statical two dimensional problems.

Notation

x, y, z : Rectangular co-ordinates
L : Length of plate or stiffener
B : Frame space
$\bar{B}$: Effective breadth
E : Young's modulus of elasticity
ν : Poisson's ratio

t_p : Thickness of plate
t_s : Thickness of web plate of stiffener

$D=\dfrac{Et_p^3}{12(1-\nu^2)}$ Flexural rigidity of plate

$\theta=\bar{\theta}\sin\dfrac{n\pi}{L}x\ e^{i\omega t}$ Rotation around x axis

$$\left.\begin{aligned} u&=\bar{u}\cos\frac{n\pi}{L}x\ e^{i\omega t}\\ v&=\bar{v}\sin\frac{n\pi}{L}x\ e^{i\omega t}\\ w&=\bar{w}\sin\frac{n\pi}{L}x\ e^{i\omega t}\end{aligned}\right\}\ \text{Displacements in direction of } x,\ y \text{ and } z\text{-axes}$$

$\lambda=\dfrac{n\pi B}{L}$

$\kappa=\left(\dfrac{\gamma t_p\omega^2}{gD}\right)^{\frac{1}{4}}\cdot B$ Eigen-value

ω : Circular frequency
γ : Weight of material of plate or stiffener per unit volume

$\mu=\dfrac{1+\nu}{2\lambda}$

a, b, c, d, e, g : Function of λ
$a'. b', c', d', e', g'$, : Function of λ and κ
M_y : Bending moment per unit length of section of plate perpendicular to y axis
V_y : Force per unit length of section of plate perpendicular to y axis

$$\left.\begin{aligned} q_{Ax}&=\bar{q}_{Ax}\cos\frac{n\pi}{L}x\ e^{i\omega t}\\ q_{Ay}&=\bar{q}_{Ay}\sin\frac{n\pi}{L}x\ e^{i\omega t}\\ q_{Az}&=\bar{q}_{Az}\sin\frac{n\pi}{L}x\ e^{i\omega t}\end{aligned}\right\}\ \text{Forces per unit length of the intersection of stiffener and plate in direction of } x,\ y \text{ and } z \text{ axes}$$

$m_{Ax}=\bar{m}_{Ax}\sin\dfrac{n\pi}{L}x\ e^{i\omega t}$ Moment per unit length of the intersection of stiffener and plate parallel to x axis

A_s : Sectional area of stiffener
m_{st} : Mass per unit length of stiffener
$\{\bar{P}_p\}=\{\bar{s}_{jk}\ \bar{s}_{kj}\ \bar{p}_{jk}\ \bar{p}_{kj}\}$
$\{\bar{U}_p\}=\{\bar{u}_j\ \bar{u}_k\ \bar{v}_j\ \bar{v}_k\}$
$\{\bar{P}_b\}=\{\bar{V}_{yjk}\ \bar{V}_{ykj}\ \bar{M}_{yjk}\ \bar{M}_{ykj}\}$
$\{\bar{U}_b\}=\{\bar{w}_j\ \bar{w}_k\ \bar{\theta}_j\ \bar{\theta}_k\}$
Subscripts and Superscript :
j, k : j-and k-side of plate
A : Intersection of stiffener and plate

p : State of plane stress of plate
b : Bending of plate
- : Amplitude of vibrating displacement or force

3 The dynamical equilibrium equations at the mth intersection of plate and stiffener

Separating the stiffener and plate at the mth intersection in Fig. 3.1, we find the slender bar of the rectangular cross-section $t_p \times t_s$. Provided that the thickness of plate and stiffener web is small as compared with other dimensions of the stiffened plate, we can consider the dynamical equilibrium equation with respect to this slender bar as follows*

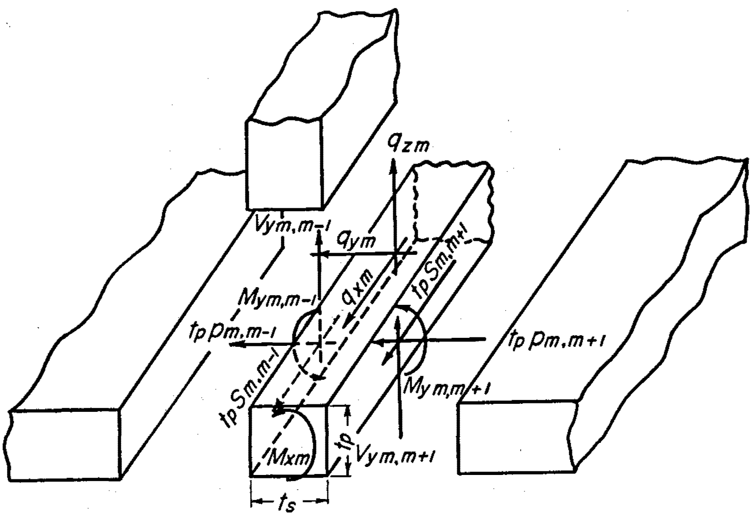


Fig. 3.1

$$\begin{bmatrix} t_p s \\ t_p p \\ V_y \\ M_y \end{bmatrix}_{m,m-1} + \begin{bmatrix} t_p s \\ t_p p \\ V_y \\ M_y \end{bmatrix}_{m,m+1} + \begin{bmatrix} q_{Ax} \\ q_{Ay} \\ q_{Az} \\ m_{Ax} \end{bmatrix}_{m} = 0 \qquad (3.1)$$

where s and p are shearing stress and normal stress, and V_y and M_y are reactive force and bending moment per unit length acting on the edge of the separated plate, respectively. q_{Ax}, q_{Ay}, q_{Az} and m_{Ax} are the forces and torque per unit length distributed along the intersection.

* On the same assumption the statical analysis of stiffened plate have been studied by T. Yanagimoto [22], E. Reissner [23], M. Holmes [24], H. Vaughan [25] and S. Shimizu [26].

If the ends of the stiffened plate are simply supported, we can take these forces in the form

$$\left.\begin{aligned} s &= \bar{s}\cos\frac{n\pi}{L}x\, e^{i\omega t} \\ p &= \bar{p}\sin\frac{n\pi}{L}x\, e^{i\omega t} \\ V_y &= \bar{V}_y\sin\frac{n\pi}{L}x\, e^{i\omega t} \\ M_y &= \bar{M}_y\sin\frac{n\pi}{L}x\, e^{i\omega t} \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} q_{Ax} &= \bar{q}_{Ax}\cos\frac{n\pi}{L}x\, e^{i\omega t} \\ q_{Ay} &= \bar{q}_{Ay}\sin\frac{n\pi}{L}x\, e^{i\omega t} \\ q_{Az} &= \bar{q}_{Az}\sin\frac{n\pi}{L}x\, e^{i\omega t} \\ m_{Ax} &= \bar{m}_{Ax}\sin\frac{n\pi}{L}x\, e^{i\omega t} \end{aligned}\right\}. \tag{3.3}$$

Substituting for the expressions (3.2) and (3.3) in Eq. (3.1), we obtain

$$\begin{bmatrix} \bar{s} \\ \bar{p} \\ \bar{V}_y \\ \bar{M}_y \end{bmatrix}_{m,m-1} + \begin{bmatrix} \bar{s} \\ \bar{p} \\ \bar{V}_y \\ \bar{M}_y \end{bmatrix}_{m,m+1} = -\begin{bmatrix} \bar{q}_{Ax}/t_p \\ \bar{q}_{Ay}/t_p \\ \bar{q}_{Az} \\ \bar{m}_{Ax} \end{bmatrix}_m \tag{3.4}$$

or

$$\begin{bmatrix} \{\bar{P}_p\}_{kj} \\ \{\bar{P}_b\}_{kj} \end{bmatrix}_{m,m-1} + \begin{bmatrix} \{\bar{P}_p\}_{jk} \\ \{\bar{P}_b\}_{jk} \end{bmatrix}_{m,m+1} = -\{\bar{Q}_s\}_m \tag{3.4$'$}$$

where

$$\begin{aligned} &\{\bar{P}_p\}_{jk} = \{\bar{s}\ \bar{p}\}_{jk}, \quad \{\bar{p}_p\}_{kj} = \{\bar{s}\ \bar{p}\}_{kj}, \\ &\{\bar{P}_b\}_{jk} = \{\bar{V}_y\ \bar{M}_y\}_{jk}, \quad \{\bar{p}_p\}_{kj} = \{\bar{V}_y\ \bar{M}_y\}_{kj}, \\ &\{\bar{Q}_s\} = \{\bar{q}_{Ax}/t_p\ \bar{q}_{Ay}/t_p\ \bar{q}_{Az}\ \bar{m}_{Ax}\}. \end{aligned}$$

By substituting Eqs. (A.1.13.), (A.2.21) and (A.3.22) (these equations are shown in the appendixes) into Eq. (3.4)$'$, we obtain the basic equation of the vibrating stiffened plate,

$$\begin{bmatrix} [k_p]_{21} & O \\ O & [k_b]_{21} \end{bmatrix}\begin{bmatrix} \{\bar{U}_p\} \\ \{\bar{U}_b\} \end{bmatrix}_{m,m-1} + \left[\begin{bmatrix} [k_p]_{22}+[k_p]_{11} & O \\ O & [k_b]_{22}+[k_b]_{11} \end{bmatrix} + [k_s]\right] \times$$

$$\times \begin{bmatrix} \{\bar{U}_p\} \\ \{\bar{U}_b\} \end{bmatrix}_m + \begin{bmatrix} [k_p]_{12} & O \\ O & [k_b]_{12} \end{bmatrix}\begin{bmatrix} \{\bar{U}_p\} \\ \{\bar{U}_b\} \end{bmatrix}_{m\cdot m+1} = \begin{bmatrix} O \\ O \end{bmatrix} \tag{3.5}$$

Using above equation for each intersection, we obtain

$$\begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & & & \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} & \boldsymbol{K}_{23} & & \\ & \boldsymbol{K}_{32} & \boldsymbol{K}_{33} & \boldsymbol{K}_{34} & \\ & & \ddots & & \\ & & & \boldsymbol{K}_{n,\,n-1} & \boldsymbol{K}_{nn} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \\ \boldsymbol{U}_3 \\ \vdots \\ \boldsymbol{U}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \\ \vdots \\ \boldsymbol{O} \end{bmatrix} \tag{3.6}$$

where the following notations are introduced:

$$\boldsymbol{K}_{n,\,n-1} = \begin{bmatrix} k_{21} & k_{23} & o & o \\ k_{41} & k_{43} & o & o \\ o & o & k'_{21} & k'_{23} \\ o & o & k'_{41} & k'_{43} \end{bmatrix} \begin{bmatrix} E/B \\ E/B \\ D \\ D \end{bmatrix}$$

$$\boldsymbol{K}_{m,\,m+1} = \begin{bmatrix} k_{12} & k_{14} & o & o \\ k_{32} & k_{34} & o & o \\ o & o & k'_{12} & k'_{14} \\ o & o & k'_{32} & k'_{34} \end{bmatrix} \begin{bmatrix} E/B \\ E/B \\ D \\ D \end{bmatrix}$$

$$\boldsymbol{K}_{mm} = \begin{bmatrix} (2k_{22}E/B+B_{11}/t_p) & B_{12}/t_p & B_{13}/t_p & B_{14}/t_p \\ B_{21}/t_p & (2k_{44}E/B+B_{22}/t_p+D_{22}/t_p) & B_{23}/t_p & (B_{24}+D_{24})/t_p \\ B_{31} & B_{32} & (2k'_{22}D+B_{33}+D_{33}) & (B_{34}+D_{34}) \\ B_{41} & (B_{42}+D_{42}) & (B_{43}+D_{43}) & (2k'_{44}D+B_{44}+D_{44}) \end{bmatrix}$$

$$\boldsymbol{U}_m = \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\theta} \end{bmatrix}_m = \begin{bmatrix} \{\bar{U}_p\} \\ \\ \{\bar{U}_b\} \end{bmatrix}_m$$

It should be noticed the matrices $\boldsymbol{K}_{11}$, $\boldsymbol{K}_{12}$, $\boldsymbol{K}_{n,\,n-1}$, $\boldsymbol{K}_{nn}$ are determined by the boundary conditions at the intersections m=1 and n.

The set of equations described by (3.6) is a set of algebraic, linear, homogeneous equations. By Cramer's rule, it follows that there can be other than trivial values of the $\boldsymbol{U}_m$ only if the determinant of the coefficients of $\boldsymbol{U}_m$ vanishes. This implies the necessity of

$$\begin{vmatrix} K_{11} & K_{12} & & & \\ K_{21} & K_{22} & K_{23} & & \\ & K_{32} & K_{33} & K_{34} & \\ & & \ddots & & \\ & & & K_{n,\,n-1} & K_{nn} \end{vmatrix} = 0 \tag{3.7}$$

where K_{11}, $K_{12}\cdots\cdots$ represent the element of the matrix $\boldsymbol{K}_{11}$, $\boldsymbol{K}_{12}\cdots\cdots$, respectively.

4 The formulas of stress distribution in the stiffened plate

For example, when the edges of plating, which are parallel to the stiffener, are supported freely, taking, the slope θ_n at supported edge unity in Eq. (3.6), we obtain

$$\begin{bmatrix} \boldsymbol{K}'_{11} & \boldsymbol{K}'_{12} & & & \\ \boldsymbol{K}'_{21} & \boldsymbol{K}_{22} & \boldsymbol{K}_{23} & & \\ & \cdot & & & \\ & & \ddots & & \\ & & \boldsymbol{K}_{n-1,n-2} & \boldsymbol{K}_{n-1,n-1} & \boldsymbol{K}'_{n-1,n} \\ & & & \boldsymbol{K}'_{n,n-1} & \boldsymbol{K}'_{n,n} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \\ \vdots \\ \boldsymbol{U}_{n-1} \\ \boldsymbol{U}'_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \\ \vdots \\ \boldsymbol{A}_{n-1} \\ \boldsymbol{O} \end{bmatrix} \tag{4.1}$$

where

$$\boldsymbol{K}'_{n-1,n} = \begin{bmatrix} k_{12} & k_{14} \\ k_{32} & k_{34} \\ o & o \\ o & o \end{bmatrix}$$

$$\boldsymbol{K}'_{n,n-1} = \begin{bmatrix} k_{21} & k_{23} & o & o \\ k_{41} & k_{43} & o & o \end{bmatrix}$$

$$\boldsymbol{K}'_{n,n} = \begin{bmatrix} k_{22} & k_{24} \\ k_{42} & k_{44} \end{bmatrix}$$

$$\boldsymbol{A}_{n-1} = \{o \quad o \quad k'_{14} \quad k'_{34}\}$$

$$\boldsymbol{U}_n' = \{\bar{u} \quad \bar{v}\}_n$$

Substituting the eigen-value obtained from the characteristic equation (3.7) into Eq. (4.1), and solving this equation, we obtain the displacements of intersection $\{\boldsymbol{U}_1\ \boldsymbol{U}_2 \cdots\cdots \boldsymbol{U}_{n-1}\ \boldsymbol{U}'_n\}$ for each eigen-value.

According to the Appendix 1, the dynamical stress components of the mid-surface of the plating are expressed as follows

$$\left.\begin{aligned} \sigma_x &= [\phi''][k_p]\{\bar{U}_p\} \sin \frac{n\pi}{L} x\, e^{i\omega t} \\ \sigma_y &= -(\lambda/B)^2 [\phi][k_p]\{\bar{U}_p\} \sin \frac{n\pi}{L} x\, e^{i\omega t} \\ \tau_{xy} &= -(\lambda/B) [\phi'][k_p]\{\bar{U}_p\} \cos \frac{n\pi}{L} x\, e^{i\omega t} \end{aligned}\right\} . \tag{4.2}$$

Substituting for $\bar{U}_p$ the above calculated value, (it should be noticed that the displacement matrix U is represented by $\{\bar{U}_p\ \bar{U}_b\}$), we can calculat the distribution of the mid-surface stress during vibration.

According to the expression (A·2·17) which is shown in Appendix 2, the transverse bending moment is expressed as follows

$$[\bar{M}_y] = -D\left[[f''_b] - \nu\left(\frac{\lambda}{B}\right)^2[f_b]\right]\{\bar{U}_b\}. \tag{4.3}$$

Substituting for $\bar{U}_b$ the above calculated value, we can find the bending stress distribution in the plating during vibration as follows

$$[\bar{\sigma}] = \left(\frac{6D}{t^2_p}\right)\left[[f''_b] - \nu\left(\frac{\lambda}{B}\right)^2[f_b]\right]\{\bar{U}_b\}. \tag{4.4}$$

5 Effective breadth of the stiffened plating during vibration

The vibration problem of stiffened plate is approximately calculated by assuming it as grillage or coupled vibration of plate and stiffeners, and it is usual to assume a certain effective breadth of plate combined with stiffener.

F. H. Schroeder [21] has studied the natural vibration of T-section, ⊓-section and box beam using the Rayleigh method supposing approximately that the plane stress σ_x in the flange distributed in a parabolic form across the width. However, the dynamical plane stress distribution in the plating stiffened by multi-stiffeners during vibration may be complicated, therefore it is necessary to make it clear, in order to determine rationally the effective breadth of the vibrating stiffened plate.

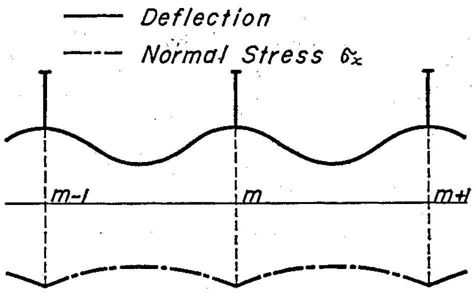
To study the plane stress distribution across frame spacing qualitatively, we consider a semi-infinite long stiffened plate, which has the same T-section stiffener and spacing, and assume four cases of modes of vibration as shown in Fig. 5.1 ~ Fig. 5.4.

In case 1, the deflections of each stiffener are the same and the transverse slopes of each stiffener are zero. In case 2, the deflections of the stiffeners have values equal in magnitude but opposite in direction, and the transverse slopes of each stiffener are zero. In case 3, the deflections of the stiffeners are zero, and the transverse slopes of the stiffeners are in the same direction. In case 4, the deflections of the stiffeners are zero, and the transverse slopes of the stiffeners have values equal in magnitude but opposite in direction.

Substituting the boundary conditions of each case shown in Fig. 5.1 ~ Fig. 5.4 into Eq. (3.6), we obtain the following equilibrium equations at the mth intersection for each case:

Case 1 :

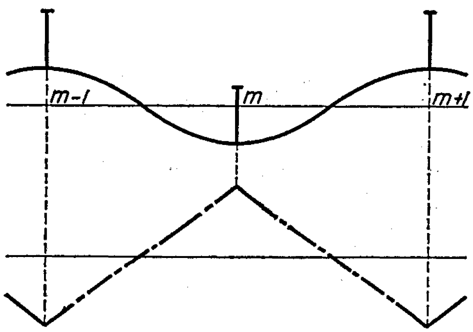
$$\bar{u}_{m-1} = \bar{u}_m = \bar{u}_{m+1},\quad \bar{v}_{m-1} = \bar{v}_m = \bar{v}_{m+1} = 0,$$
$$\bar{w}_{m-1} = \bar{w}_m = \bar{w}_{m+1},\quad \bar{\theta}_{m-1} = \bar{\theta}_m = \bar{\theta}_{m+1} = 0,$$



$U_{m-1}=U_m=U_{m+1}$, $V_{m-1}=V_m=V_{m+1}=0$
$W_{m-1}=W_m=W_{m+1}$, $\theta_{m-1}=\theta_m=\theta_{m+1}=0$

Case 1

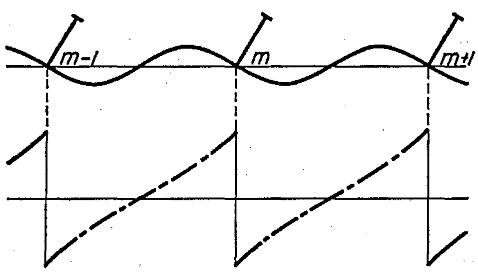
Fig. 5.1



$-U_{m-1}=U_m=-U_{m+1}$, $V_{m-1}=V_m=V_{m+1}=0$
$-W_{m-1}=W_m=-W_{m+1}$, $\theta_{m-1}=\theta_m=\theta_{m+1}=0$

Case 2

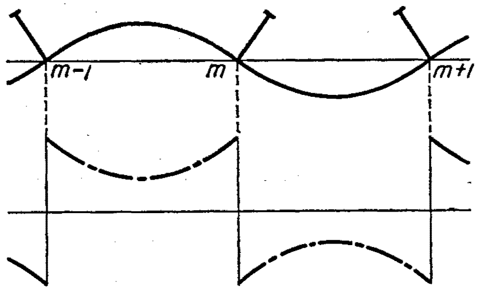
Fig. 5.2



$U_{m-1}=U_m=U_{m+1}=0$, $V_{m-1}=V_m=V_{m+1}$
$W_{m-1}=W_m=W_{m+1}=0$, $\theta_{m-1}=\theta_m=\theta_{m+1}$

Case 3

Fig. 5.3



$U_{m-1}=U_m=U_{m+1}=0\ ,\ V_{m-1}=-V_m=V_{m+1}$
$W_{m-1}=W_m=W_{m+1}=0\ ,-\theta_{m-1}=\theta_m=-\theta_{m+1}$

Case 4

Fig. 5.4

$$\begin{bmatrix} 2(E/B)(k_{11}+k_{12})+B_{11}/t_p & B_{13}/t_p \\ B_{31} & 2D(k'_{11}+k'_{12})+B_{33}+D_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_m \\ \bar{w}_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 2:

$$-\bar{u}_{m-1}=\bar{u}_m=-\bar{u}_{m+1},\quad \bar{v}_{m-1}=\bar{v}_m=\bar{v}_{m+1}=0,$$
$$-\bar{w}_{m-1}=\bar{w}_m=-\bar{w}_{m+1},\quad \bar{\theta}_{m-1}=\bar{\theta}_m=\bar{\theta}_{m+1}=0,$$

$$\begin{bmatrix} 2(E/B)(k_{11}-k_{12})+B_{1}/t_p & B_{13}/t_p \\ B_{31} & 2D(k'_{11}-k'_{12})+B_{33}+D_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_m \\ \bar{w}_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 3:

$$\bar{u}_{m-1}=\bar{u}_m=\bar{u}_{m+1}=0,\quad \bar{v}_{m-1}=\bar{v}_m=\bar{v}_{m+1},$$
$$\bar{w}_{m-1}=\bar{w}_m=\bar{w}_{m+1}=0,\quad \bar{\theta}_{m-1}=\bar{\theta}_m=\bar{\theta}_{m+1},$$

$$\begin{bmatrix} 2(E/B)(k_{33}+k_{34})+(B_{22}+D_{22})/t_p & (B_{24}+D_{24})/t_p \\ B_{42}+D_{42} & 2D(k'_{33}+k'_{34})+B_{44}+D_{44} \end{bmatrix} \begin{bmatrix} \bar{v}_m \\ \bar{\theta}_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 4:

$$\bar{u}_{m-1}=\bar{u}_m=\bar{u}_{m+1},\quad \bar{v}_{m-1}=-\bar{v}_m=\bar{v}_{m+1},$$
$$\bar{w}_{m-1}=\bar{w}_m=\bar{w}_{m+1}=0,\quad -\bar{\theta}_{m-1}=\bar{\theta}_m=-\bar{\theta}_{m+1},$$

$$\begin{bmatrix} 2(E/B)(k_{33}-k_{34})+(B_{22}+D_{22})/t_p & (B_{24}+D_{24})/t_p \\ B_{42}+D_{42} & 2D(k'_{33}-k'_{34})+B_{44}+D_{44} \end{bmatrix} \begin{bmatrix} \bar{v}_m \\ \bar{\theta}_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting the determinant of the stiffness matrix of above equations zero, we find the characteristic equations of vibration as shown in Table 5.1.

The longitudinal stress σ_x in the plate at the one frame space during vibration is obtained by using Eq. (4.2),

$$\sigma_x = [\phi''][k_p]\{\bar{U}_p\}\ sin\frac{n\pi}{L}x\ e^{i\omega t} \tag{5.1}$$

Table 5.1 Characteristic equation for each case

Case 1	$\begin{vmatrix} 2(k_{11}+k_{12})+\bar{B}_{xx}\lambda^2 & \bar{B}_{xz}\lambda^3 \\ \bar{B}_{zx}\lambda^3 & 2(k'_{11}+k'_{12})+\bar{B}_{zz}\lambda^4-\bar{D}_{zz}\kappa^4 \end{vmatrix}=0$
Case 2	$\begin{vmatrix} 2(k_{11}-k_{12})+\bar{B}_{xx}\lambda^2 & \bar{B}_{xz}\lambda^3 \\ \bar{B}_{zx}\lambda^3 & 2(k'_{11}-k'_{12})+\bar{B}_{zz}\lambda^4-\bar{D}_{zz}\kappa^4 \end{vmatrix}=0$
Case 3	$\begin{vmatrix} 2(k_{33}+k_{34})+\bar{B}_{yy}\lambda^4-\bar{D}_{yy}\kappa^4 & \bar{B}_{y\theta}\lambda^4-\bar{D}_{y\theta}\kappa^4 \\ \bar{B}_{\theta y}\lambda^4-\bar{D}_{\theta y}\kappa^4 & 2(k'_{33}+k'_{34})+\bar{B}_{\theta\theta}\lambda^4+\bar{K}_c\lambda^2-\bar{D}_{\theta\theta}\kappa^4 \end{vmatrix}=0$
Case 4	$\begin{vmatrix} 2(k_{33}-k_{34})+\bar{B}_{yy}\lambda^4-\bar{D}_{yy}\kappa^4 & \bar{B}_{y\theta}\lambda^4-\bar{D}_{y\theta}\kappa^4 \\ \bar{B}_{\theta y}\lambda^4-\bar{D}_{\theta y}\kappa^4 & 2(k'_{33}-k'_{34})+\bar{B}_{\theta\theta}\lambda^4+\bar{K}_c\lambda^2-\bar{D}_{\theta\theta}\kappa^4 \end{vmatrix}=0$

$$\bar{B}_{xx}=\frac{A_s}{t_pB},\quad \bar{B}_{xz}=\frac{A_sz_A}{t_pB^2},\quad \bar{B}_{yy}=\frac{I_z}{t_pB^3},\quad \bar{B}_{y\theta}=\frac{(z_A-z_S)I_z}{t_pB^4},\quad \bar{B}_{zx}=\frac{EA_sz_A}{D},$$
$$\bar{B}_{zz}=\frac{E\,(I_y+A_sz^2{}_A)}{DB},\quad \bar{B}_{\theta y}=\frac{EI_z(z_A-z_S)}{DB^2},\quad \bar{B}_{\theta\theta}=\frac{EI_z(z_A-z_S)^2}{BD^3},\quad \bar{K}_c=\frac{C}{DB},$$
$$\bar{D}_{yy}=\frac{D}{Et_pB^2}\cdot\frac{m_{st}}{m_b},\quad \bar{D}_{y\theta}=\frac{Dz_A}{Et_pB^3}\cdot\frac{m_{st}}{m_b},\quad \bar{D}_{\theta y}=\frac{z_A}{B}\cdot\frac{m_{st}}{m_b},\quad \bar{D}_{\theta\theta}=\frac{\Theta}{m_b\cdot B^2},$$
$$m_b=\frac{\gamma t_pB}{g}$$

Substituting for $\{\bar{U}_p\}$ above described values for each case, we obtain the dynamical plane stress σ_x for each case, as follows

$$\text{Case 1}:\ \sigma_x=[\phi''][k_p]\{\bar{u}_m\ \ \bar{u}_m\ \ o\ \ o\}\ sin\frac{n\pi}{L}x\ e^{i\omega t}$$

$$\text{Case 2}:\ \sigma_x=[\phi''][k_p]\{\bar{u}_m-\bar{u}_m\ \ o\ \ o\}\ sin\frac{n\pi}{L}x\ e^{i\omega t}$$

$$\text{Case 3}:\ \sigma_x=[\phi''][k_p]\{o\ \ o\ \ \bar{v}_m\ \ \bar{v}_m\}\ sin\frac{n\pi}{L}x\ e^{i\omega t}$$

$$\text{Case 4}:\ \sigma_x=[\phi''][k_p]\{o\ \ o\ \ \bar{v}_m-\bar{v}_m\}\ sin\frac{n\pi}{L}x\ e^{i\omega t}$$

The distribution of σ_x for $\lambda=1$ $(L/B=\pi)$ is shown in Fig. 5.1 ~ Fig. 5.2 for each case.

The shearing stress in the intersection of plate and stiffener is obtained by using Eq. (A.1.8),

$$\{\bar{P}_p\}=[k_p]\{\bar{U}_p\} \tag{5.2}$$

Substituting the expression (A.1.9) (A.1.10) and (A.1.11) into Eq. (5.2), we obtain

$$\begin{bmatrix}\bar{s}_{jk}\\ \bar{s}_{kj}\end{bmatrix}=\begin{bmatrix}k_{11} & k_{12} & k_{13} & k_{14}\\ k_{12} & k_{11} & k_{14} & k_{13}\end{bmatrix}\{\bar{u}_j\ \ \bar{u}_k\ \ \bar{v}_j\ \ \bar{v}_k\} \tag{5.2$'$}$$

Let us represent the shearing stress at the right and left intersection of the mth ones s_L and s_R, respectively, we can express them in the following form by using Eq. (5.2)$'$,

$$\begin{bmatrix} \bar{s}_L \\ \bar{s}_R \end{bmatrix}_m = \begin{bmatrix} \bar{s}_{m,\, m-1} \\ \bar{s}_{m,\, m+1} \end{bmatrix}$$

$$= \begin{bmatrix} k_{12} & k_{11} & o & k_{14} & k_{13} & o \\ o & k_{11} & k_{12} & o & k_{13} & k_{14} \end{bmatrix} \{\bar{u}_{m-1}\ \bar{u}_m\ \bar{u}_{m+1}\ \bar{v}_{m-1}\ \bar{v}_m\ \bar{v}_{m+1}\} \qquad (5.3)$$

Substituting the mode conditions for each case in Eq. (5.3), we find the relation of the shearing stresses at the right and left side of the (m−1)th, mth and (m+1)th intersection, as follows

$$\text{Case 1 :} \begin{bmatrix} \bar{s}_L \\ \bar{s}_R \end{bmatrix}_{m-1} = \begin{bmatrix} \bar{s}_L \\ \bar{s}_R \end{bmatrix}_m = \begin{bmatrix} \bar{s}_L \\ \bar{s}_R \end{bmatrix}_{m+1}$$

$$\text{Case 2 :} \begin{bmatrix} -\bar{s}_L \\ -\bar{s}_R \end{bmatrix}_{m-1} = \begin{bmatrix} \bar{s}_L \\ \bar{s}_R \end{bmatrix}_m = \begin{bmatrix} -\bar{s}_L \\ -\bar{s}_R \end{bmatrix}_{m+1}$$

$$\text{Case 3 :} \begin{bmatrix} -\bar{s}_L \\ \bar{s}_R \end{bmatrix}_{m-1} = \begin{bmatrix} -\bar{s}_L \\ \bar{s}_R \end{bmatrix}_m = \begin{bmatrix} -\bar{s}_L \\ \bar{s}_R \end{bmatrix}_{m+1}$$

$$\text{Case 4 :} \begin{bmatrix} \bar{s}_L \\ -\bar{s}_R \end{bmatrix}_{m-1} = \begin{bmatrix} -\bar{s}_L \\ \bar{s}_R \end{bmatrix}_m = \begin{bmatrix} \bar{s}_L \\ -\bar{s}_R \end{bmatrix}_{m+1}$$

Hence the positive directions of these shearing stresses are same at *j*- and *k*-side of the plating, as shown in Fig. A. 1. 1, in case 1, the shearing stresses on the each intersection are all in the same direction, and the stress of σ_x distributed parabolically across the width, as indicated in Fig. 5. 1. On the contrary, in case 2, the directions of shearing stress of plating on the right and left sides of the joint are the same, but their directions are opposite at the mth and m+1 th intersections.

Therefore we find that the peaks of σ_x appear in alternative at each intersection, and σ_x is equal to zero on the middle of the frame space.

In case 3, we see that the directions of shearing stress at both sides of each panel are opposite and the distribution of σ_x takes zero at the middle of frame space and maximum values at the both sides of each panel, but they are in the opposite direction, as shown in Fig. 5. 3.

In case 4, the directions of shearing stress at both sides of each panel are the same, but those in two panels are opposite, therefore, the direction of σ_x repeats alternately tension and compression, and takes parabolic form in each panel, in Fig. 5. 4.

From the above results, we see that the distribution of the stress σ_x in the stiffened plate is considerably different from the case which is subjected to lateral load statically.

Now, let us apply the same definition of the effective breadth to the vibrating stiffened plate as the statically and laterally loaded stiffened plate, and then we find

$$\bar{B}_m = \frac{\int_{-B/2}^{0} \sigma_x dy + \int_{0}^{B/2} \sigma_x dy}{\sigma_{xm}} \tag{5.4}$$

where $\bar{B}_m$ is effective breadth, σ_{xm} is the normal stress of the web of stiffener at the mth intersection.

By taking the displacement component of the x-direction in the form

$$u = \bar{u} \cos n\pi \frac{x}{L} e^{i\omega t}$$

we find the following expression for the denominator of Eq. (5.4),

$$\sigma_{xm} = E\frac{\partial u_m}{\partial x} = -E\bar{u}_m \frac{\lambda}{B} \sin \frac{n\pi}{L} x \ e^{i\omega t} \tag{5.5}$$

Substituting Eq. (5.5) and Eq. (5.1) into the denominator and the numerator of Eq. (5.4), respectively, we obtain

$$\left|\frac{\bar{B}}{B}\right| = \frac{\int_{-B/2}^{0} [\phi''] [k_p] \{\bar{U}_p\} dy + \int_{0}^{B/2} [\phi''] [k_p] \{\bar{U}_p\} dy}{E\bar{u}_m \lambda} \tag{5.6}$$

By using Eq. (5.6) for the previous mentioned four cases, we obtain the formula of effective breadth for each case as shown in Table 5.2.

In Table 5.2 the numerical values of $\bar{B}/B$ are calculated for $\lambda=1$,

Table 5.2 Effective breadth for each case

	$\bar{B}/B$		$\lambda=1$
Case 1	$\frac{2}{\lambda^2}(k_{11}+k_{12})$	$\frac{4}{\lambda} \cdot \frac{\cosh\lambda-1}{(3-\nu)(1+\nu)\sinh\lambda-(1+\nu)^2\lambda}$ $\lambda\to 0 : \frac{1}{1-\nu^2}$	0.8921
Case 2	$\frac{2}{\lambda^2}\{(k_{11}-k_{12})+2k_A(k_{11}-k_{12})-2k_B(k_{13}+k_{14})\}$	$\frac{2}{\lambda} \cdot \frac{4\cosh^2\frac{\lambda}{2}-4\cosh\frac{\lambda}{2}+(1+\nu)\lambda\sinh\frac{\lambda}{2}}{(\nu-3)(\nu+1)\sinh\lambda-(1+\nu)^2\lambda}$ $\lambda\to 0 : \frac{1}{2}\left(\frac{1+\nu/2}{1+\nu}\right)$	0.4310
Case 3	*indeterminate*	*indeterminate*	——
Case 4	*indeterminate*	*indeterminate*	——

$$k_A = \frac{\lambda\cosh(\lambda/2)-2\sinh(\lambda/2)}{\lambda(\sinh\lambda-\lambda)}, \quad k_B = \frac{\lambda\sin(\lambda/2)}{\lambda(\sinh\lambda-\lambda)}$$

and the formulas in the third column are obtained by substituting k_{ij}s which are shown by Eq. (A.1.12) (Appendix 1) into the expression in the second column.

The expression for case 1 is in complete agreement with the formula of H. A. Schade's[28] "multiple web" subjected to the statical sinusoidal load.

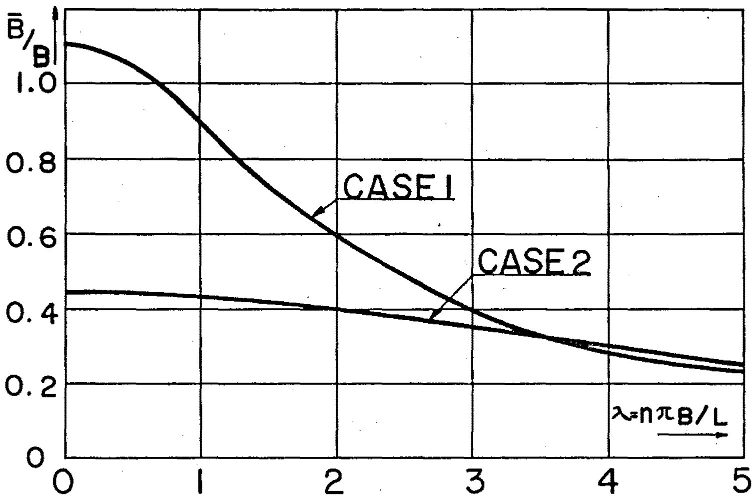


Fig. 5.5

Fig. 5.5 presents $\bar{B}/B$ plotted the $\lambda = \dfrac{n\pi B}{L}$ for case 1 and 2. These curve show that the effective breadth is equal to approximately the frame space B and the half space $B/2$ for case 1 and 2, respectively, provided the aspect ratio of the panel L/B is large, and the mode of longitudinal direction is low (it means that n is small).

In case 3 and 4, it is obvious that the effective breadth is indeterminate, because the integration in numerator of Eq. (5.6) makes zero and the $\bar{u}_m$ in the denominator is also equal to zero.

However, it is seen that the characteristic equations for case 3 and 4 in Table 1 do not involve the term B_{zz} which represents of the bending stiffness of stiffener, therefore, we can calculate the vibration of the stiffened plate without consideration of the bending stiffness of stiffener for such modes as the stiffeners do not deflect laterally but rotate about the intersection of plate and stiffener which correspond to case 3 and 4.

For this reason, when we calculate the vibration of stiffened plate by the approximate method using the effective breadth for stiffener, even though the effective breadth is indeterminate, eigen-values can be obtained regardless of the effective breadth as far as case 3 and 4.

The approximate method will be shown in another author's paper.

6 Numerical calculation

We calculate the eigen-values, modes, bending stresses and mid-surface stress of a rectangular plate simply supported around and stiffened by an inverted angle at its centre. The ends of the stiffener are restrained for torsional rotation but free to warp. Their dimensions are shown in Fig. 6.1. The effective length and breadth of the plate are 628×400 mm, and its thickness is 1.65 mm. Then the aspect ratio L/B is equal to π. The stiffener has 50×3.3 mm web and 15×3.3 mm flange, and its length is 628 mm.

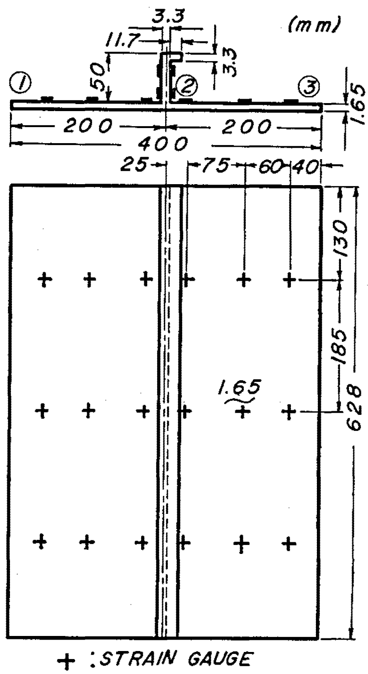


Fig. 6.1

Since ① and ③ edges of plate are simply supported, the bending moment per unit length about the edges M_y is equal to zero. Therefore the equilibrium equations are shown as follows

$$\begin{bmatrix} K'_{11} & K'_{12} & O \\ K'_{21} & K_{22} & K'_{23} \\ O & K'_{32} & K'_{33} \end{bmatrix} \begin{bmatrix} U'_1 \\ U_2 \\ U'_3 \end{bmatrix} = \begin{bmatrix} O \\ O \\ O \end{bmatrix}$$

where

$$K'_{11} = \begin{bmatrix} k_{22} & k_{42} & o \\ k_{42} & k_{44} & o \\ o & o & k'_{44} \end{bmatrix} \qquad K'_{12} = \begin{bmatrix} k_{21} & k_{41} & o & o \\ k_{41} & k_{43} & o & o \\ o & o & -k_{41} & k_{43} \end{bmatrix}$$

$$\boldsymbol{K}'_{21}=\begin{bmatrix} k_{21} & k_{41} & o \\ k_{41} & k_{43} & o \\ o & o & k'_{41} \\ o & o & k'_{43} \end{bmatrix} \begin{bmatrix} E/B \\ E/B \\ D \end{bmatrix}$$

$$\boldsymbol{K}_{22}=\begin{bmatrix} \{2k_{22}(E/B)+B_{11}/t_p\} & B_{12}/t_p & B_{13}/t_p & B_{14}/t_p \\ B_{21}/t_p & \{2k_{44}(E/B)+B_{22}/t_p+D_{22}/t_p\} & B_{23}/t_p & (B_{24}+D_{24})/t_p \\ B_{31} & B_{32} & (2k'_{22}D+B_{33}+D_{33}) & (B_{34}+D_{34}) \\ B_{41} & (B_{42}+D_{42}) & (B_{43}+D_{43}) & (2k'_{44}D+B_{44}+D_{44} \end{bmatrix}$$

$$\boldsymbol{K}'_{23}=\begin{bmatrix} k_{21} & k_{41} & o \\ k_{41} & k_{43} & o \\ o & o & k'_{41} \\ o & o & k'_{43} \end{bmatrix} \begin{bmatrix} E/B \\ E/B \\ D \end{bmatrix}, \qquad \boldsymbol{K}_{32}=\begin{bmatrix} k_{21} & k_{41} & o & o \\ k_{41} & k_{43} & o & o \\ o & o & k'_{41} & k'_{43} \end{bmatrix} \begin{bmatrix} E/B \\ E/B \\ D \\ D \end{bmatrix}$$

$$\boldsymbol{K}'_{33}=\begin{bmatrix} k_{22} & k_{42} & o \\ k_{42} & k_{44} & o \\ o & o & k_{44} \end{bmatrix} \begin{bmatrix} E/B \\ E/B \\ D \end{bmatrix}$$

$$\boldsymbol{U}'_1=\{\bar{u}_1 \quad \bar{v}_1 \quad \bar{\theta}_1\} \qquad \boldsymbol{U}_2=\{\bar{u}_2 \quad \bar{v}_2 \quad \bar{w}_2 \quad \bar{\theta}_2\} \qquad \boldsymbol{U}'_3=\{\bar{u}_3 \quad \bar{v} \quad \bar{\theta}_3\}$$

The characteristic equation is obtained by equating the determinant of the stiffness matrix of Eq. (6.1) to zero

$$\begin{vmatrix} K'_{11} & K'_{12} & O \\ K'_{21} & K_{22} & K'_{23} \\ O & K'_{32} & K'_{33} \end{vmatrix} = O \tag{6.2}$$

Taking the slope θ_3 at supported edge ③ as unity, the following equation is obtained by using Eq. (4.1)

$$\begin{bmatrix} \boldsymbol{K}'_{11} & \boldsymbol{K}'_{12} & \boldsymbol{O} \\ \boldsymbol{K}'_{21} & \boldsymbol{K}_{22} & \boldsymbol{K}''_{23} \\ \boldsymbol{O} & \boldsymbol{K}''_{32} & \boldsymbol{K}''_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \\ \boldsymbol{U}''_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{A}_2 \\ \boldsymbol{O} \end{bmatrix} \tag{6.3}$$

where

$$\mathbf{K}''_{23}=\begin{bmatrix} k_{21} & k_{41} \\ k_{41} & k_{43} \\ o & o \\ o & o \end{bmatrix}, \qquad \boldsymbol{K}''_{32}=\begin{bmatrix} k_{21} & k_{41} & o & o \\ k_{41} & k_{43} & o & o \end{bmatrix}$$

$$\boldsymbol{K}''_{33}=\begin{bmatrix} k_{22} & k_{42} \\ k_{42} & k_{44} \end{bmatrix}, \qquad \boldsymbol{A}_2=\{o \quad o \quad k'_{41} \quad k'_{43}\}$$

$$\boldsymbol{U}''_3=\{\bar{u}_3 \quad \bar{v}_3\}$$

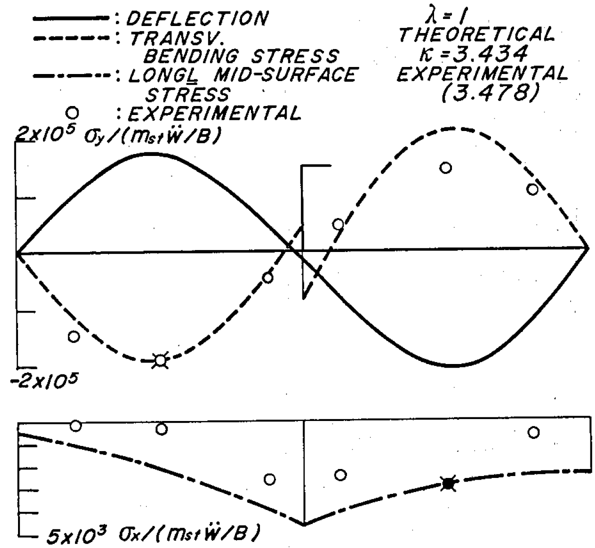


Fig. 6.2.1

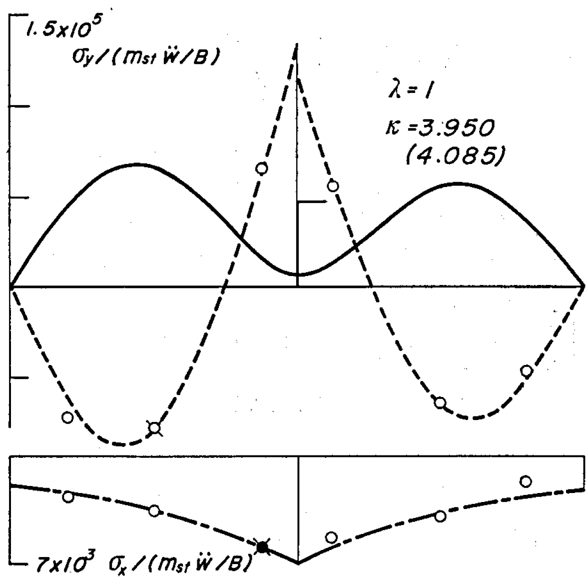


Fig. 6.2.2

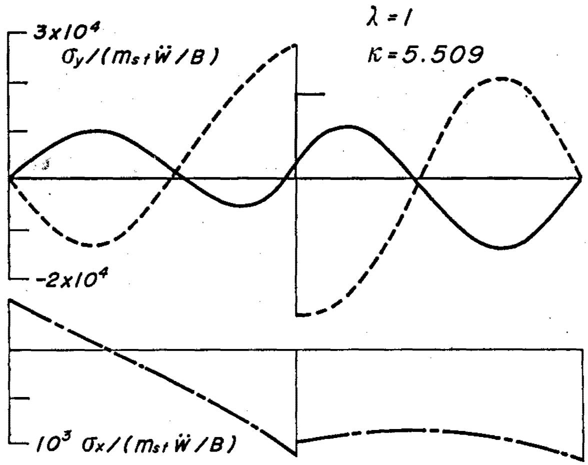


Fig. 6.2.3

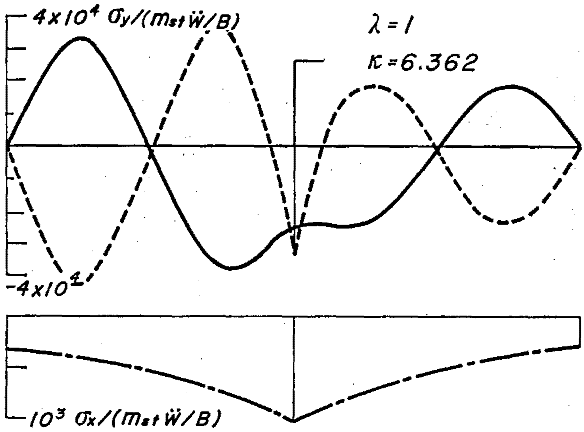


Fig. 6.2.4

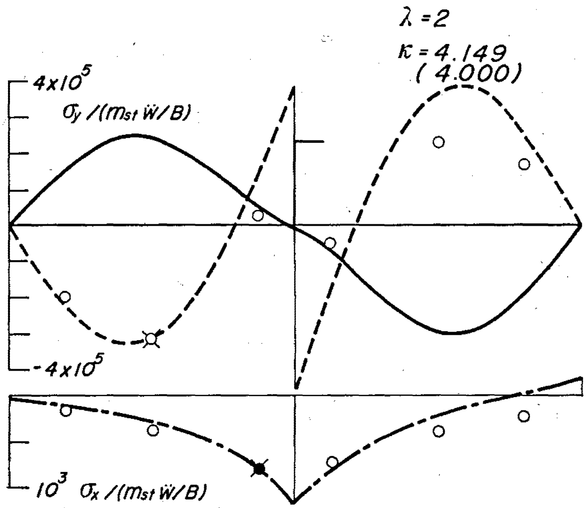


Fig. 6.3.1

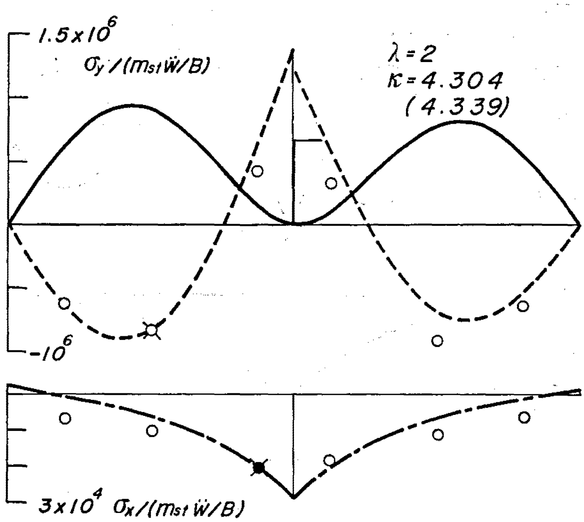


Fig. 6.3.2

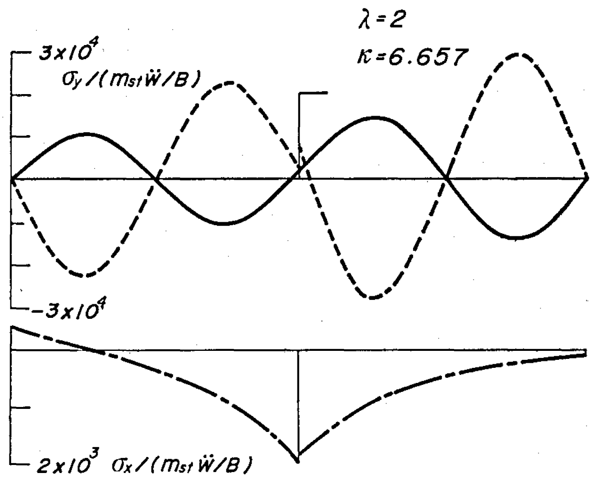


Fig. 6.3.3

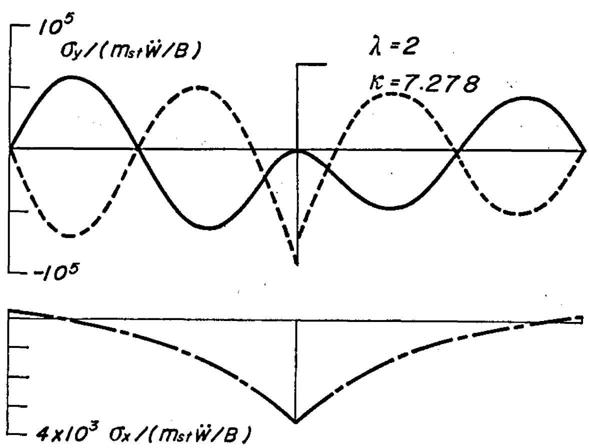


Fig. 6.3.4

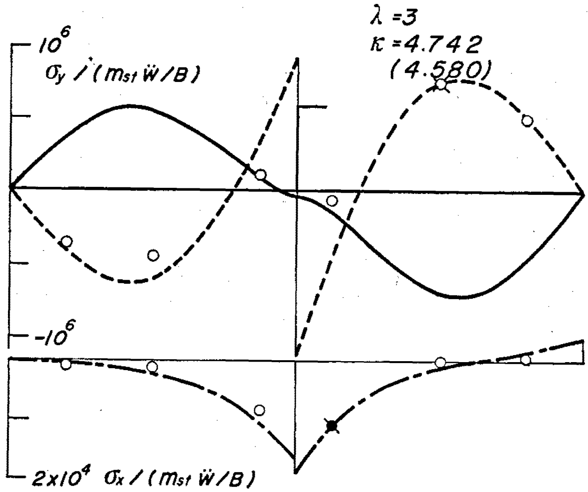


Fig. 6.4.1

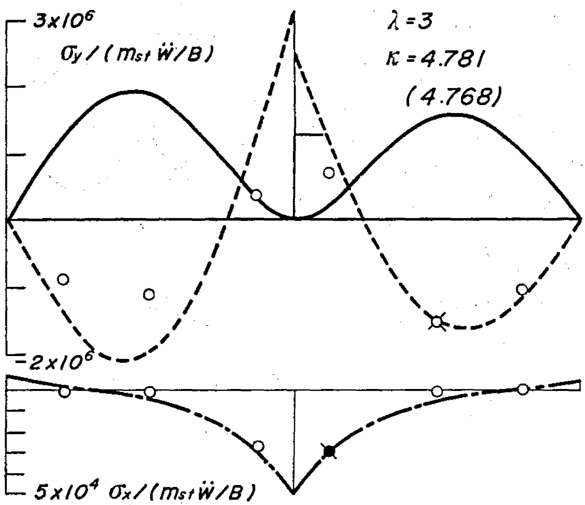


Fig. 6.4.2

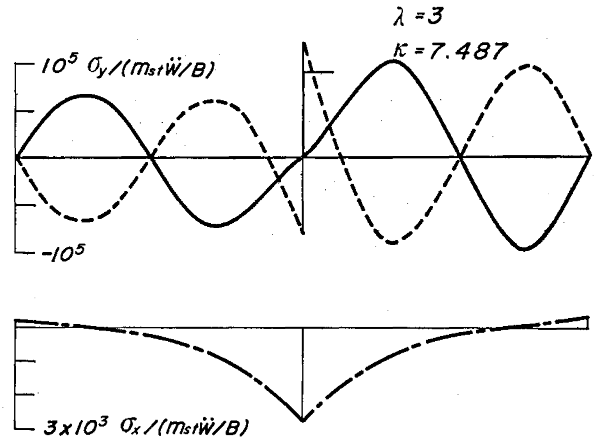


Fig. 6.4.3

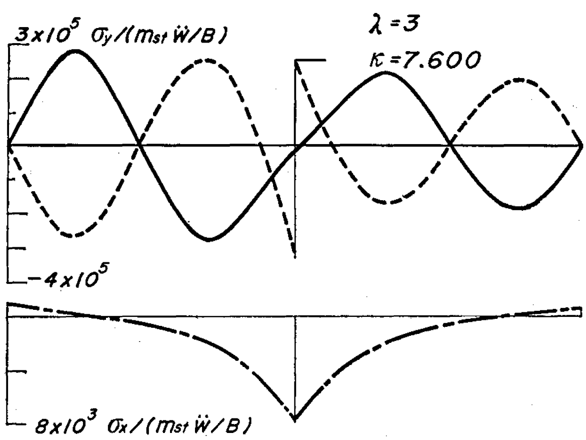


Fig. 6.4.4

Using the eigen-values obtained from the characteristic equation (6.2), we can solve Eq. (6.3) about the displacement U of each intersection for the eigen-value of vibration.

The modes of the stiffened plate for each eigen-value are obtained by using Eq. (A.2.2) and (A.2.16), that is,

$$\left.\begin{aligned} & w=\psi \sin\frac{n\pi}{L}x\, e^{i\omega t}=\bar{w}\sin\frac{n\pi}{L}x\, e^{i\omega t} \\ & [\psi]=[f_b]\ \{\bar{U}_b\} \end{aligned}\right\} \qquad (6.4)$$

The bending moment M_y induced in the plate during vibration are repre-

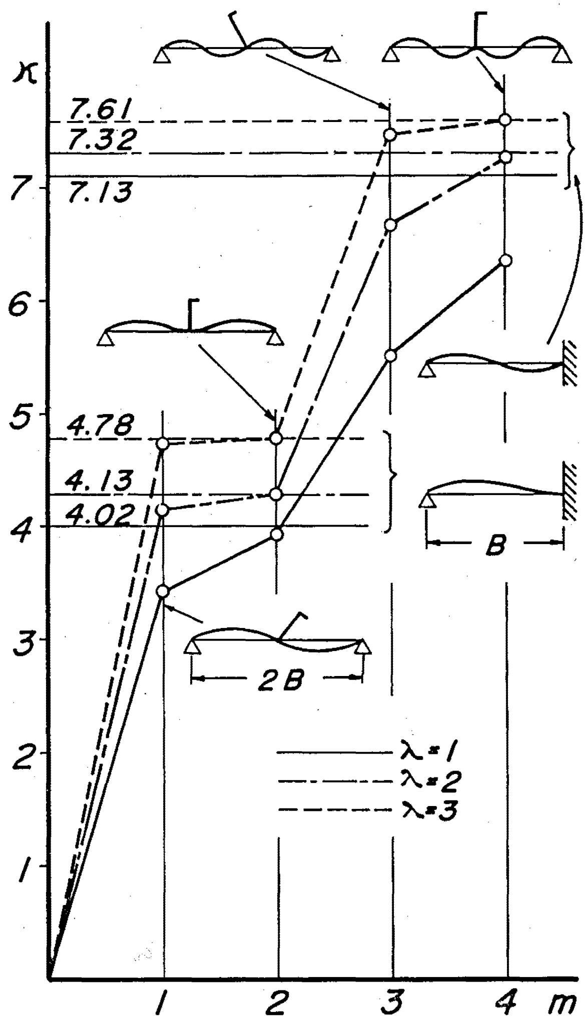


Fig. 6.5

sented by Eqs. (A. 2. 8) and (A. 2. 17), as follows

$$\left.\begin{aligned} M_y &= \bar{M}_y \sin\frac{n\pi}{L}x\, e^{i\omega t} \\ \bar{M}_y &= -D\left[[f''_b]-\nu\left(\frac{\lambda}{B}\right)^2[f_b]\right]\{\bar{U}_b\} \end{aligned}\right\} \tag{6.5}$$

Substituting for $\{\bar{U}_b\}$ of Eqs. (6. 4) and (6. 5) the above calculated values, we obtain the modes and the bending stresses of the stiffened plate during vibration, and they are shown in the Fig. 6. 2~Fig. 6. 4 by the full lines and dotted lines, respectively, for each longitudinal mode λ. The bending stress curves are plotted by the ratio $\sigma_y/(m_{st}\frac{\partial^2 w}{\partial t^2}/B)$ which represents the ratio of the bending stress σ_y and the mean value across the width of plate B of the inertia force of stiffener induced during vibration.

Giving a number to the eigen-value κs in order of smaller value, $m=1$, 2, 3, …… for each longitudinal nodal number, the relation between κ and m is shown in Fig. 6. 5. The parallel lines to the abscissa represent the eigen-values of the rectangular plate with three edges simply supported, and the fourth edge, which corresponds to the intersection of plate and stiffener, built in.

We see that eigen-values of the stiffened plate are almost the same in these special cases, except $\lambda=1$, and therefore the plate is nearly fixed at the intersection. We may recognize the phenomenon also in Fig. 6. 2~6. 4 which show the bending stresses at the intersection are pretty high.

In the case $\lambda=1$, the apparent length of stiffener l_1 is equal to the actual length of stiffener L, but in the case $\lambda=2$ and $\lambda=3$ the apparent length l_2 and l_3 are equal to $L/2$ and $L/3$, respectively. Therefore, the apparent bending and torsional rigidity EI/l_n and GJ/l_n are the smallest in the case $\lambda=1$, and we see the effectiveness of the stiffener during vibration in the case $\lambda=1$ is smaller than those in the case $\lambda=2$ and $\lambda=3$.

Consequently the eigen-values in the case $\lambda=1$ are far away from the values of built-in.

The deflections and the bending stresses of above calculated values are not absolute, because they are obtained during the free vibration.

Let us assume the fatigue limit of the materials of the stiffened plate is equal to 22 kg/mm², and we find the ratio of maximum deflection of plate and width of plate B, corresponding to the maximum bending stress in the plate gets to their fatigue limit.

Eliminating $\{\bar{U}_b\}$ from Eqs. (6. 4) and (6. 5) we find

$$(w/B) = \frac{[f_b]}{\left[[f''_b]-\nu\left(\frac{\lambda}{B}\right)^2[f_b]\right]}\frac{M_y}{BD}$$

Denoting the fatigue limit σ_F, we may represent

$$(M_y)_{max} = \frac{t^2{}_p}{6}\sigma_F$$

Table 6.1

The ratio of the maximum deflection w_{max} and the frame space B, when the maximum bending stress on the plate gets to the fatigue limit $\sigma_F=22$kg/mm². The values in the table represent the intersection of the plate and the stiffener, expect the * mark.

$\lambda=1$		$\lambda=2$		$\lambda=3$	
κ	w_{max}/B	κ	w_{max}/B	κ	w_{max}/B
3.434	2.23×10^{-2}*	4.149	1.42×10^{-2}	4.742	1.02×10^{-2}
3.950	1.16×10^{-2}	4.304	1.03×10^{-2}	4.781	9.33×10^{-3}
5.509	5.92×10^{-3}	6.657	5.74×10^{-3}*	7.487	3.86×10^{-3}
6.362	7.04×10^{-3}*	7.278	3.33×20^{-3}	7.600	3.21×10^{-3}

Substituting $(M_y)_{max}$ into above expression, we obtain

$$(w/B)_{max}=\frac{[f_b]_{max}}{\left[[f''_b]-\nu\left(\frac{\lambda}{B}\right)^2[f_b]\right]_{max}}\quad\frac{t_p^2}{6BD}\cdot\sigma_F \tag{6.6}$$

Using Eq. (6.6), we calculate $(w/B)_{max}$ to the fourth mode for each λ, and show their values in Table 6.1.

We see that the bending stress on the plate gets to the fatigue limit when $(w/B)_{max}$ is equal $2\times10^{-2}\sim10^{-3}$. However, the assumed fatigue limit of the material 22 kg/mm² is the value of mild steel under an ideal condition, and therefore, under an actual condition, this value will decrease, provided that the shape factor of the welded part as well as the change in quality of materials due to welding and the influence of corrosion are considered.

Thus since the maximum bending stress on the stiffened plate will get to the fatigue limit when the value of $(w/B)_{max}$ is less than 10^{-2}, it is conceivable that the fatigue fracture of the stiffened plate may be caused by local vibration of ship, especially when the vibrating bending stresses are superposed on the residual stress or the bending stress due to the hydrostatic loads.

Using Eq. (4.2), we calculate the mid-surface stress σ_x, that is

$$\sigma_x=[\phi''][k_p]\{\bar{U}_p\}\sin\frac{n\pi}{L}x\, e^{i\omega t}$$

Substituting for $\{\bar{U}_p\}$ the solutions of Eq. (6.3) (it should be noticed that $\boldsymbol{U}=\{\bar{U}_p\}\{\bar{U}_b\}$), we obtain the distribution of stress σ_x for each λ. The results are shown in Fig. 6.2~6.4 by chain lines, where the ratio $\sigma_x/(m_{st}\frac{\partial^2 w}{\partial t^2}/B)$ is plotted in the same way as the curves of bending stress.

We see that the peak of normal stress is found on the intersection of plate and stiffener due to shear lag for each mode.

We calculate the effective breadth of the stiffened plate, using Eq. (5.4),

$$\bar{B}=\frac{\int_{①}^{②}\sigma_x dy+\int_{②}^{③}\sigma_x dy}{\sigma_{x②}} \tag{6.7}$$

Using now Eqs. (5.1) and (5.2), we obtain the normal stress σ_x as follows

$$\sigma_x=[\phi''] \{\bar{P}_p\} \sin\frac{n\pi}{L}x\, e^{i\omega t} \tag{6.8}$$

Because the shearing stress s_{12}, s_{32} are equal to zero at the free supported edges, substituting Eq. (5.5) and Eq. (6.8) into the denominator and the numerator of Eq. (6.7), respectively, we obtain

$$\frac{\bar{B}}{B}=\frac{\bar{s}_{21}-\bar{s}_{23}}{E\bar{u}_2\lambda^2/B}=\frac{k_{33}(k^2{}_{12}-k^2{}_{11})+k_{11}(k_{13}+k_{14})^2+2k_{13}k_{14}(k_{12}-k_{11})}{(k_{11}k_{33}-k^2{}_{13})\lambda^2} \tag{6.9}$$

Substituting for k_{ij} the expression (A.1.12) in the Appendix 1,

$$\frac{\bar{B}}{B}=\frac{e+g}{cg+de-2ab-4\{(1+\nu)/(2\lambda)\}\{(1+\nu)/(2\lambda)-a-b\}}\cdot\frac{1}{\lambda^2} \tag{6.10}$$

Further, we rewrite Eq. (6.9) using the expression a, b, c, ········ shown in the Appendix 1, Eq. (A.1.4), as follows

$$\frac{\bar{B}}{B}=\frac{2}{\lambda}\frac{\sinh 2\lambda+2\lambda}{(1+\nu)(3-\nu)\cosh 2\lambda+(\nu^2-2\nu+5)+2\lambda^2(1+\nu)^2} \tag{6.10}$$

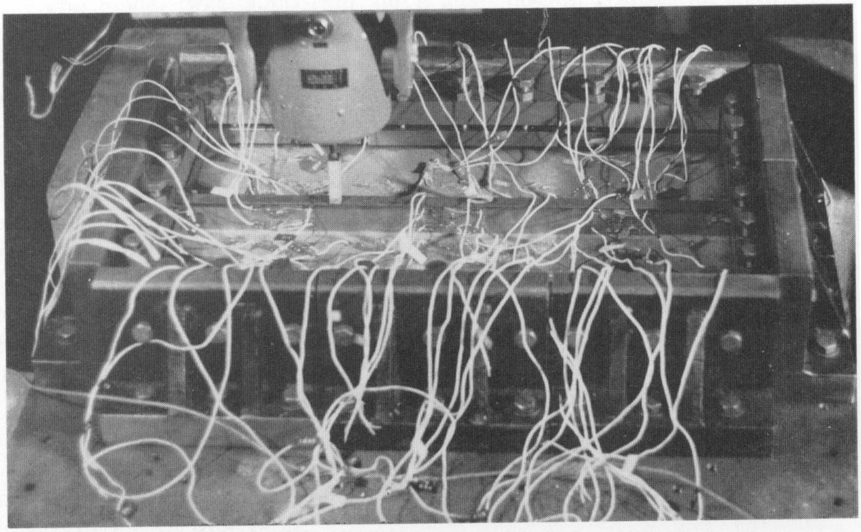
The expression (6.10) agrees with the formula of H. A. Schade's [28] "single web" subjected to sine load. It should be noted that in this case the effective breadth has no relation to the shape of the cross-section of stiffener. Since the effective breadth is only the function of λ, we can calculate it statically for the higher mode to the longitudinal direction, taking the aspect ratio to the half wave length.

7 Experiment

To assess the accuracy of the preceding theory, an experiment was made by using such a model as a rectangular plate simply supported around, stiffened by an inverted angle at its centre, and the ends of the stiffener were restrained for torsional rotation but free to warp. The experimental results obtained by measuring the natural frequencies and dynamical stresses were compared with the theoretical ones.

Specimen : The dimensions of the specimen are shown in Fig. 6.1. The effective length and breadth of the plate are 628×400 mm ($L/B=\pi$), and its thickness is 1.65 mm.

The angle has 50×3.3 mm web and 15×3.3 mm flange and its length is 628 mm. It was machined out of a solid steel bar, and attached by bolting and soldering at the centre of the plate.



Phot. 7.1 The specimen and the testing frame.

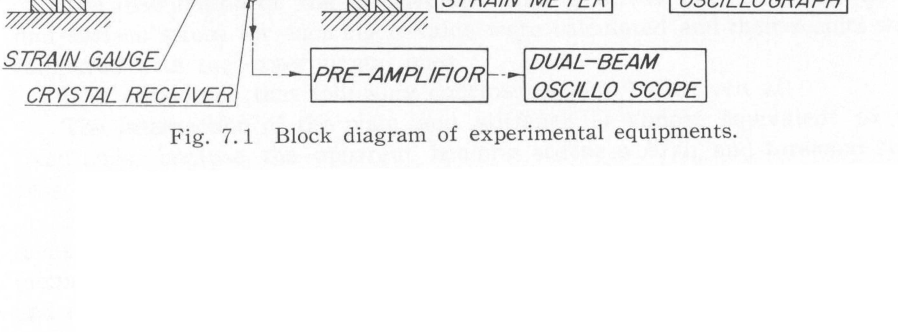


Fig. 7.1 Block diagram of experimental equipments.

(Facing p. 190)

The model was supported on a steel testing frame, as shown in Phot. 7.1, and the boundary conditions for the plate were satisfied by putting it between the two rollers which were set horizontally around the edges of plate, and for the stiffener both ends of web were held between two vertical rollers.

The block-diagram of the experimental equipments is shown in Fig. 7.1. The vibration of the stiffener was excited by an electromagnetic type exciter, and resonant frequencies were determined by the dual beam oscilloscope charged by the outputs of two crystal pick-ups.

The two-elements electrical wire resistance strain guages were stuck on both surfaces of the plate and stiffener at the same position, as shown in Fig. 6.1. The bending stresses and the mid-surface stress σ_x are obtained by the measured strains.

The results for each eigen-values are plotted in Fig. 6.2~6,4 with the theoretical curves.

Let us correspond with the experimental and theoretical results to the mark ¤ or ✖ in Fig. 6.2~6.5, and another experimental values are plotted in proportion to this value.

We see the experimental results agree fairly well with theoretical ones. The pretty high bending stress appears at the intersection.

8 Conclusions

The basic equations for the vibration of stiffened plate were introduced by combining the plane stress theory and the equation of lateral vibration of plate.

As an example, a rectangular plate was simply supported around, stiffened by an inverted angle at its centre, and ends of the stiffener were restrained for torsional rotation but free to warp.

The distribution of the transverse bending stress and the longitudinal mid-surface stress for each eigen-value were calculated and their results were compared with the experimental ones.

It is considered that following conclusions may be arrived at;

The intersection of the plate and stiffener is almost equivalent to the fixed ends, because the apparent bending stiffness EI/l_n and torsional stiffness GJ/l_n increase with the longitudinal mode number of the stiffener.

Furthermore, the inertia forces due to the lateral and angular displacement of the stiffener act on the intersection as the concentrated force and moment. Therefore, the higher bending stress appears at the intersection, and eigen-values and modes of the stiffened plate are nearly equal to those of the plate supported freely at three edges and fixed at the fourth edge corresponding to the intersection.

If the fatigue limit of the material of the stiffened plate is 22 kg/mm^2, the maximum transverse bending stress in the plate gets to this fatigue limit, and the ratio of maximum deflection to the width of plate B is about $2 \times 10^{-2} \sim 10^{-3}$ for each eigen-value.

The above fatigue limit of the materials is the value of mild steel under an ideal condition, but under an actual condition, this value will decrease, provided that the shape factor of the welded part, the change in quality of materials due to welding and the influence of corrosion are considered.

Thus the maximum bending stress at the intersection may get to a fatigue limit for the relative small vibrating deflection. Therefore, the fatigue fracture of the stiffened plate may be caused by local vibration of ship, especially when the vibrating bending stress is superposed on the residual stress or the bending stress due to the hydrostatic load.

The effective breadth of the stiffened plate during vibration is generally equal to the Schade's formulas of the case subjected to the sinusoidal load statically, but such particular cases as the stiffeners do not deflect but rotate about the intersection or the deflections of the stiffeners have value equal in magnitude but opposite in direction are excluded.

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Appendix 1 Stiffness matrix for the state of plane stress of plate

We define x, y as coordinate on each plate, as shown in Fig. A. 1. 1 where L and B are the length and breadth of plate, and both sides of the plate $y=-B/2$ and $y=+B/2$ are represented by j- and k-side, respectively.

Now, j- and k-side of plate are subjected to the following shearing stress and normal stress as shown in Fig. A. 1. 1.

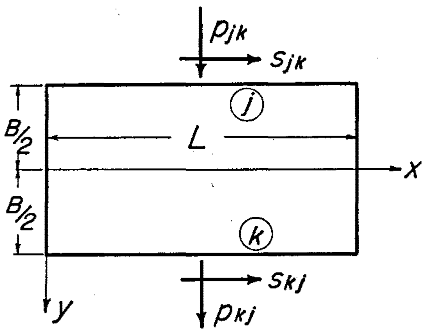

Fig. A·1·1

j-side $(y=-B/2)$ shearing stress $s_{jk}=\bar{s}_{jk}cos\dfrac{n\pi}{L}x$, normal stress $p_{jk}=\bar{p}_{jk}sin\dfrac{n\pi}{L}x$ (A. 1. 1)$_1$

k-side $(y=+B/2)$ shearing stress $s_{kj}=\bar{s}_{kj}cos\dfrac{n\pi}{L}x$, normal stress $p_{kj}=\bar{p}_{kj}sin\dfrac{n\pi}{L}x$ (A. 1. 1)$_2$

and the displacement components in the x and y direction at the j- and k-side of plate are represented by

$$\left.\begin{aligned} u_j&=\bar{u}_j cos\frac{n\pi}{L}x \\ v_j&=\bar{v}_j sin\frac{n\pi}{L}x \\ u_k&=\bar{u}_k cos\frac{n\pi}{L}x \\ v_k&=\bar{v}_k sin\frac{n\pi}{L}x \end{aligned}\right\} \qquad \text{(A. 1. 2)}$$

The relation between the displacement and stress is represented by the same form as the author has induced in the previous paper [26], as follows

$$\begin{bmatrix} \bar{u}_j \\ \bar{u}_k \\ \bar{v}_j \\ \bar{v}_k \end{bmatrix} = \frac{B}{E} \begin{bmatrix} (c+d) & -(c-d) & (a+b-\mu) & (a-b) \\ -(c-d) & (c+d) & -(a-b) & -(a+b-\mu) \\ (a+b-\mu) & -(a-b) & (e+g) & (e-g) \\ (a-b) & -(a+b-\mu) & (e-g) & (e+g) \end{bmatrix} \begin{bmatrix} \bar{s}_{jk} \\ \bar{s}_{kj} \\ \bar{p}_{jk} \\ \bar{p}_{kj} \end{bmatrix} \tag{A. 1. 3}$$

where

$$\left.\begin{aligned} a &= \frac{sinh\lambda}{\lambda(sinh\lambda-\lambda)} & d &= \frac{2cosh^2\lambda/2}{\lambda(sinh\lambda+\lambda)} \\ b &= \frac{sinh\lambda}{\lambda(sinh\lambda+\lambda)} & e &= \frac{2cosh^2\lambda/2}{\lambda(sinh\lambda-\lambda)} \\ c &= \frac{2sinh^2\lambda/2}{(sinh\lambda-\lambda)} & g &= \frac{2sinh^2\lambda/2}{\lambda(sinh\lambda+\lambda)} \end{aligned}\right\} \tag{A. 1. 4}$$

and,

$$\lambda = \frac{n\pi B}{L} \tag{A. 1. 5}$$

$$\mu = \frac{(1+\nu)}{\lambda} \tag{A. 1. 6}$$

We rewrite Eq. (A. 1. 3) in a simpler form as follows

$$\{\bar{U}_p\} = [f_p]\ \{\bar{P}_p\} \tag{A. 1. 7}$$

or

$$\{\bar{P}_p\} = [k_p]\ \{\bar{U}_p\} \tag{A. 1. 8}$$

where

$$\{\bar{U}_p\} = \{\bar{u}_j \quad \bar{u}_k \quad \bar{v}_j \quad \bar{v}_k\} \tag{A. 1. 9}$$

$$\{\bar{P}_p\} = \{\bar{s}_{jk} \quad \bar{s}_{kj} \quad \bar{p}_{jk} \quad \bar{p}_{kj}\} \tag{A. 1. 10}$$

The matrix $[k_p]$ is a stiffness matrix obtained by inversing flexibility matrix $[f_p]$, as follows

$$[k_p] = \frac{E}{B} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \tag{A. 1. 11}$$

where

$$\left.\begin{aligned} k_{11}=k_{22} &= \frac{1}{\Delta}\{2(ag+be)-\mu(g+e)\} \\ k_{12}=k_{21} &= \frac{1}{\Delta}\{2(ag-be)-\mu(g-e)\} \\ -k_{13}=-k_{31}=k_{24}=k_{42} &= \frac{1}{\Delta}\{2\mu^2-3\mu(a+b)+4ab\} \\ -k_{14}=-k_{41}=k_{23}=k_{32} &= \frac{1}{\Delta}\mu(a-b) \\ k_{33}=k_{44} &= \frac{1}{\Delta}\{2(bc+ad)-\mu(c+d)\} \\ k_{34}=k_{43} &= \frac{1}{\Delta}\{2(bc-ad)-\mu(c-d)\} \\ \Delta &= 4\mu(\mu-2a)(\mu-2b) \end{aligned}\right\} \tag{A. 1. 12}$$

We rewrite Eq. (A. 1. 8) as follows

$$\left.\begin{array}{l}\{\bar{P}_p\}_{jk}=[k_p]_{11}\{\bar{U}_p\}_{jk}+[k_p]_{12}\{\bar{U}_p\}_{kj}\\ \{\bar{P}_p\}_{kj}=[k_p]_{21}\{\bar{U}_p\}_{jk}+[k_p]_{22}\{\bar{U}_p\}_{kj}\end{array}\right\}\qquad (A. 1. 13)$$

where

$$\{\bar{P}_p\}_{jk}=\{\bar{s}_{jk}\quad \bar{p}_{jk}\}\ ,\qquad \{\bar{P}_p\}_{kj}=\{\bar{s}_{kj}\quad \bar{p}_{kj}\}\ ,$$

$$\{\bar{U}_p\}_{jk}=\{\bar{u}_j\quad \bar{v}_j\}\ ,\qquad \{\bar{U}_p\}_{kj}=\{\bar{u}_k\quad \bar{v}_k\}\ ,$$

$$[k_p]_{11}=\frac{E}{B}\begin{bmatrix} k_{11} & k_{13}\\ k_{31} & k_{33}\end{bmatrix},\qquad [k_p]_{12}=\frac{E}{B}\begin{bmatrix} k_{12} & k_{14}\\ k_{32} & k_{34}\end{bmatrix},$$

$$[k_p]_{21}=\frac{E}{B}\begin{bmatrix} k_{21} & k_{23}\\ k_{41} & k_{43}\end{bmatrix},\qquad [k_p]_{22}=\frac{E}{B}\begin{bmatrix} k_{22} & k_{24}\\ k_{42} & k_{44}\end{bmatrix},$$

and $[k_p]_{12}=[k_p]^T{}_{21}$.

The stress and displacement components are represented by the following equations.

$$\left.\begin{array}{l}\sigma_x=[\phi'']\,[k_p]\,\{\bar{U}_p\}\ \sin\dfrac{n\pi}{L}x\\[2mm] \sigma_y=-(\lambda/B)^2\,[\phi]\,[k_p]\,\{\bar{U}_p\}\ \sin\dfrac{n\pi}{L}x\\[2mm] \tau_{xy}=-(\lambda/B)\,[\phi']\,[k_p]\,\{\bar{U}_p\}\ \cos\dfrac{n\pi}{L}x\end{array}\right\}\qquad (A. 1. 14)$$

$$\left.\begin{array}{l}u=-\dfrac{B}{E\lambda}\Big[[\phi'']+\nu(\lambda/B)^2[\phi]\Big]\,[k_p]\,\{\bar{U}_p\}\ \cos\dfrac{n\pi}{L}x\\[2mm] v=-\dfrac{1}{E}\Big[(1+\nu)[\phi']+[\varDelta\phi']\Big]\,[k_p]\,\{\bar{U}_p\}\ \sin\dfrac{n\pi}{L}x\end{array}\right\}\qquad (A. 1. 15)$$

where

$$[\phi]=\Big[(f_1+f_2)\ \ (f_1-f_2)\ \ (f_3+f_4)\ \ (f_3-f_4)\Big]\qquad (A. 1. 16)$$

$$[\phi'']=(\lambda/B)^2[\phi]-[\varDelta\phi'']\qquad (A. 1. 17)$$

$$\left.\begin{array}{l}f_1=\dfrac{-(\lambda\cdot y/B)\cosh(\lambda/2)\sinh(\lambda\cdot y/B)+(\lambda/2)\sinh(\lambda/2)\cosh(\lambda\cdot y/B)}{\lambda^2(\sinh\lambda+\lambda)}\cdot B^2\\[3mm] f_2=\dfrac{(\lambda\cdot y/B)\sinh(\lambda/2)\cosh(\lambda\cdot y/B)-(\lambda/2)\cosh(\lambda/2)\sinh(\lambda\cdot y/B)}{\lambda^2(\sinh\lambda-\lambda)}\cdot B^2\\[3mm] f_3=\dfrac{\lambda\cdot y/B)\sinh(\lambda/2)\sinh(\lambda\cdot y/B)-\{\sinh(\lambda/2)+(\lambda/2)\cosh(\lambda/2)\cosh(\lambda\cdot y/B)}{\lambda^2(\sinh\lambda+\lambda)}\cdot B^2\\[3mm] f_4=\dfrac{-(\lambda\cdot y/B)\cosh(\lambda/2)\cosh(\lambda\cdot y/B)+\{(\lambda/2)\sinh(\lambda/2)+\cosh(\lambda/2)\}\sinh(\lambda\cdot y/B)}{\lambda^2(\sinh\lambda-\lambda)}\cdot B^2\end{array}\right\}$$

(A. 1. 18)

$$[\varDelta\phi'']=[(\varDelta f_1''+\varDelta f_2'')\quad(\varDelta f_1''-\varDelta f_2'')\quad(\varDelta f_3''+\varDelta f_4'')\quad(\varDelta f_3''-\varDelta f_4'')]\qquad (A. 1. 19)$$

$$\left.\begin{aligned}
\varDelta f_1'' &= \frac{2cosh(\lambda/2)cosh(\lambda\cdot y/B)}{sinh\lambda+\lambda} \\
\varDelta f_2'' &= \frac{-2sinh(\lambda/2)sinh(\lambda\cdot y/B)}{sinh\lambda-\lambda} \\
\varDelta f_3'' &= \frac{-2sinh(\lambda/2)cosh(\lambda\cdot y/B)}{sinh\lambda+\lambda} \\
\varDelta f_4'' &= \frac{2cosh(\lambda/2)sinh(\lambda\cdot y/B)}{sinh\lambda-\lambda}
\end{aligned}\right\} \qquad (A.1.20)$$

Appendix 2 Stiffness matrix for the lateral vibration of square plate

The differential equation of lateral vibration of square plate is given by the equation

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=-\frac{\gamma t_p}{gD}\frac{\partial^2 w}{\partial t^2} \qquad (A.2.1)$$

Since both edges of the plate $x=0$, $x=L$ are simply supported, we put the deflection of plate

$$w=\phi(y)sin\frac{n\pi}{L}x \quad e^{i\omega t}=\bar{w}\,sin\frac{n\pi}{L}x \quad e^{i\omega t} \qquad (A.2.2)$$

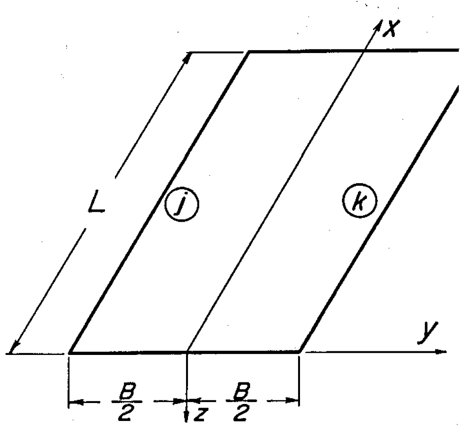

Fig. A·2·1

where

$$\bar{w}=\phi(y) \qquad (A.2.3)$$

and ω is circular frequency.

Substituting Eq. (A.2.2) into Eq. (A.2.1), we obtain an ordinary differential equation respect to y, as follows

$$\psi^{N} - 2\left(\frac{\lambda^2}{B^2}\right)\psi'' + \frac{1}{B^4}\ (\lambda^4-\kappa^4)\ \psi=0 \tag{A. 2. 4}$$

where

$$\lambda=\frac{n\pi B}{L} \qquad \kappa=\left\{\left(\frac{\gamma t_p}{gD}\right)\omega^2\right\}^{1/4}\cdot B \tag{A. 2. 5}$$

The general solution of the differential equation (A. 2. 3) is expressed as follows

(1) $\lambda^2-\kappa^2<0$

$$\psi\ (y)= A_1\ sinh\frac{\alpha y}{B}+A_2\ cosh\frac{\alpha y}{B}+A_3\ sin\frac{\beta y}{B}+A_4 cos\frac{\beta y}{B} \tag{A. 2. 6}$$

(2) $\lambda^2-\kappa>0$

$$\psi\ (y)\ = A_1'\ sinh\frac{\alpha y}{B}+A_2' cosh\frac{\alpha y}{B}+A_3'\ sinh\frac{\beta y}{B}+A'_4 cosh\frac{\beta y}{B} \tag{A. 2. 6)'}$$

where the following notations are introduced:

$$\alpha=\sqrt{\lambda^2+\kappa^2}\ , \qquad \beta=\sqrt{\lambda^2-\kappa^2}\ . \tag{A. 2. 7}$$

The constants of integration $A_1\cdots\cdots A_4$ or $A'_1\cdots\cdots A'_4$ may be determined from the both boundary conditions on the both edges of plate $y=\mp B/2$.

Now, the bending moment per unit length about an axis is parallel to x axis and the reactive force along the edge perpendicular to y axis, as represented as follows

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$V_y= -D\left\{\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right\}$$

where,

$$D=\frac{Et^3{}_p}{12(1-\nu^2)}$$

Substituting the expression (A. 2. 2) into the above equations, we obtain the following equations:

$$M_y=\bar{M}_y\ sin\frac{n\pi}{L}x\ e^{i\omega t} \tag{A. 2. 8}$$

$$V_y=\bar{V}_y\ sin\frac{n\pi}{L}x\ e^{i\omega t} \tag{A. 2. 9}$$

where,

$$\bar{M}_y= -D\left\{\psi''-\nu\left(\frac{\lambda}{B}\right)^2\psi\right\} \tag{A. 2. 10}$$

$$\bar{V}_y = -D\left\{\psi''' - (2-\nu)\left(\frac{\lambda}{B}\right)^2\psi'\right\} \qquad (A.2.11)$$

Differentiating Eq. (A. 2. 2) with respect to y, we obtain the slope of the deflection surface along the axis which is parallel to x axis

$$\theta = \frac{\partial w}{\partial y} = \psi' \sin\frac{n\pi}{L}x\, e^{i\omega t} = \bar{\theta}\sin\frac{n\pi}{L}x\, e^{i\omega t} \qquad (A.2.12)$$

where

$$\bar{\theta} = \psi' \ . \qquad (A.2.13)$$

Now w_j, w_k, θ_j *and* θ_k denoting the deflections and slopes along the j-side ($y=-B/2$) and k-side ($y=+B/2$) of plate, the expressions of these deflection and slope are obtained by substituting $y=-B/2$, $y=+B/2$ into the expression (A. 2. 2) and (A. 2. 13), using the Eq. (A. 2. 6) or Eq. (A. 2. 6)′, as follows

(1) $\lambda^2-\kappa^2<0$

$$\begin{bmatrix}\bar{w}_j\\ \bar{w}_k\\ \bar{\theta}_j\\ \bar{\theta}_k\end{bmatrix} = \begin{bmatrix} -\sinh\alpha/2 & \cosh\alpha/2 & -\sin\beta/2 & \cos\beta/2\\ \sinh\alpha/2 & \cosh\alpha/2 & \sin\beta/2 & \cos\beta/2\\ \alpha/B\cosh\alpha/2 & -\alpha/B\sinh\alpha/2 & \beta/B\cos\beta/2 & \beta/B\sin\beta/2\\ \alpha/B\cosh\alpha/2 & \alpha/B\sinh\alpha/2 & \beta/B\cos\beta/2 & -\beta/B\sin\beta/2 \end{bmatrix}\begin{bmatrix}A_1\\ A_2\\ A_3\\ A_4\end{bmatrix} \qquad (A.2.14)$$

(2) $\lambda^2-\kappa^2>0$

$$\begin{bmatrix}\bar{w}_j\\ \bar{w}_k\\ \bar{\theta}_j\\ \bar{\theta}_k\end{bmatrix} = \begin{bmatrix} -\sinh\alpha/2 & \cosh\alpha/2 & -\sinh\beta/2 & \cosh\beta/2\\ \sinh\alpha/2 & \cosh\alpha/2 & \sinh\beta/2 & \cosh\beta/2\\ \alpha/B\cosh\alpha/2 & -\alpha/B\sinh\alpha/2 & \beta/B\cosh\beta/2 & -\beta/B\sinh\beta/2\\ \alpha/B\cosh\alpha/2 & \alpha/B\sinh\alpha/2 & \beta/B\cosh\beta/2 & \beta/B\sinh\beta/2 \end{bmatrix}\begin{bmatrix}A_1\\ A'_2\\ A'_3\\ A'_4\end{bmatrix} \qquad (A.2.14)'$$

Solving Eq. (A. 2. 14) or (A. 2 14)′ with respect to $A_1 \cdots\cdots A_4$, or $A'_1 \cdots\cdots A'_4$ we obtain

(1) $\lambda^2-\kappa^2<0$

$$\left.\begin{aligned}
A_1 &= \frac{1}{\Delta_2}\left\{(\bar{w}_j-\bar{w}_k)\frac{\beta}{B}\cos\frac{\beta}{2} + (\bar{\theta}_j+\bar{\theta}_k)\sin\frac{\beta}{2}\right\}\\
A_2 &= \frac{1}{\Delta_1}\left\{(\bar{w}_j+\bar{w}_k)\frac{\beta}{B}\sin\frac{\beta}{2} - (\bar{\theta}_j-\bar{\theta}_k)\cos\frac{\beta}{2}\right\}\\
A_3 &= \frac{1}{\Delta_1}\left\{-(w_j-w_k)\frac{\alpha}{B}\cosh\frac{\alpha}{2} - (\theta_j+\theta_k)\sinh\frac{\alpha}{2}\right\}\\
A_4 &= \frac{1}{\Delta_2}\left\{(w_j+w_k)\frac{\alpha}{B}\sinh\frac{\alpha}{2} + (\theta_j-\theta_k)\cosh\frac{\alpha}{2}\right\}\\
\Delta_1 &= 2\left(\frac{\beta}{B}\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} + \frac{\alpha}{B}\sinh\frac{\alpha}{2}\cos\frac{\beta}{2}\right)\\
\Delta_2 &= 2\left(\frac{\alpha}{B}\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} - \frac{\beta}{B}\sinh\frac{\alpha}{2}\cos\frac{\beta}{2}\right)
\end{aligned}\right\} \qquad (A.2.15)$$

(2) $\lambda^2-\kappa^2>0$

$$\left.\begin{aligned}
A_1 &= \frac{1}{\varDelta_2}\left\{(\bar{w}_j-\bar{w}_k)\frac{\beta}{B}\cosh\frac{\beta}{2}+(\bar{\theta}_j+\bar{\theta}_k)\sinh\frac{\beta}{2}\right\}\\
A_2 &= \frac{1}{\varDelta_1}\left\{-(\bar{w}_j+\bar{w}_k)\frac{\beta}{B}\sinh\frac{\beta}{2}-(\bar{\theta}_j-\bar{\theta}_k)\cosh\frac{\beta}{2}\right\}\\
A_3 &= \frac{1}{\varDelta_2}\left\{-(\bar{w}_j-\bar{w}_k)\frac{\alpha}{B}\cosh\frac{\alpha}{2}-(\bar{\theta}_j+\bar{\theta}_k)\sinh\frac{\alpha}{2}\right\}\\
A_4 &= \frac{1}{\varDelta_1}\left\{(\bar{w}_j+\bar{w}_k)\frac{\alpha}{B}\sinh\frac{\alpha}{2}+(\bar{\theta}_j-\bar{\theta}_k)\cosh\frac{\alpha}{2}\right\}\\
\varDelta_1 &= 2\left(\frac{\alpha}{B}\sinh\frac{\alpha}{2}\cosh\frac{\beta}{2}-\frac{\beta}{B}\cosh\frac{\alpha}{2}\sinh\frac{\beta}{2}\right)\\
\varDelta_2 &= 2\left(\frac{\alpha}{B}\cosh\frac{\alpha}{2}\sinh\frac{\beta}{2}-\frac{\beta}{B}\sinh\frac{\alpha}{2}\cosh\frac{\beta}{2}\right)
\end{aligned}\right\} \qquad \text{(A. 2. 15)}'$$

Substituting Eq. (A. 2. 15) or Eq. (A. 2. 15)′ into expression (A. 2. 3), we obtain

$$\bar{w}=(f_{b1}+f_{b2})\bar{w}_j+(f_{b1}-f_{b2})\bar{w}_k+(f_{b3}+f_{b4})\bar{\theta}_j+(f_{b3}-f_{b4})\bar{\theta}_k \qquad \text{(A. 2. 3)}'$$

where,

(1) $\lambda^2-\kappa^2<0$

$$f_{b1}=\frac{1}{\varDelta_1}\left\{\frac{\beta}{B}\sin\frac{\beta}{2}\cosh\frac{\alpha y}{B}+\frac{\alpha}{B}\sinh\frac{\alpha}{2}\cos\frac{\beta y}{B}\right\}$$

$$f_{b2}=\frac{1}{\varDelta_2}\left\{\frac{\beta}{B}\cos\frac{\beta}{2}\sinh\frac{\alpha y}{B}-\frac{\alpha}{B}\cosh\frac{\alpha}{2}\sin\frac{\beta y}{B}\right\}$$

$$f_{b3}=\frac{1}{\varDelta_2}\left\{\sin\frac{\beta}{2}\sinh\frac{\alpha y}{B}-\sinh\frac{\alpha}{2}\sin\frac{\beta y}{B}\right\}$$

$$f_{b4}=\frac{1}{\varDelta_1}\left\{-\cos\frac{\beta}{2}\cosh\frac{\alpha y}{B}+\cosh\frac{\alpha}{2}\cos\frac{\beta y}{B}\right\}$$

(2) $\lambda^2-\kappa^2>0$

$$f_{b1}=\frac{1}{\varDelta_1}\left(-\frac{\beta}{B}\sinh\frac{\beta}{2}\cosh\frac{\alpha y}{B}+\frac{\alpha}{B}\sinh\frac{\alpha}{2}\cosh\frac{\beta y}{B}\right)$$

$$f_{b2}=\frac{1}{\varDelta_2}\left(\frac{\beta}{B}\cosh\frac{\beta}{2}\sinh\frac{\alpha y}{B}-\frac{\alpha}{B}\cosh\frac{\alpha}{2}\sinh\frac{\beta y}{B}\right)$$

$$f_{b3}=\frac{1}{\varDelta_2}\left(-\sinh\frac{\beta}{2}\sinh\frac{\alpha y}{B}-\sinh\frac{\alpha}{2}\sinh\frac{\beta y}{B}\right)$$

$$f_{b4}=\frac{1}{\varDelta_1}\left(-\cosh\frac{\beta}{2}\cosh\frac{\alpha y}{B}+\cosh\frac{\alpha}{2}\cosh\frac{\beta y}{B}\right)$$

Now, Eq. (A. 2. 3)′ may be expressed in the matrix form

$$[\psi]=[f_b]\,\{\bar{U}_b\} \qquad \text{(A. 2. 16)}$$

where

$$[f_b]=[f_{b1}+f_{b2} \quad f_{b1}-f_{b2} \quad f_{b3}+f_{b4} \quad f_{b3}-f_{b4}]$$

$$\{\bar{U}_b\}=\{\bar{w}_j\ \bar{w}_k\ \bar{\theta}_j\ \bar{\theta}_k\}$$

Substituting Eq. (A. 2. 16) into Eqs. (A. 2. 10) and (A. 2. 11), we obtain

$$\bar{M}_y= -D\left([f''_b]-\nu\left(\frac{\lambda}{B}\right)^2[f_b]\right)\{\bar{U}_b\} \qquad \text{(A. 2. 17)}$$

$$\bar{V}_y= -D\left([f'''_b]-(2-\nu)\left(\frac{\lambda}{B}\right)^2[f_b]\right)\{\bar{U}_b\} \qquad \text{(A. 2. 18)}$$

The reactive forces and the bending moments along the j- and k-side of the plate are obtained by substituting $y=-B/2$ and $y=B/2$ into the Eqs. (A. 2. 18) and (A 2. 17), as follows

$$\begin{bmatrix}\bar{V}_{yjk}\\ \bar{V}_{ykj}\\ \bar{M}_{yjk}\\ \bar{M}_{ykj}\end{bmatrix}=D\begin{bmatrix}-e' & h' & -(c'_k-\Lambda_1) & -d'\\ h' & -e' & d' & (c'_k-\Lambda_1)\\ (c'+\Lambda_2) & -d' & a' & b'\\ d' & -(c'+\Lambda_2) & b' & a'\end{bmatrix}\begin{bmatrix}\bar{w}_j\\ \bar{w}_k\\ \bar{\theta}_j\\ \bar{\theta}_k\end{bmatrix}$$

It should be noted that the positive directions of forces and displacements along the j- and k-side are shown in Fig A. 2. 2. The elements of the stiffness matrix of the Eq. (A. 2. 19) are given by the following formulas,

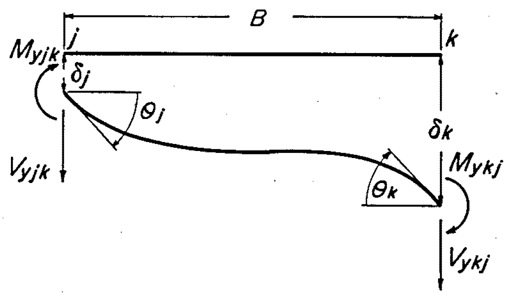

Fig . A·2·2

(1) $\lambda^2-\kappa^2<0$

$$a'=\frac{\alpha^2+\beta^2}{2B^2}\left(\frac{\sin\beta/2\ \sinh\alpha/2}{\Lambda_1}+\frac{\cos\beta/2\ \cosh\alpha/2}{\Lambda_2}\right)$$

$$b'=\frac{\alpha^2+\beta^2}{2B^2}\left(\frac{\sin\beta/2\ \sinh\alpha/2}{\Lambda_1}-\frac{\cos\beta/2\ \cosh\alpha/2}{\Lambda_2}\right)$$

$$c'=\frac{\alpha\beta}{2B^2}\left(\frac{\Lambda_2}{\Lambda_1}-\frac{\Lambda_1}{\Lambda_2}\right), \qquad d'=\frac{\alpha\beta}{2B^2}\left(\frac{\Lambda_2}{\Lambda_1}+\frac{\Lambda_1}{\Lambda_2}\right)$$

$$e'=\frac{\alpha\beta(\alpha^2+\beta^2)}{2B^4}\left(\frac{\cos\beta/2\ \cosh\alpha/2}{\Lambda_1}-\frac{\sin\beta/2\ \sinh\alpha/2}{\Lambda_2}\right)$$

$$h'=\frac{\alpha\beta(\alpha^2+\beta^2)}{2B^4}\left(\frac{\cos\beta/2\ \cosh\alpha/2}{\Lambda_1}+\frac{\sin\beta/2\ \sinh\alpha/2}{\Lambda_2}\right)$$

$$c'_k=c'+\frac{(\alpha^2-\beta^2)}{B^2},\ \Lambda_1=(2-\nu)\frac{\lambda^2}{B^2},\ \Lambda_2=\nu\frac{\lambda^2}{B^2}$$

(2) $\lambda^2-\kappa^2>0$

$$a' = \frac{\alpha^2-\beta^2}{2B^2}\left(\frac{\sinh\alpha/2 \quad \sinh\beta/2}{\Delta_2}+\frac{\cosh\alpha/2 \quad \cosh\beta/2}{\Delta_1}\right)$$

$$b' = \frac{\alpha^2-\beta^2}{2B^2}\left(\frac{\sinh\alpha/2 \quad \sinh\beta/2}{\Delta_2}+\frac{\cosh\alpha/2 \quad \cosh\beta/2}{\Delta_1}\right)$$

$$c' = \frac{\alpha\beta}{2B^2}\left(\frac{\Delta_1}{\Delta_2}+\frac{\Delta_2}{\Delta_1}\right), \qquad d' = \frac{\alpha\beta}{2B^2}\left(\frac{\Delta_1}{\Delta_2}-\frac{\Delta_2}{\Delta_1}\right)$$

$$e' = \frac{\alpha\beta(\alpha^2-\beta^2)}{2B^4}\left(\frac{\cosh\alpha/2 \quad \cosh\beta/2}{\Delta_2}+\frac{\sinh\alpha/2 \quad \sinh\beta/2}{\Delta_1}\right)$$

$$h' = \frac{\alpha\beta(\alpha^2-\beta^2)}{2B^4}\left(\frac{\cosh\alpha/2 \quad \cosh\beta/2}{\Delta_2}-\frac{\sinh\alpha/2 \quad \sinh\beta/2}{\Delta_1}\right) .$$

Now representing Eq. (A. 2. 19) as follows

$$\{\bar{P}_b\} = [k_b'] \ \{\bar{U}_b\} \tag{A. 2. 19)'$$

where

$$\{\bar{P}_b\} = \{\bar{V}_{yjk} \ \bar{V}_{ykj} \ \bar{M}_{yjk} \ \bar{M}_{ykj}\}$$

$$[k'_b] = D\begin{bmatrix} -e' & h' & -(c'_k-\Lambda_1) & -d' \\ h' & -e' & d' & (c'_k-\Lambda_1) \\ (c'+\Lambda_2) & -d' & a' & b' \\ d' & -(c'+\Lambda_2) & b' & a' \end{bmatrix}$$

$$= D\begin{bmatrix} k'_{11} & k'_{12} & k'_{13} & k'_{14} \\ k'_{21} & k'_{22} & k'_{23} & k'_{24} \\ k'_{31} & k'_{32} & k'_{33} & k'_{34} \\ k'_{41} & k'_{42} & k'_{43} & k'_{44} \end{bmatrix} \tag{A. 2. 20}$$

and each element of matrix $[k'_{ij}]$ have the following relation:

$$k'_{11}=k'_{22}, \ k'_{12}=k'_{21}, \ -k'_{13}=k'_{24}, \ k'_{33}=k'_{44},$$
$$k'_{34}=k'_{43}, \ -k'_{14}=k'_{41}=k'_{23}=-k'_{32}, \ k'_{31}=-k'_{42}$$

Further we rewrite Eq. (A. 2. 19)′ as follows

$$\left.\begin{aligned} \{\bar{P}_b\}_{jk} &= [k'_b]_{11}\{\bar{U}_b\}_{jk} + [k'_b]_{12}\{\bar{U}_b\}_{kj} \\ \{\bar{P}_b\}_{kj} &= [k'_b]_{21}\{\bar{U}_b\}_{jk} + [k'_b]_{22}\{\bar{U}_b\}_{kj} \end{aligned}\right\} \tag{A. 2. 21}$$

where

$$\{\bar{P}_b\}_{jk} = \{\bar{V}_{yjk} \ \bar{M}_{yjk}\}, \quad \{\bar{P}_b\}_{kj} = \{\bar{V}_{ykj} \ \bar{M}_{ykj}\},$$
$$\{\bar{U}_b\}_{jk} = \{\bar{w}_j \ \bar{\theta}_j\}, \quad \{\bar{U}_b\}_{kj} = \{\bar{w}_k \ \bar{\theta}_k\}$$

$$[k'_b]_{11} = D\begin{bmatrix} k'_{11} & k'_{13} \\ k'_{31} & k'_{33} \end{bmatrix}, \quad [k'_b]_{12} = D\begin{bmatrix} k'_{12} & k'_{14} \\ k'_{32} & k'_{34} \end{bmatrix}$$

$$[k'_b]_{21} = D\begin{bmatrix} k'_{21} & k'_{23} \\ k'_{41} & k'_{43} \end{bmatrix}, \quad [k'_b]_{22} = D\begin{bmatrix} k'_{22} & k'_{24} \\ k'_{42} & k'_{44} \end{bmatrix}$$

and $[k'_b]_{12} = [k'_b]^T{}_{21}$.

Appendix 3 Stiffness matrix of the beam subjected to bending moments, torsion and shearing forces.

The coordinate system to be used here is shown in Fig. A. 3. 1, where the origin of coordinates is the centroid G of cross section, but y and x axes are not always the principal axes of the cross section.

The displacement components of the shear centre S (y_S, z_S) in the x, y and z directions are represented by u, v and w, respectively.

The rotation of the cross section about the shear centre is denoted by the angle θ.

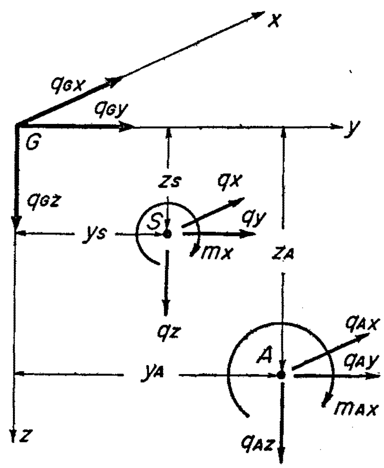

Fig. A·3·1

The components of forces per unit length acting at an arbitrary point A (y_A, z_A) on the intersection of plate and stiffener are denoted by q_{Ax}, q_{Ay}, q_{Az} and m_{Ax}, and u_A, v_A, w_A and θ_A represent the displacement components at that point.

The components of inertia forces per unit length acting at centroid G of cross section are expressed by q_{Gx}, q_{Gy} and q_{Gz}, as shown in Fig. A. 3. 1. We obtain the components of forces per unit length acting at shear centre S, q_x, q_y, q_z and m_x, as follows

$$\left.\begin{aligned} q_x\ (x,t) &= q_{Ax}(x,t) + q_{Gx}(x,t) \\ q_y\ (x,t) &= q_{Ay}(x,t) + q_{Gy}(x,t) \\ q_z\ (x,t) &= q_{Az}(x,t) + q_{Gz}(x,t) \\ m_x(x,t) &= m_{Ax}(x,t) - (z_A - z_S)q_{Ay}(x,t) \\ &\quad + (y_A - y_S)q_{Az}(x,t) + m_{sx}(x,t) \end{aligned}\right\} \qquad \text{(A. 3. 1.)}$$

Now, we start with the statical state of a thin-walled elastic beam subjected to bending moment, torsion and shearing forces, as shown Fig. A. 3. 2, and the differential equations of a beam in such load conditions have been induced by the author [26], as follows

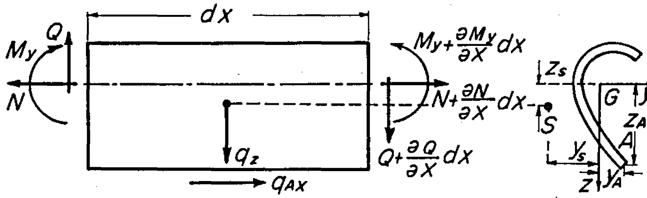

Fig. A•3•2

$$\frac{\partial q_{Ax}}{\partial x} = B_1 \frac{\partial^3 u_A}{\partial x^3} + B_z q_z + B_y q_y \tag{A. 3. 2}$$

$$\frac{\partial^4 v_A}{\partial x^4} EI_z + \frac{\partial^4 w_A}{\partial x^4} EI_{yz} + \{(z_A - z_S) EI_z - (y_A - y_S) EI_{yz}\} \frac{\partial^4 \theta_A}{\partial x^4} - B_1 y_A \frac{\partial^3 u_A}{\partial x^3}$$
$$= B_z y_z q_z + (1 + B_y y_A) q_y \tag{A. 3. 3}$$

$$\frac{\partial^4 w_A}{\partial x^4} EI_y + \frac{\partial^4 v_A}{\partial x^4} EI_{yz} + \{(z_A - z_S) EI_{yz} - (y_A - y_S) EI_y\} \frac{\partial^4 \theta}{\partial x^4} - B_1 z_A \frac{\partial^3 u_A}{\partial x^3}$$
$$= B_y z_A q_y + (1 + B_z z_A) q_z \tag{A. 3. 4}$$

$$\frac{\partial^4 \theta_A}{\partial x^4} C_1 - \frac{\partial^2 \theta_A}{\partial x^2} C = m_{Ax} - (z_A - z_S) q_x + (y_A - y_S) q_z \tag{A. 3. 5}$$

In these expressions I_y and I_z represent the moment of inertia of the cross section about y and z axes, respectively, and I_{yz} represents the product of inertia of the cross section and C_1 and C are the warping and torsional rigidity, and

$$\left.\begin{aligned} B_1 &= \frac{-EA_s (I_z I_y - I^2_{yz})}{(I_z I_y - I^2_{yz}) + A_s (I_z z^2_A - 2I_{yz} y_A z_A + I_y y^2_A)} \\ B_y &= \frac{-A_s (I_y y_A - I_{yz} z_A)}{(I_z I_y - I^2_{yz}) + A_s (I_z z^2_A - 2I_{yz} y_A z_A + I_y y^2_A)} \\ B_z &= \frac{-A_s (I_y z_A - I_{yz} y_A)}{(I_z I_y - I^2_{yz}) + A_s (I_z z^2_A - 2I_{yz} y_A z_A + I_y y^2_A)} \end{aligned}\right\} \tag{A. 3. 6}$$

The transition from the statical differential equations (A. 3. 2), (A. 3. 3), (A. 3. 4) and (A. 3. 5) to the dynamical equation of motion may be made by equating q_y, q_z, m_x to the intensity of the inertia forces in the beam, but the longitudinal component of the inertia force may be neglected, on the assumption of the previous section. Considering a bar with arbitrary cross section, Fig. A. 3. 3, we denote the displacement component at any point B (y_B, z_B) on the cross section, v_B , w_B and θ_B , the deflection of the point B are expressed as follows

$$\left.\begin{aligned} v_B &= v - (z_B - z_S)\,\theta \\ w_B &= w + (y_B - y_S)\,\theta \end{aligned}\right\} \quad (A.3.7)$$

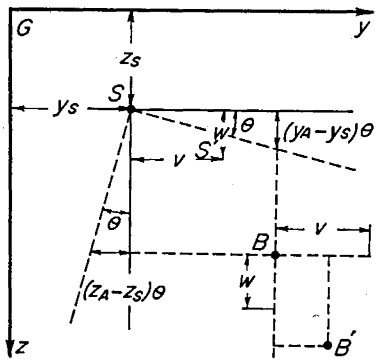

Fig. A·3·3

Using Eq. (A. 3. 7) the components of the inertia force acting on the point B on the cross section are expressed as follows [27]

$$p_y = -\frac{\gamma t_s}{g}\frac{\partial^2 v_B}{\partial t^2} = -\frac{\gamma t_s}{g}\left\{\frac{\partial^2 v}{\partial t^2} - (z_B - z_S)\frac{\partial^2\theta}{\theta t^2}\right\} \quad (A.3.8)$$

$$p_z = -\frac{\gamma t_s}{g}\frac{\partial^2 w_B}{\partial t^2} = -\frac{\gamma t_s}{g}\left\{\frac{\partial^2 w}{\partial t^2} + (y_B - y_S)\frac{\partial^2\theta}{\partial t^2}\right\} \quad (A.3.9)$$

where γ is the weight of material of the stiffener per unit volume and t_s is the thickness of the stiffener wall.

Integrating Eq. (A. 3. 8) and (A. 3. 9) along the centre line of the cross section, we obtain the inertia forces acting on the centre of gravity of the cross section

$$q_{Gy} = \int_s p_y ds = -m_{st}\left(\frac{\partial^2 v}{\partial t^2} - z_S\frac{\partial^2\theta}{\partial t^2}\right) \quad (A.3.10)$$

$$q_{Gz} = \int_s p_z ds = -m_{st}\left(\frac{\partial^2 w}{\partial t^2} - y_S\frac{\partial^2\theta}{\partial t^2}\right) \quad (A.3.11)$$

where $m_{st} = \gamma A_s/g$ is the mass per unit length of the stiffener. The inertia moment of the element m_{sx} about the shear centre S of the cross section is

$$m_{sx} = \int_s \{-p_y(z_B - z_S) + p_z(y_B - y_S)\,ds\}$$

$$= -m_{st}\left(z_S\frac{\partial^2 v}{\theta t^2} - y_S\frac{\partial^2 w}{\partial t^2}\right) - (\Theta_G + r^2{}_S m_{st})\frac{\partial^2\theta}{\partial t^2} \quad (A.3.12)$$

where

$$\Theta_G = \int \frac{\gamma}{A_s} (z^2_B + y^2_B)\, dA_s, \quad r^2_S = z^2_S + y^2_S$$

Replacing the suffix B with A in Eqs. (A. 3. 7), we obtain the expressions between the displacement of shear centre S and the point A which is the intersection of plate and stiffener, as follows

$$\begin{aligned} v &= v_A + (z_A - z_S)\, \theta_A \\ w &= w_A - (y_A - y_S)\, \theta_A \end{aligned} \qquad \text{(A. 3. 13)}$$

Substituting the expressions (A. 3. 13) into Eqs. (A. 3. 10), (A. 3. 11) and (A. 3. 12), we have

$$q_{Gy} = -\, m_{st} \left(\frac{\partial^2 v_A}{\partial t^2} + z_A \frac{\partial^2 \theta_A}{\partial t^2} \right) \qquad \text{(A. 3. 14)}$$

$$q_{Gz} = -\, m_{st} \left(\frac{\partial^2 w_A}{\partial t^2} - y_A \frac{\partial^2 \theta_A}{\partial t^2} \right) \qquad \text{(A. 3. 15)}$$

$$m_{sx} = -\, m_{st} \left(z_A \frac{\partial^2 v_A}{\partial t^2} - \frac{y_A \partial^2 w_A}{\partial t^2} \right) - \Theta_A \frac{\partial^2 \theta_A}{\partial t^2} \qquad \text{(A. 3. 16)}$$

where

$$\Theta_A = \Theta_G + m_{st}\, (z^2_A + y^2_A)$$

Substituting Eqs. (A. 3. 14), (A. 3. 15) and (A. 3. 16) into Eq. (A. 3. 1), we obtain

$$\left.\begin{aligned} q_{Ax}\,(x, t) &= q_x\,(x, t) \\ q_{Ay}\,(x, t) &= q_y\,(x, t) + m_{st} \left(\frac{\partial^2 v_A}{\partial t^2} - z_A \frac{\partial^2 \theta_A}{\partial t^2} \right) \\ q_{Az}\,(x, t) &= q_z\,(x, t) + m_{st} \left(\frac{\partial^2 w_A}{\partial t^2} - y_A \frac{\partial^2 \theta_A}{\partial t^2} \right) \\ m_{Ax}(x, t) &= m_x(x, t) + m_{st} \left(z_A \frac{\partial^2 v_A}{\partial t^2} - y_A \frac{\partial^2 w_A}{\partial t^2} \right) + \Theta_A \frac{\partial^2 \theta_A}{\partial t^2} \end{aligned}\right\} \qquad \text{(A. 3. 17)}$$

Eliminating q_y, q_z, m_x from Eqs. (A. 3. 2), (A. 3. 3), (A. 3. 4), (A. 3. 5) and (A. 3. 17), we find the following expressions for the components of the dynamical forces per unit length acting on the intersection of plate and stiffener:

$$
\begin{pmatrix} \dfrac{\partial q_{Ax}}{\partial x} \\ q_{Ay} \\ q_{Az} \\ m_{Ax} \end{pmatrix}
=
\begin{pmatrix} -B_{xx} & -B_{xy} & -B_{xz} & -B_{x\theta} \\ B_{yx} & B_{yy} & B_{yz} & B_{y\theta} \\ B_{zx} & B_{zy} & B_{zz} & B_{z\theta} \\ B_{\theta x} & B_{\theta y} & B_{\theta z} & B_{\theta\theta} \end{pmatrix}
\begin{pmatrix} \dfrac{\partial^3 u_A}{\partial x^3} \\ \dfrac{\partial^4 v_A}{\partial x^4} \\ \dfrac{\partial^4 w_A}{\partial x^4} \\ \dfrac{\partial^4 \theta_A}{\partial x^4} \end{pmatrix}
+
\begin{pmatrix} O \\ O \\ O \\ -C\dfrac{\partial^2 \theta_A}{\partial x^2} \end{pmatrix}
$$

$$
+ \begin{pmatrix} O & O & O & O \\ O & m_{st} & O & z_A m_{st} \\ O & O & m_{st} & -y_A m_{st} \\ O & z_A m_{st} & -y_A m_{st} & \Theta_A \end{pmatrix}
\begin{pmatrix} \dfrac{\partial^2 u_A}{\partial t^2} \\ \dfrac{\partial^2 v_A}{\partial t^2} \\ \dfrac{\partial^2 w_A}{\partial t^2} \\ \dfrac{\partial^2 \theta_A}{\partial t^2} \end{pmatrix}
\qquad \text{(A. 3. 18)}
$$

where

$$
\begin{aligned}
B_{xx} &= EA_s \\
B_{xy} &= B_{yx} = EA_s y_A \\
B_{xz} &= B_{zx} = EA_s z_A \\
B_{x\theta} &= B_{\theta x} = EA_s \{(z_A - z_S)\, y_A - (y_A - y_S)\, z_A\} \\
B_{yy} &= E\,(I_z + A_s y^2{}_A) \\
B_{yz} &= B_{zy} = E\,(I_{yz} + A_s y_A z_A) \\
B_{y\theta} &= B_{\theta y} = E\,\{(z_A - z_S)\,(I_z + A_s y^2{}_A) - (y_A - y_S)\,(I_{yy} + A_s y_A z_A)\} \\
B_{zz} &= E\,(I_y + A_s z^2{}_A) \\
B_{z\theta} &= B_{\theta z} = E\,\{(z_A - z_S)\,(I_{yz} + A_s y_A z_A) - (y_A - y_S)\,(I_y + A_s z^2{}_A)\} \\
B_{\theta\theta} &\doteq C_1 + E\,\{(z \;\; - z_S)^2\,(I_z + A_s y^2{}_A) + (y_A - y_S)^2\,(I_y + A_s z^2{}_A) \\
&\quad - 2\,(z_A - z_S)\,(y_A - y_S)\,(I_{yz} + A_s z_A z_S)\}
\end{aligned}
$$

On the previous assumption, the ends of the stiffener are free to warp and rotate with respect to y- and z-axes instead of rotating around x-axis. In such a case, the end conditions are

$$
\left.
\begin{aligned}
& v_A = w_A = \theta_A = 0 \\
& \frac{\partial^2 v_A}{\partial x^2} = \frac{\partial^2 w_A}{\partial x^2} = \frac{\partial^2 \theta}{\partial x^2} = 0
\end{aligned}
\right\} \quad \text{for } x = 0 \text{ and } x = L \;.
$$

All these conditions will be satisfied by taking the solution of Eqs. (A. 3. 18) in the form

$$\left.\begin{aligned} u_A &= \bar{u}_A \cos \frac{n\pi}{L} x\, e^{i\omega t} \\ v_A &= \bar{v}_A \sin \frac{n\pi}{L} x\, e^{i\omega t} \\ w_A &= \bar{w}_A \sin \frac{n\pi}{L} x\, e^{i\omega t} \\ \theta_A &= \bar{\theta}_A \sin \frac{n\pi}{L} x\, e^{i\omega t} \end{aligned}\right\} \tag{A. 3. 19}$$

and

$$\left.\begin{aligned} q_{Ax} &= \bar{q}_{Ax} \cos \frac{n\pi}{L} x\, e^{i\omega t} \\ q_{Ay} &= \bar{q}_{Ay} \sin \frac{n\pi}{L} x\, e^{i\omega t} \\ q_{Az} &= \bar{q}_{Az} \sin \frac{n\pi}{L} x\, e^{i\omega t} \\ m_x &= \bar{m}_x \sin \frac{n\pi}{L} x\, e^{i\omega t} \end{aligned}\right\} \tag{A. 3. 20}$$

Substituting the expressions (A. 3. 19) and (A. 3. 20) into Eq. (A. 3. 18), we obtain

$$\begin{bmatrix} \bar{q}_{Ax} \\ \bar{q}_{Ay} \\ \bar{q}_{Az} \\ \bar{m}_x \end{bmatrix} = \begin{bmatrix} B_{xx}\,(\lambda/B)^2 & B_{xy}\,(\lambda/B)^3 & B_{xz}\,(\lambda/B)^3 & B_{x\theta}\,(\lambda/B)^3 \\ B_{yx}\,(\lambda/B)^3 & B_{yy}\,(\lambda/B)^4 & B_{yz}\,(\lambda/B)^4 & B_{y\theta}\,(\lambda/B)^4 \\ B_{zx}\,(\lambda/B)^3 & B_{zy}\,(\lambda/B)^4 & B_{zz}\,(\lambda/B)^4 & B_{z\theta}\,(\lambda/B)^4 \\ B_{\theta x}\,(\lambda/B)^3 & B_{\theta y}\,(\lambda/B)^4 & B_{\theta z}\,(\lambda/B)^4 & B_{\theta\theta}\,(\lambda/B)^4 + C\,(\lambda/B)^2 \end{bmatrix} \times$$

$$\times \begin{bmatrix} \bar{u}_A \\ \bar{v}_A \\ \bar{w}_A \\ \bar{\theta}_A \end{bmatrix} + \begin{bmatrix} O & O & O & O \\ O & -(D/B^3)K_M\kappa^4 & O & -(D/B^2)K_{Gz}\kappa^4 \\ O & O & -(D/B^3)K_M\kappa^4 & (D/B^2)K_{Gy}\kappa^4 \\ O & -(D/B^2)K_{Gz}\kappa^4 & (D/B^2)K_{Gy}\kappa^4 & -(D/B)K_R\kappa^4 \end{bmatrix} \begin{bmatrix} \bar{u}_A \\ \bar{v}_A \\ \bar{x}_A \\ \bar{\theta}_A \end{bmatrix} \tag{A. 3. 21}$$

where

$$K_M = \frac{m_{st}}{\left(\frac{\gamma t_p}{g}\right) B}, \qquad K_{Gz} = \frac{z_A m_{st}}{\left(\frac{\gamma t_p}{g}\right) B^2},$$

$$K_{Gy} = \frac{y_A m_{st}}{\left(\frac{\gamma t_p}{g}\right) B^2}, \qquad K_R = \frac{\Theta_A}{\left(\frac{\gamma t_p}{g}\right) B^3}$$

We rewrite Eq. (A. 3. 21) into a simple matrix form like

$$\{\bar{Q}_s\} = [k_s]\,\{\bar{U}_A\} \tag{A. 3. 22}$$

where

$$\{\bar{Q}_s\} = \{\bar{q}_{Ax} \quad \bar{q}_{Ay} \quad \bar{q}_{Az} \quad \bar{m}_{Ax}\}$$

$$\{\bar{U}_A\} = \{\bar{u}_A \quad \bar{v}_A \quad \bar{w}_A \quad \bar{\theta}_A\}$$

$$[k_s] = [\boldsymbol{B}+\boldsymbol{D}] = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22}+D_{22} & B_{23} & B_{24}+D_{24} \\ B_{31} & B_{32} & B_{33}+D_{33} & B_{34}+D_{34} \\ B_{41} & B_{42} & B_{43}+D_{43} & B_{44}+D_{44} \end{bmatrix} \tag{A. 3. 23}$$

The matrix given by Eq. (A. 3. 23) is a stiffness matrix for stiffener, and the elements B_{ij} $(i = 1, 2, \cdots 4, j = 1, 2, \cdots 4)$ show the geometrical properties of cross section of stiffener and D_{ij} means the dynamical properties of stiffener associated with plate.