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CALCULATIONS OF WATER LEVEL IN A TIDAL LAKE*

By

Jun-ichi OKABE**

The time-variation of the water level in a tidal lake is governed approximately by a pair of the differential equations

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = B\sqrt{2g}\sqrt{H-h} \qquad \text{(when } H \geqq h\text{)},$$

and

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = -B\sqrt{2g}\sqrt{h-H} \qquad \text{(when } h \geqq H\text{)},$$

where A and B denote respectively the area of the lake, and the effective area of its mouth leading to the outer sea, i.e. breadth $\times$ depth of the mouth $\times$ some constant representing *vena contracta*. A and B are assumed invariable, irrespective of the change of the water level. H is the level of the outer sea, h that of the lake, g the acceleration due to gravity, and t the time. Assuming H is sinusoidal with regard to t, we can transform the equations into the non-dimensional forms

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\sigma} = \frac{\pi}{2}\rho\sqrt{\cos\frac{\pi}{2}\sigma-\zeta} \qquad \text{(when } \cos\frac{\pi}{2}\sigma \geqq \zeta\text{)},$$

and

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\sigma} = -\frac{\pi}{2}\rho\sqrt{\zeta-\cos\frac{\pi}{2}\sigma} \qquad \text{(when } \zeta \geqq \cos\frac{\pi}{2}\sigma\text{)},$$

where ζ and σ come from h and t respectively and ρ is a certain constant.

Integral curves are worked out numerically starting from five kinds of initial values for the case when $\rho = 0.5$. We can observe that the curves converge steadily to a final form independent of initial values. It is concluded that in order to calculate accurately the periodic solution of the pair of the fundamental differential equations mentioned above, which we define as the exact solution of our equations, we have only to start from an arbitrary value and integrate the equations numerically by, for example, RUNGE and KUTTA's method. In due course of time we shall arrive at a state that the solution is practically periodic having the period 4 in terms of the variable σ.

Turning to the approximate solution, the method of moments are employed so as to satisfy the condition that the zeroth order moment of the pair of the governing differential equations is zero as well as the condition at

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the crest of ζ. It is proved that the solution constructed in this manner makes also the first order moment vanish automatically.

Finally for the cases when $\rho = 1.0$, 0.5, and 0.25, which correspond to $A/B = 3.04$, 6.09 and 12.18 $(\times 10^4)$ respectively, supposing that the amplitude and the period of the undulation of the outer sea are 1 m and 12 hours respectively, the *exact*, and the *approximate*, solutions are worked out and compared with each other. It is concluded that agreement between them is quite satisfactory.

In the appendix, a system of the differential equations describing the up-and-down motion of the water level in a double-tidal lake which has an outer and an inner seas on both sides. For a particular case, the solution is shown to converge once again to a limiting periodic function.

1. Equation of motion of water level

Assuming A, the area of a tidal lake, is kept invariable in the course of an up-and-down motion of the water level within it, let $h(t)$ and $H(t)$ be used to denote the water level of the lake and that of the outer sea, respectively, measured from the mean level, t being the time; see Figure 1. Further, let b, d, and e stand respectively for the breadth, the depth, and the coefficient of efflux, of the mouth of the lake. They are naturally supposed to vary with every change of h and of H, but let us consider for simplicity only the case when the product $b \cdot d \cdot e$ might be approximated to a sufficient accuracy by a constant B, the so-called 'effective area' of the mouth, on the understanding that the variable portion of the product be so small compared with the invariable. Finally, we shall neglect the time of propagation of the current, both inward and outward, in the lake, and so it will be presumed

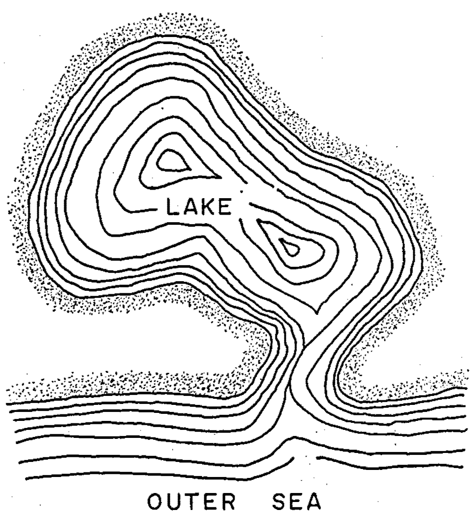

Fig. 1. Tidal lake.

in the following sections that the level moves upward or downward uniformly throughout the whole area of the lake.

After all of these simplifications, let us suppose that the motion of the water level in the lake is governed by a pair of the differential equations

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = B\sqrt{2g}\sqrt{H-h} \quad \text{when} \quad H \geqq h,$$

and

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = -B\sqrt{2g}\sqrt{h-H} \quad \text{when} \quad h \geqq H, \tag{1. 1}$$

g being the acceleration due to gravity. At the instant when $h = H$, h takes the value maximum or minimum and concurrently the equation changes its form from one to the other. Speaking specifically, this is the characteristic of this problem, and at the same time forms the first obstacle.

The second difficulty, on the other hand, lies in the following circumstances. When $H(t)$ is given as a known function of t, we have to solve this equation, or more precisely, the pair of these equations, in order to evaluate h for various values of t. However, in doing so we cannot designate the initial condition at $t = 0$. Instead, all we know beforehand is that the function h which we are now striving for should be a function having the same period as that of H, the tidal level of the outer sea. Speaking mathematically, we are now interested solely in the periodic solution of the differential equation (1. 1). *

Choosing a typical length L of the problem, let us introduce non-dimensional quantities η, ζ, τ, and κ defined respectively by

$$H \equiv L\eta, \quad h \equiv L\zeta, \tag{1. 2}$$

and

$$t \equiv \sqrt{L}\,\tau/\sqrt{g}, \qquad B/A \equiv \kappa/\sqrt{2}, \tag{1. 3}$$

then (1. 1) may be transformed into the forms

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = \kappa\sqrt{\eta-\zeta} \quad \text{when} \quad \eta \geqq \zeta,$$

and

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = -\kappa\sqrt{\zeta-\eta} \quad \text{when} \quad \zeta \geqq \eta. \tag{1. 4}$$

Since H is supposed to increase and decrease sinusoidally with the period of 12 hours, we may write approximately, leaving terms of higher harmonics out of consideration,

$$\eta = \cos\alpha\tau, \tag{1. 5}$$

provided we choose the amplitude of the tide in the outer sea as L and imagine that τ would be measured from an appropriate origin of time, α being some constant. Then by putting

$$\alpha\tau \equiv \frac{\pi}{2}\sigma, \quad \text{and} \quad \frac{\kappa}{\alpha} \equiv \rho, \tag{1. 6}$$

* In what follows, the term the 'equation' or the 'differential equation' will be frequently employed for convenience to designate the 'pair of differential equations'.

(1.5) and (1.4) may be rewritten respectively as follows:

$$\eta = \cos\frac{\pi}{2}\sigma \tag{1.7}$$

$$\frac{d\zeta}{d\sigma} = \frac{\pi}{2}\rho\sqrt{\cos\frac{\pi}{2}\sigma-\zeta} \quad \text{when} \quad \cos\frac{\pi}{2}\sigma \geqq \zeta,$$

and

$$\frac{d\zeta}{d\sigma} = -\frac{\pi}{2}\rho\sqrt{\zeta-\cos\frac{\pi}{2}\sigma} \quad \text{when} \quad \zeta \geqq \cos\frac{\pi}{2}\sigma. \tag{1.8}$$

The purpose of this note is twofold : *first,* to show a method of computing exactly the periodic solution of our differential equation (1.8), and *second,* to provide those who want to treat the problem at the expense of the least labor possible with some practical informations on construction and accuracy of the approximate solution. On the other hand, however, it is not our aim primarily at present to scrutinize validity or accuracy of assumptions concerning physical phenomena underlying equation (1.8).

2. Exact solutions

In proceeding to integrate equation (1.8) numerically starting from $\sigma = 0$, the appropriately chosen origin of time, we are confronted with the difficulty as mentioned already: we are not in a position to designate the initial value of ζ, $\zeta(0)$ say, in advance; on the contrary $\zeta(0)$ has to be determined in such a way that the integral curve emerging from it would be a periodic function of σ, the non-dimensional time, the period being 4 in terms of it. In other words, if we should start from an arbitrary $\zeta(0)$, we could not expect necessarily to come back to the original value after carrying out the integration throughout one period from $\sigma = 0$ to 4.

It is not the object of this note to study mathematically the periodic property of equation (1.8) and the environmental condition under which the periodicity of the solution is possible.[1] Only, it is found out empirically that, as far as our examples are concerned (see section 4 of this paper), if we perform numerical integration of the equation starting from a value, assumed somewhat arbitrarily (as to a choice of $\zeta(0)$, a proposition is made in the same section), an integral curve is observed to converge gradually but steadily to a limiting curve having the period of 4; which we define as the exact solution of the problem. * In fact, after a few of undulations of ζ, or, speaking more precisely, after working out the integral curve over a few of crests and troughs throughout, we can notice that a value of ζ on the curve, $\zeta(\sigma_0)$ say, does not differ sensibly from that at which we arrive through numerical integration covering one more period, i.e. $\zeta(\sigma_0+4)$. In this way, in a

* It would be pertinent to add here in order to avoid unnecessary confusions in terminology that as the word 'exact' we mean the exact solution of equation (1.8) and do not necessarily the accurate answer of the whole problem of calculating the water level in a tidal lake.

practical sense at least, we are enabled to calculate the exact solution of the equation (1. 8) with an accuracy as much as we please. The numerical discussions will be found in section 4.

With a view to visualizing the convergence of integral curves, Figure 2 and

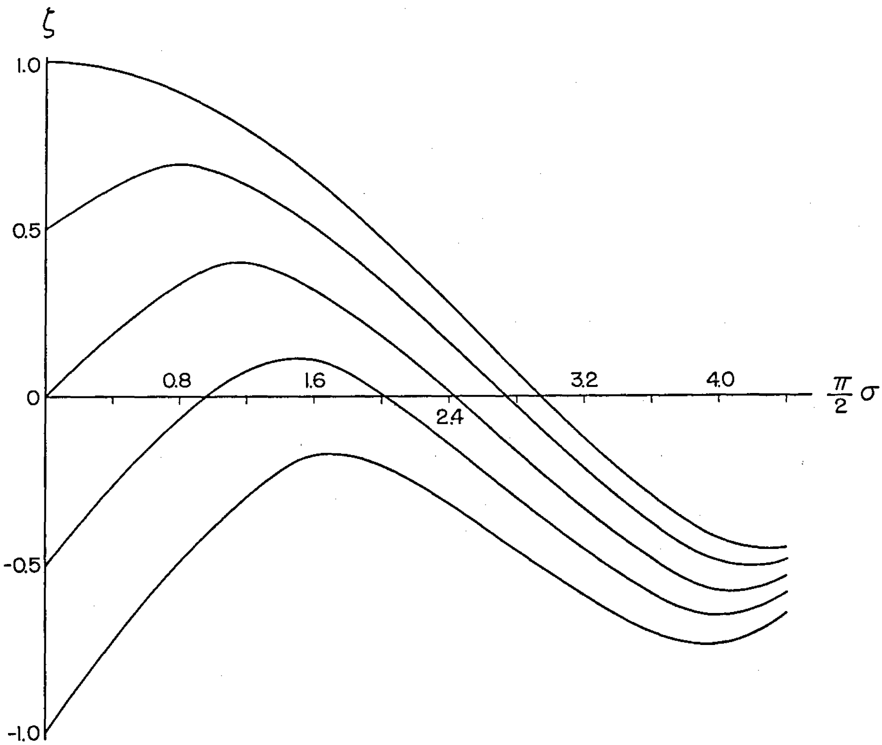

Fig. 2. Five integral curves of equation (1. 8).

Table 1 were both prepared which show the behavior of the solutions of equation (1. 8) originating from five kinds of the initial value ($\zeta(0) = 1.0$, 0.5, 0, -0.5 and -1.0) for the case when $\rho = 0.5$. It will be readily observed that the difference between the largest and the smallest solutions (in other words, A$-$E) reduces to 1/10 of its initial value after the lapse of time $\pi\sigma/2 = 4.4$.

3. Approximate solutions

Turning now to an approximate solution of equation (1. 8), since it would be most plausible to assume, from a physical point of view, that ζ is a function having the same period as that of η, let us put

$$\zeta = \lambda \cos\left(\frac{\pi}{2}\sigma - \mu\right), \tag{3.1}$$

where λ and μ are both constants, denoting respectively the amplitude, and the time-lag, of the tidal motion in the lake. * We might justify the expression (3.1) partly by analyzing ζ into FOURIER series and neglecting the higher terms upon the supposition that they are dominated by the leading term. From equation (1.8), we should have

$$-\lambda \sin\left(\frac{\pi}{2}\sigma - \mu\right) \sim \rho \sqrt{\cos\frac{\pi}{2}\sigma - \lambda\cos\left(\frac{\pi}{2}\sigma - \mu\right)} \tag{3.2}$$

when
$$\cos\frac{\pi}{2}\sigma \geqq \lambda \cos\left(\frac{\pi}{2}\sigma - \mu\right), \tag{3.3}$$

and alternatively

$$\lambda \sin\left(\frac{\pi}{2}\sigma - \mu\right) \sim \rho \sqrt{\lambda\cos\left(\frac{\pi}{2}\sigma - \mu\right) - \cos\frac{\pi}{2}\sigma} \tag{3.4}$$

in the range
$$\lambda\cos\left(\frac{\pi}{2}\sigma - \mu\right) \geqq \cos\frac{\pi}{2}\sigma. \tag{3.5}$$

In so far as the expression (3.1) is not the exact solution of the differential equation, neither (3.2) nor (3.4) cannot be identities; with a view to making it clear we have made use of $\sim$ in place of $=$. All we can do is to determine the two constants λ and μ in such a way that the both-hand sides of (3.2) as well as of (3.4) would be as close as possible in the whole domain of σ ranging from 0 to 4. Incidentally, by 'as close as possible' we mean that the equality relations in both expressions would hold good at points as many as possible in the pertaining interval of σ and besides that even at places where the equality is invalidated the differences between the both-hand sides would be as small as possible. Now, however, putting

$$\sigma = 2 + \sigma' \tag{3.6}$$

in (3.4) and (3.5), we have

$$-\lambda \sin\left(\frac{\pi}{2}\sigma' - \mu\right) \sim \rho \sqrt{\cos\frac{\pi}{2}\sigma' - \lambda\cos\left(\frac{\pi}{2}\sigma' - \mu\right)}, \tag{3.7}$$

and
$$\cos\frac{\pi}{2}\sigma' \geqq \lambda\cos\left(\frac{\pi}{2}\sigma' - \mu\right), \tag{3.8}$$

respectively, and so we know that the values of λ and μ determined by the condition (3.2) should be the same as those which are to be obtained from (3.4). Accordingly, let us confine ourselves only to condition (3.2) in what follows.

Now that we want to compute the values of λ and μ, two conditions

* It should be noted that, in view of equation (1.5), λ is the non-dimensional amplitude defined as a fraction of that of the undulation of the outer sea.

become necessary. Of course it is possible in principle to assume ζ in a form containing a number of parameters more than two by taking terms of higher harmonics into consideration and to determine all of them by the aid of the same number of conditions, but then the amount of necessary calculations will become formidable, and we think it would be more advisable to take the alternative explained already in the preceding section in some detail when such a high degree of accuracy is required.

Construction of the approximate solution is outlined in the following three paragraphs:—

i) First, as stated already, the water level inside the lake is known to reveal its maxima or minima when both of the levels of the outer sea and of the lake become equal to each other: namely since the left-hand side of (3.2) vanishes at the instant

$$\frac{\pi}{2}\sigma = \mu, \tag{3.9}$$

the differential equation will be satisfied at the same instant by the function (3.1), provided that the relation

$$\lambda = \cos\mu \tag{3.10}$$

exists between λ and μ. Further, it will be clear from this equality that

$$1 \geqq \lambda, \tag{3.11}$$

which means that the amplitude of the tidal motion in the lake should be smaller than that of the outer sea, as is naturally understood from a physical point of view. Of course an arbitrary number of similar conditions could be supplemented, if we required that the differential equation should be satisfied at various values of σ appropriately chosen, but we are not going to take this way.

ii) In order to derive the second condition, let us employ the 'method of moments' which was invented originally by YAMADA and used successfully by himself and other people toward solving a number of problems.[2] Taking the square of both-hand sides of (3.2) for convenience of sebsequent computations, we write

$$\lambda^2 \sin^2(\frac{\pi}{2}\sigma-\mu) \sim \rho^2 \sqrt{1-2\lambda\cos\mu+\lambda^2}\,\sin(\frac{\pi}{2}\sigma+\nu), \tag{3.12}$$

where

$$\sin\nu \equiv \frac{1-\lambda\cos\mu}{\sqrt{1-2\lambda\cos\mu+\lambda^2}}, \text{ and } \cos\nu \equiv \frac{-\lambda\sin\mu}{\sqrt{1-2\lambda\cos\mu+\lambda^2}}. \tag{3.13}$$

Since the range of σ in which the inequality (3.3) holds good is given by

$$2-\frac{2}{\pi}\nu \geqq \sigma \geqq -\frac{2}{\pi}\nu, \tag{3.14}$$

we shall integrate the both-hand sides of (3.12) with respect to σ from $-2\nu/\pi$

to $2-2\nu/\pi$, or one half of the period of the motion of the water level, and equate to each other, with a view to making our differential equation satisfied 'in the mean'. The relation obtained in this way should be less stringent than the original differential equation. However, at any rate, it assures that the residual R defined as

$$R \equiv \lambda^2 \sin^2(\frac{\pi}{2}\sigma-\mu)-\rho^2 \left\{\cos\frac{\pi}{2}\sigma-\lambda\cos(\frac{\pi}{2}\sigma-\mu)\right\}$$

$$= \lambda^2 \sin^2(\frac{\pi}{2}\sigma-\mu)-\rho^2\sqrt{1-2\lambda\cos\mu+\lambda^2}\ \sin(\frac{\pi}{2}\sigma+\nu) \tag{3.15*}$$

would change its sign at least once in the range given by (3.14) and that moreover the area (the integral, in other words) of the positive R would be equal to that of the negative R. Upon carrying out the integration, we arrive without difficulty at the result

$$\pi\lambda^2 = 4\rho^2\sqrt{1-2\lambda\cos\mu+\lambda^2}, \tag{3.16**}$$

or

$$\frac{\pi^2}{16\rho^4}\lambda^4+2\lambda\cos\mu-\lambda^2-1 = 0. \tag{3.17}$$

This is taken as the second condition. Substituting (3.10) into (3.17), we are led to the equation

$$k\lambda^4+\lambda^2-1 = 0, \tag{3.18}$$

where we write

$$k \equiv \pi^2/16\rho^4. \tag{3.19}$$

From equation (3.18), we readily obtain

$$\lambda = \sqrt{\frac{\sqrt{1+4k}-1}{2k}}, \tag{3.20}$$

and so from (3.10) we have also

$$\mu = \cos^{-1}\sqrt{\frac{\sqrt{1+4k}-1}{2k}}. \tag{3.21}$$

In order to have a general idea of the approximate solution constructed in this manner, let us consider the limiting cases

$$k\to 0, \quad \text{and} \quad k\to\infty. \tag{3.22}$$

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* In the range where

$$\lambda\cos(\frac{\pi}{2}\sigma-\mu) \geqq \cos\frac{\pi}{2}\sigma, \quad \text{or} \quad 4-\frac{2}{\pi}\nu \geqq \sigma \geqq 2-\frac{2}{\pi}\nu$$

is valid, we put

$$R \equiv \lambda^2 \sin^2(\frac{\pi}{2}\sigma-\mu)-\rho^2\left\{\lambda\cos(\frac{\pi}{2}\sigma-\mu)-\cos\frac{\pi}{2}\sigma\right\}$$

$$= \lambda^2 \sin^2(\frac{\pi}{2}\sigma-\mu)+\rho^2\sqrt{1-2\lambda\cos\mu+\lambda^2}\ \sin(\frac{\pi}{2}\sigma+\nu). \tag{3.15'}$$

** The same result is obtained by integrating R given by the preceding footnote from $2-2\nu/\pi$ to $4-2\nu/\pi$.

Combining (1.3), (1.6), and (3.19), we know that these cases correspond to the extremities when the effective area of the mouth is very large and negligibly small respectively compared with the area of the lake itself. For each of these cases we have from (3.20) and (3.21) that

$$\lambda \to 1, \quad \mu \to 0 \quad \text{as} \quad k \to 0,$$
and
$$\lambda \to 0, \quad \mu \to \pi/2 \quad \text{as} \quad k \to \infty. \tag{3.23}$$

Namely, in the former the amplitude of the tidal motion of the water level in a lake is found to be identical with that of the outer sea and at the same time no phase lag is observed. On the contrary, in the latter, the up-and-down motion of the level in a lake is negligible and also the phase lag $\pi/2$ takes place. All of these might be mentioned quite plausible. Furthermore, with a view to gaining a better understanding of behaviors of λ, μ, and A/B, these three quantities are computed for various values of the parameter k. The result is reproduced in Figure 3, λ, μ, and $\sqrt{LA/B}$ (L being the amplitude measured in m of the tidal motion of the outer sea) on the ordinate and k on the abscissa.

iii) The method of moments, in general, consists in the following computation procedure. Confining ourselves to the present problem, let us assume in the first place that the unknown variable ζ be expressed approximately by the form containing N indeterminate constants. Substitute this expression in the differential equation whose solution we are now striving for, and formulate the remainder R defined by the difference between the both-hand sides; see (3.15). Multiply R by σ^i ($i = 0, 1, 2, \ldots, N-1$) (or more generally, a sequence of orthogonal functions) and integrate it with respect to σ from $-2\nu/\pi$ to $2-2\nu/\pi$ (the range of the independent variable pertinent to the differential equation). From N relationships thus obtained, we are now enabled to determine N constants, and accordingly also the approximate expression for ζ. By increasing N indefinitely, we shall arrive at the exact solution of our problem. In ii) we took $i = 0$ (the so-called moment of the zeroth order); by this means it is ensured that R changes its sign at least *once* in the range of σ corresponding to one half of the period of the motion of the water level. Taking the cases $i = 0$ and 1 simultaneously, we may expect moreover that the change of the sign of R should occur at least *twice* in the same range of σ.* However, from

$$\int_{-2\nu/\pi}^{2-2\nu/\pi} R\sigma \, \mathrm{d}\sigma = 0, \tag{3.24}$$

or
$$\int_{-2\nu/\pi}^{2-2\nu/\pi} \left\{ \lambda^2 \sin^2(\frac{\pi}{2}\sigma - \mu) - \rho^2 \sqrt{1-2\lambda\cos\mu+\lambda^2}\, \sin(\frac{\pi}{2}\sigma+\nu) \right\} \sigma \, \mathrm{d}\sigma$$

* In this case, we have to discard the condition (3.10) imposed at the points of the maxima and the minima of the water level, since only two indeterminate constants are included in the assumed form for ζ; see (3.1). Incidentally, it should be noticed that forcing an approximate solution to satisfy the differential equation exactly at particular points might decrease eventually the overall accuracy of that solution.[3)]

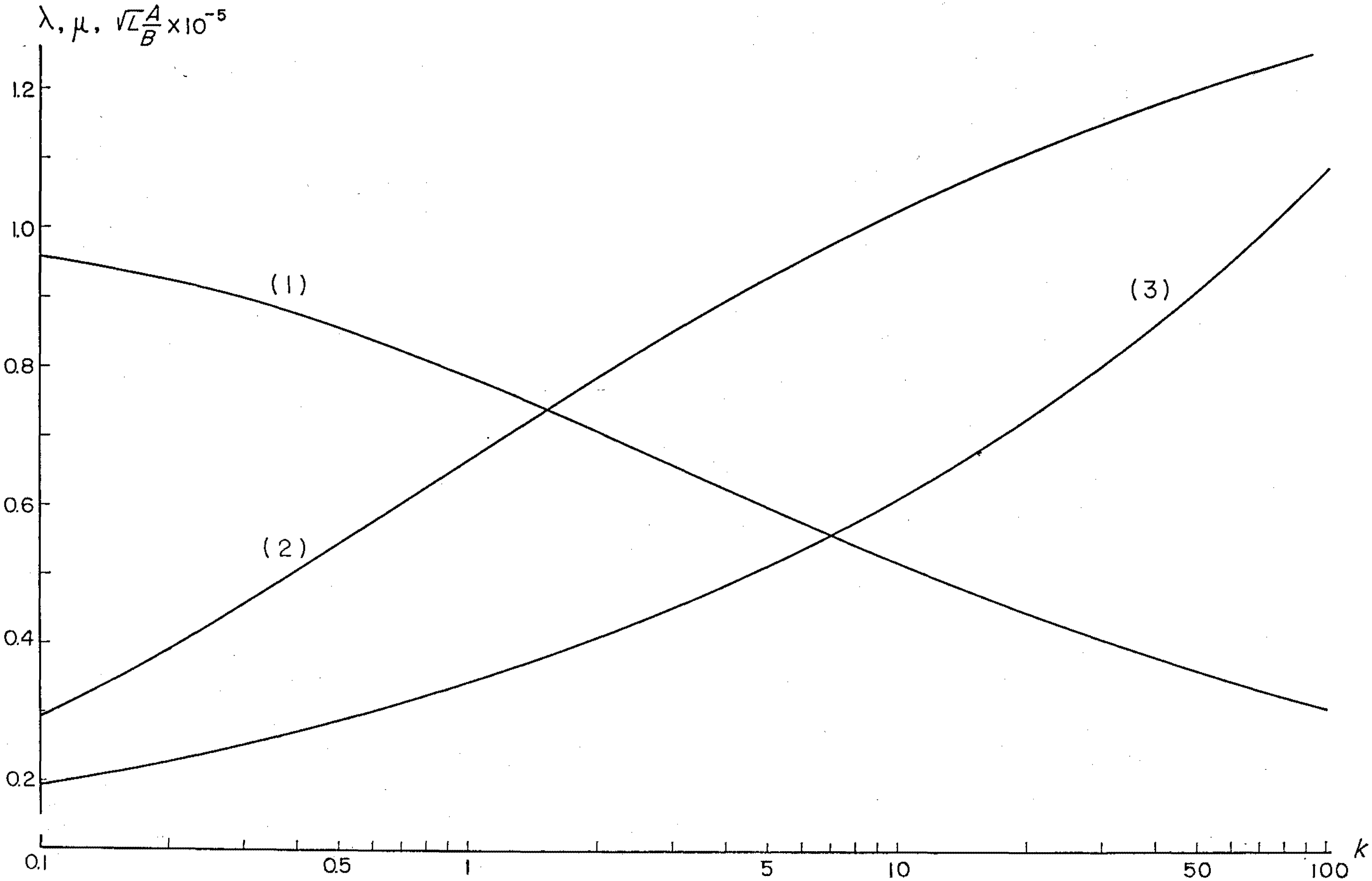

Fig. 3. (1) λ, (2) μ, and (3) $\sqrt{L}A/B$ (L in m) as functions of k.

$$= 0 \tag{3.25}$$

(see (3.14) and (3.15)), we obtain after some computations the result

$$\pi\lambda^2\{\pi-2\nu+\sin 2(\mu+\nu)\} = 4\rho^2\sqrt{1-2\lambda\cos\mu+\lambda^2}\,(\pi-2\nu), \tag{3.26}$$

but taking into consideration (3.16), we may write alternatively

$$\sin 2(\mu+\nu) = 0; \tag{3.27}$$

this is the condition derived from the moment of the first order. From (3.27), we should have

$$\cos(\mu+\nu) = 0 \qquad \text{or} \qquad \sin(\mu+\nu) = 0. \tag{3.28}$$

Now if we go back to equation (3.13), we obtain from the above equations respectively

$$\sin\mu = 0 \qquad \text{or} \qquad \lambda = \cos\mu. \tag{3.29}$$

The first condition requires that $\mu = 0$ or π, μ being the phase lag of the motion of the water level in the lake, and so it is not appropriate to our present problem. The second one, on the other hand, is found to be nothing but (3.10).

To sum up, we are led to a very interesting conclusion. If we determine those two constants included in (3.1) by virture of the condition imposed upon the function in order to satisfy the differential equation exactly at the points of the maxima and the minima of the water level, together with the condition of vanishing of zeroth order moment of the equation, then the solution automatically reduces the first order moment to zero. So it may be said that we are now in a favorable situation.

This is well visualized in the following example. By taking

$$k = 10 \qquad (\text{or} \quad \rho \doteqdot 0.5) \tag{3.30}$$

in equation (3.19), we obtain

$$\lambda = 0.5198, \qquad \text{and} \qquad \mu = 1.0242, \tag{3.31}$$

from (3.20) and (3.21) respectively. The tidal motions in the sea and the lake, expressed by (1.7) and (3.1) respectively are shown as (1) and (2) in Figure 4. The residual R defined by (3.15) or (3.15)′ pertaining to the domain of the independent variable σ, which might be interpreted as a kind of the measure of accuracy of our approximate solution, is also drawn as (3) in the same figure. The facts that the curve of R intersects the abscissa at two points in a half period, ranging from the trough to the crest, of the tidal motion in the lake and that also at both of them R is observed to vanish as requested by the relation (3.10) are natural consequences from the conditions imposed upon our approximate solution.

4. Numerical examples

With a view to testing the accuracy attained by our approximate solutions

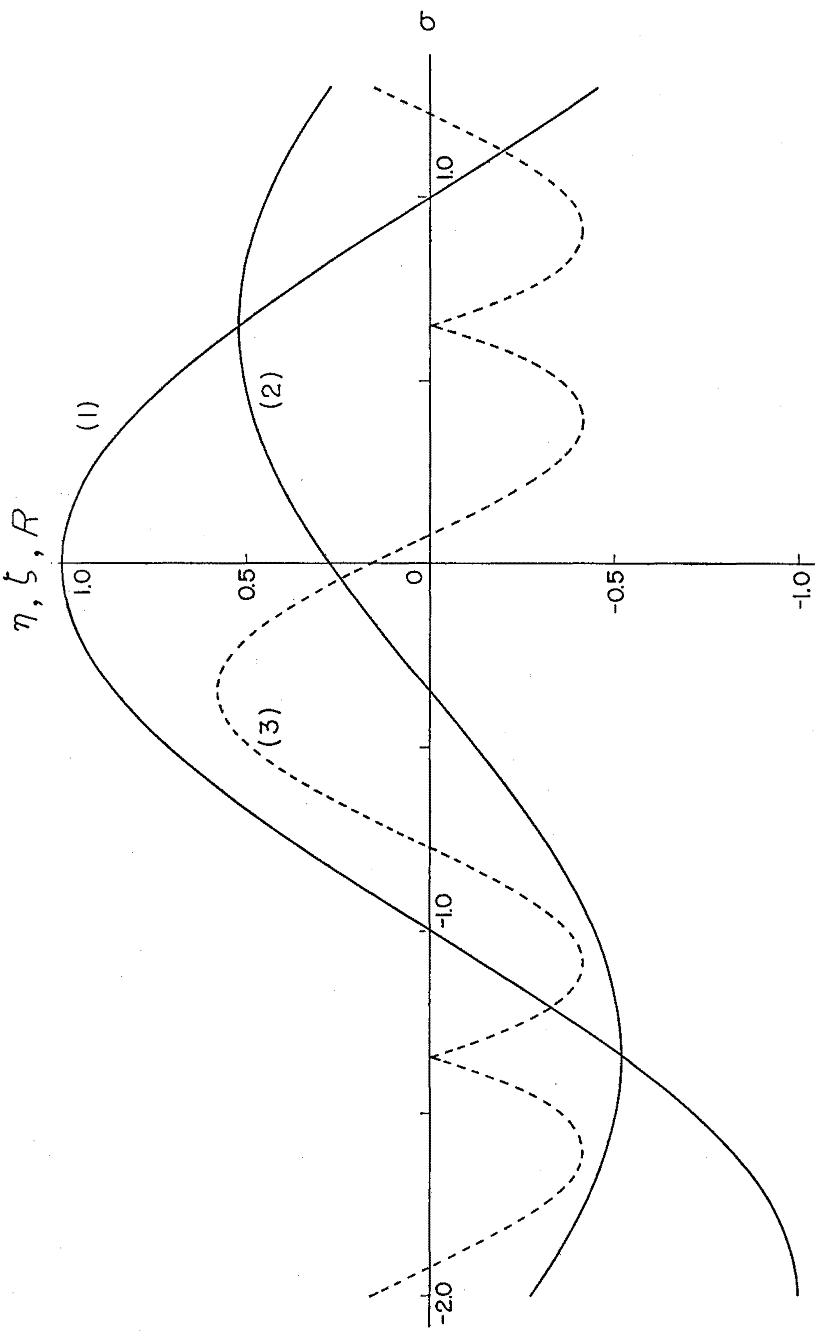

Fig. 4. (1) η, (2) ζ, and (3) R as functions of σ.

thus obtained, in the followings they are compared with the exact solutions explained in some detail previously. Three examples are taken:

$$\rho = 1, \quad 0.5, \quad \text{and} \quad 0.25; \tag{4.1}$$

see (1.6). From (3.19), therefore, the corresponding values of k are found to be

$$k = 0.61685, \quad 9.8696, \quad \text{and} \quad 157.914, \tag{4.2}$$

respectively. Supposing that the tidal motion of the outer sea is such that L, the amplitude, and T, the period, are 1 m and 12 hours, we may deduce without difficulty that in these cases the ratio $A/B \times 10^{-4}$ (A and B denoting the area of the lake and the effective area of its mouth respectively; see section 1) is found to be 3.04, 6.09, and 12.18 respectively. From the values of k mentioned above in (4.2), the approximate solutions pertinent to them can be readily constructed with the aid of the formulas (3.20), (3.21) together with (3.1). They are reproduced in Table 2 for the values of σ ranging from 0 to 4 at intervals of 0.1.

In order to work out the exact solutions, on the other hand, it should be most convenient to start from the value given by the corresponding approximate solution at $\sigma = 0$ instead of beginning the integration arbitrarily. They are

$$0.698789, \quad 0.27166, \quad \text{and} \quad 0.07647, \tag{4.3}$$

respectively for the cases $\rho = 1$, 0.5 and 0.25. RUNGE-KUTTA's method was employed in carrying out the numerical integration. The result is shown collectively in Tables 3, 4, and 5, where ζ_n ($n = 1, 2, 3, \ldots$) stands for values of $\zeta(\sigma)$ computed from equation (1.8) for the range ($\overline{4n-1} \leqq \sigma \leqq 4n$). In Figures 5, 6, and 7, on the other hand, the approximate solutions are shown side by side with the exact solutions (ζ_3, ζ_3, and ζ_4 for the cases $\rho = 1, 0.5$, and 0.25, respectively) for ready comparison.

5. Appendix — double-tidal lake

As an extension of the preceding computations, let us take up a problem concerning, so to speak, a *double-tidal lake*, i.e. a lake situated between two separate seas (an outer sea and an inner one); see Figure 8. For purposes of simplifying our analysis, however, we shall assume in what follows that the inner sea is so large that the effect of the tidal motion of the outer sea is not sensible there any longer and that the water level within the inner sea is kept invariably at the mean height of the outer sea. Writing the water levels in the outer sea and in the lake as $H(t)$ and $h(t)$ respectively as before, t being the time, and also denoting by A, B, B', and L, the area of the lake, the effective areas of its mouths leading to the outer and the inner seas, and the amplitude of the tidal motion in the outer sea, respectively, we may define in the same way as in section 1 non-dimensional quantities such as η, ζ, and τ. In addition to the parameter κ, on the other hand, however, we should have naturally

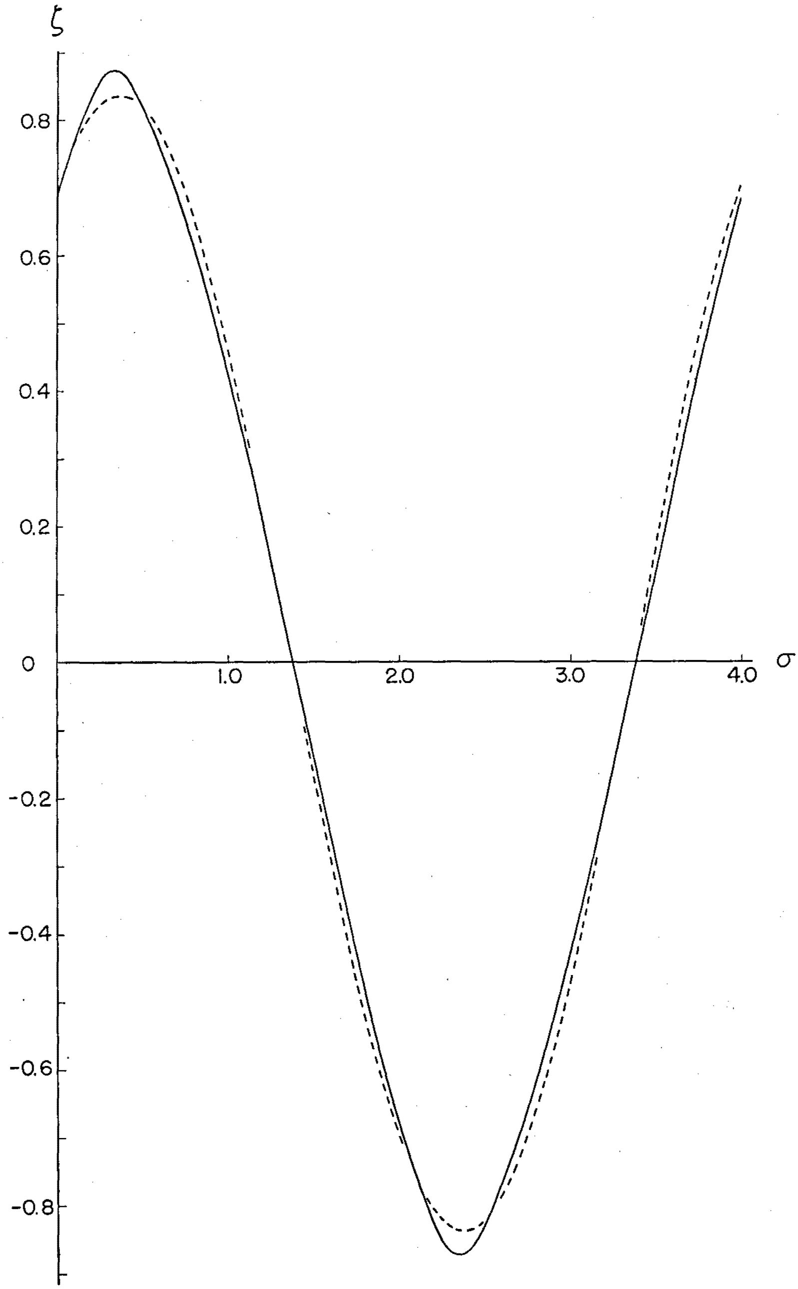

Fig. 5. Exact solution (full line) and approximate solution (dotted line) for the case $\rho = 1$.

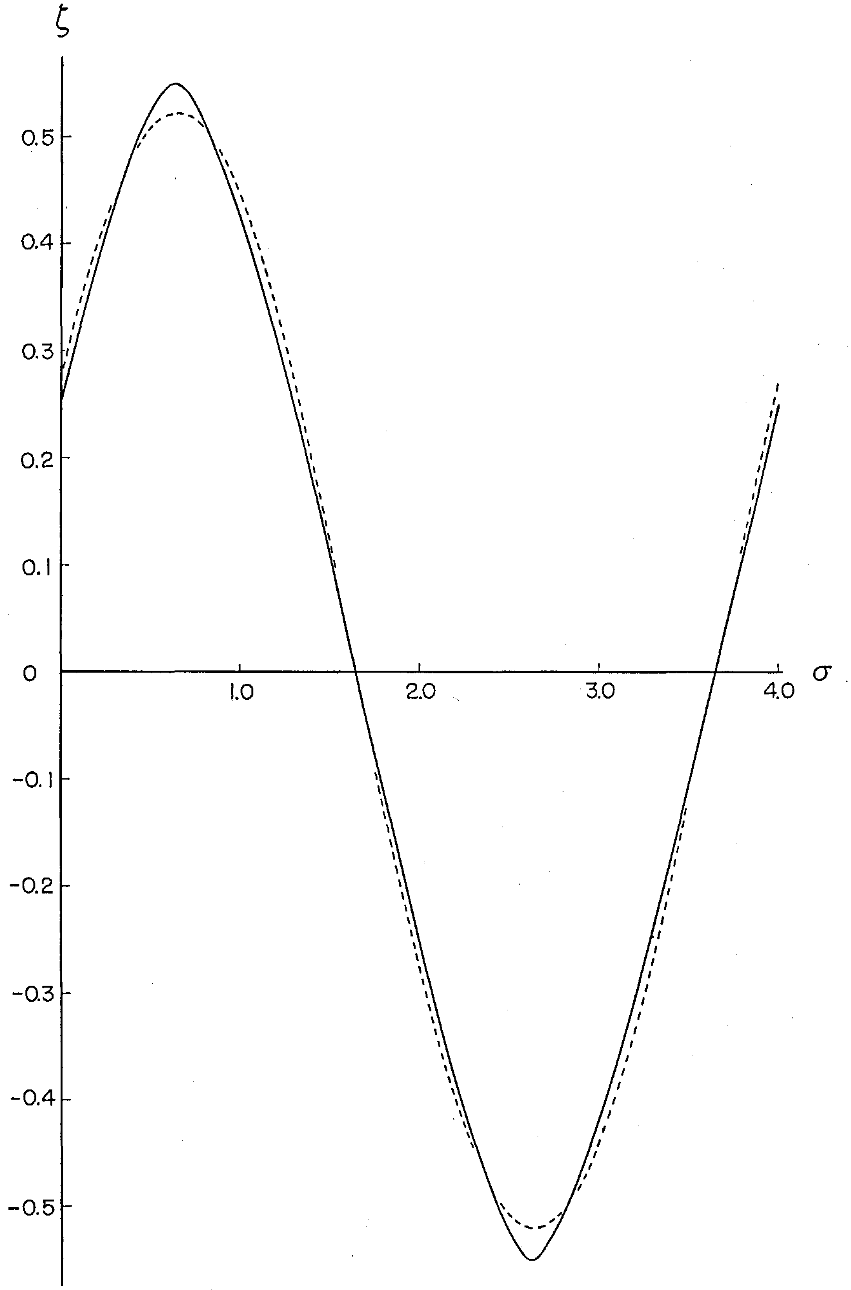

Fig. 6. Exact solution (full line) and approximate solution (dotted line) for the case $\rho = 0.5$.

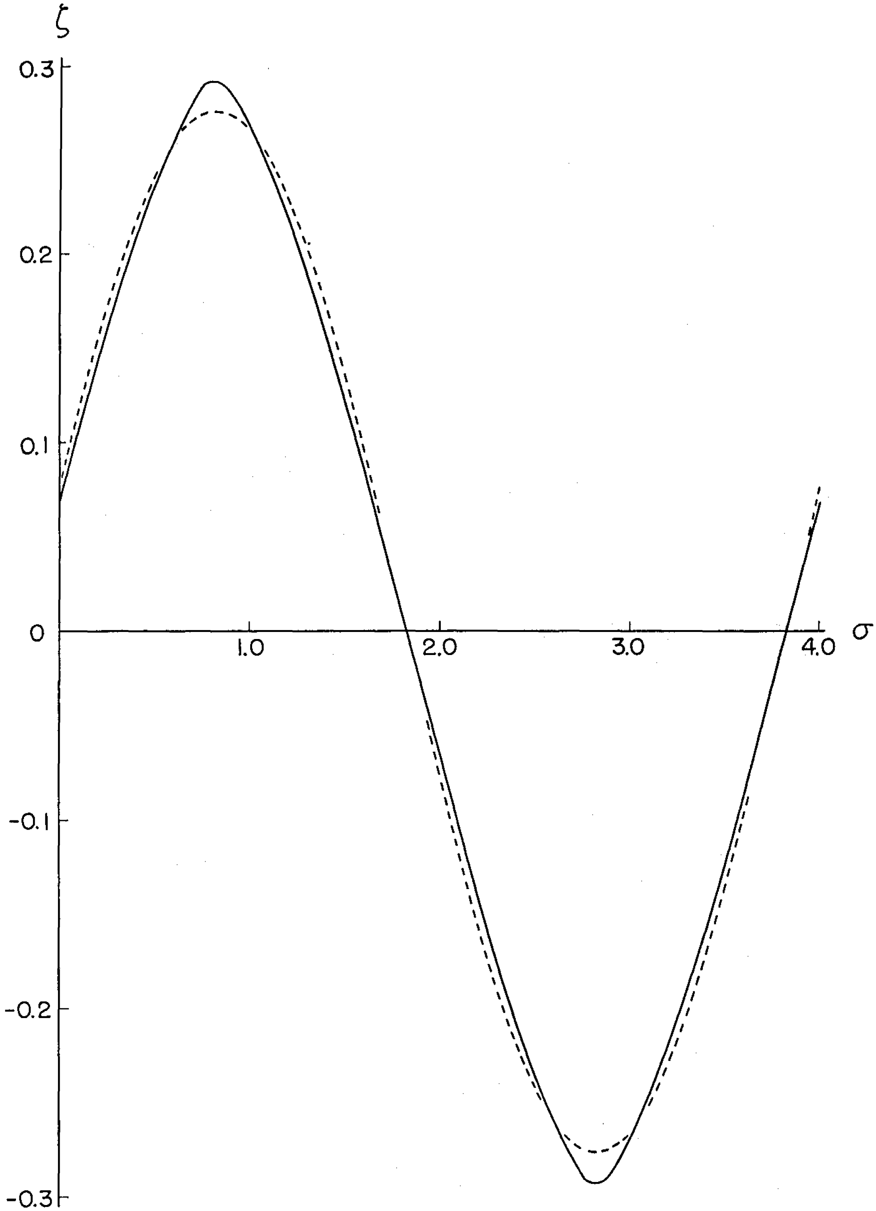

Fig. 7. Exact solution (full line) and approximate solution (dotted line) for the case $\rho = 0.25$.

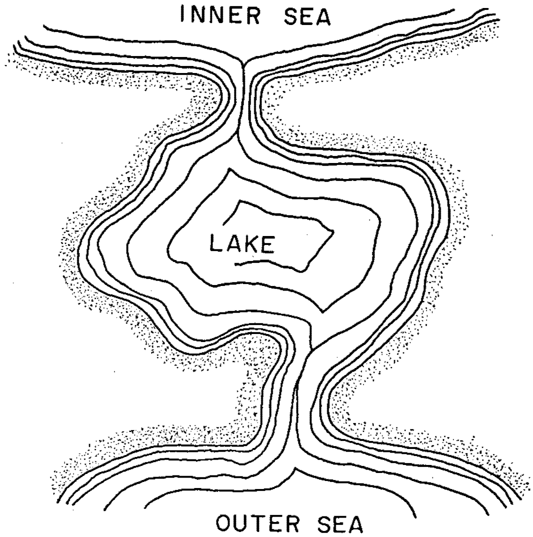

Fig. 8. Double-tidal lake.

$$\kappa' = \sqrt{2}\, B'/A. \tag{5.1}$$

Then the fundamental equation governing the tidal motion in the lake can be written approximately as follows:

$$\begin{aligned}
\frac{d\zeta}{d\tau} &= \kappa\sqrt{\eta-\zeta}+\kappa'\sqrt{-\zeta} && (\text{when } 0 \geqq \eta \geqq \zeta \text{ or } \eta \geqq 0 \geqq \zeta),\\
\frac{d\zeta}{d\tau} &= \kappa\sqrt{\eta-\zeta}-\kappa'\sqrt{\zeta} && (\text{when } \eta \geqq \zeta \geqq 0),\\
\frac{d\zeta}{d\tau} &= -\kappa\sqrt{\zeta-\eta}-\kappa'\sqrt{\zeta} && (\text{when } \zeta \geqq \eta \geqq 0 \text{ or } \zeta \geqq 0 \geqq \eta),\\
\text{and}\quad \frac{d\zeta}{d\tau} &= -\kappa\sqrt{\zeta-\eta}+\kappa'\sqrt{-\zeta} && (\text{when } 0 \geqq \zeta \geqq \eta).
\end{aligned} \tag{5.2}$$

The maxima and the minima of ζ take place obviously for the second and the fourth forms of the above equation (or the system of equations, speaking more precisely), and between the extreme value of ζ and the value of η at that point, say ζ_m and η_m respectively, there exists the relationship

$$\zeta_m = \frac{\kappa^2}{\kappa^2+\kappa'^2}\,\eta_m. \tag{5.3}$$

If we assume further that the undulation of the level in the outer sea is sinusoidal, i. e.

$$\eta = \cos\alpha\tau = \cos\frac{\pi}{2}\sigma, \tag{5.4}$$

α being some constant as before, and in the same way as (1.6) putting also

$$\kappa'/\alpha \equiv \rho' \tag{5.5}$$

we can transform (5.2) into the form

$$\frac{d\zeta}{d\sigma} = \frac{\pi}{2}\rho\sqrt{\cos\frac{\pi}{2}\sigma-\zeta} + \frac{\pi}{2}\rho'\sqrt{-\zeta}$$

$$\left(\text{when } 0 \geqq \cos\frac{\pi}{2}\sigma \geqq \zeta \text{ or } \cos\frac{\pi}{2}\sigma \geqq 0 \geqq \zeta\right),$$

$$\frac{d\zeta}{d\sigma} = \frac{\pi}{2}\rho\sqrt{\cos\frac{\pi}{2}\sigma-\zeta} - \frac{\pi}{2}\rho'\sqrt{\zeta} \quad \left(\text{when } \cos\frac{\pi}{2}\sigma \geqq \zeta \geqq 0\right),$$

$$\frac{d\zeta}{d\sigma} = -\frac{\pi}{2}\rho\sqrt{\zeta-\cos\frac{\pi}{2}\sigma} - \frac{\pi}{2}\rho'\sqrt{\zeta}$$

$$\left(\text{when } \zeta \geqq \cos\frac{\pi}{2}\sigma \geqq 0 \text{ or } \zeta \geqq 0 \geqq \cos\frac{\pi}{2}\sigma\right),$$

and $$\frac{d\zeta}{d\sigma} = -\frac{\pi}{2}\rho\sqrt{\zeta-\cos\frac{\pi}{2}\sigma} + \frac{\pi}{2}\rho'\sqrt{-\zeta} \quad \left(\text{when } 0 \geqq \zeta \geqq \cos\frac{\pi}{2}\sigma\right). \tag{5.6}$$

This might be interpreted as an extension of the pair of the equations (1.8).

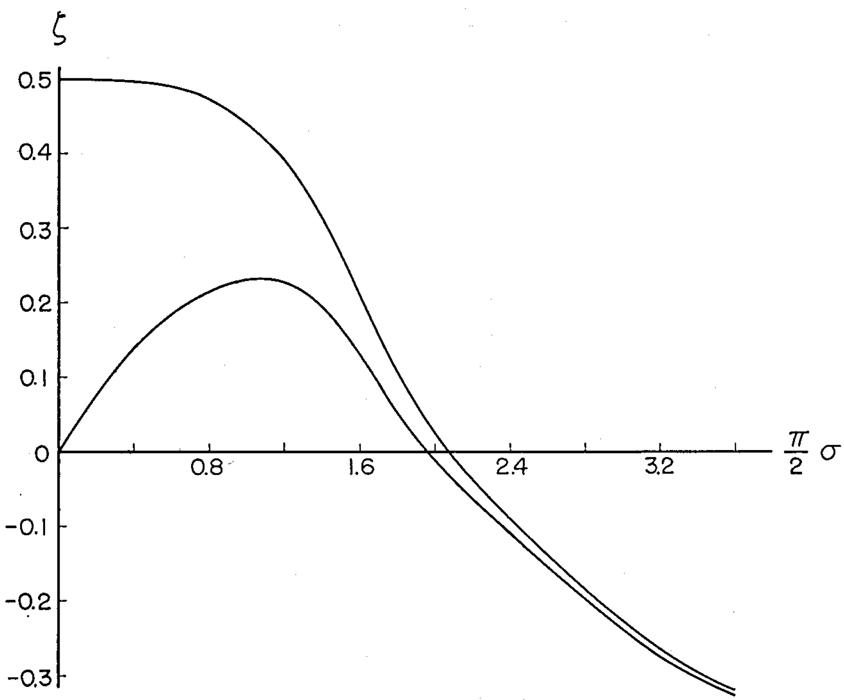

Fig. 9. Two integral curves of equation (5.6).

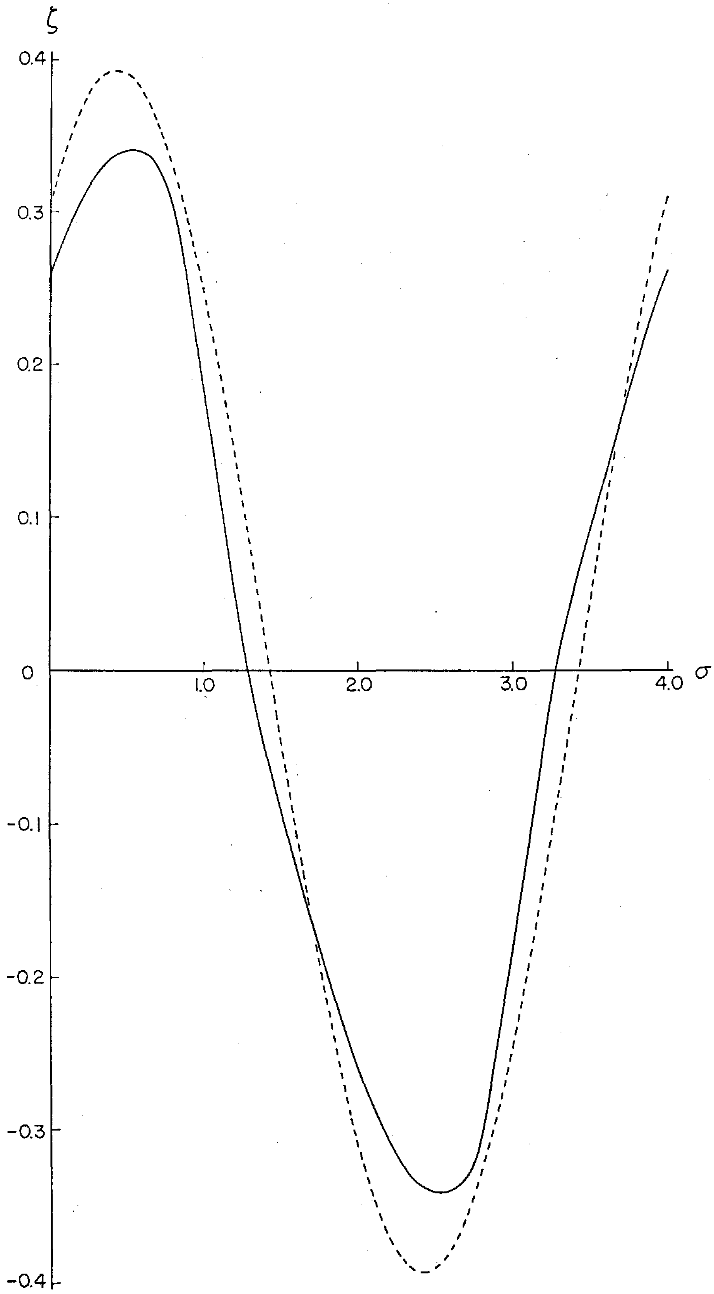

Fig. 10. Exact solution (full line) and approximate solution (dotted line) of equation (5.6) for the case $\rho = \rho' = 0.5$.

With a view to getting a *feel* (not a proof) of convergency of solutions starting from $\sigma = 0$ arbitrarily, we take up a particular case

$$\rho = \rho' = 0.5, \tag{5.7}$$

and proceed to integrate the equation numerically with the initial conditions

$$\zeta(0) = 0.5, \text{ and } 0. \tag{5.8}$$

In Figure 9 it can be seen how the two integral curves thus produced converge steadily into one limiting curve. Finally, the limiting periodic solution of the equation, which we define as the exact solution, can be worked out empirically through the same procedure as in section 2. Namely, for the case mentioned in (5.7), starting from the value suggested by a rough estimate,* we put

$$\zeta(0) = 0.30902. \tag{5.9}$$

Table 6 and Figure 10 embody the result of the computation. The notations ζ_1 and ζ_2 are used in the same manner as in section 4, and ζ_2 is found satisfactorily periodic.

References

1) With respect to a mathematical aspect of the problem, see the following: M. OHJI, "On the solutions of first-order ordinary differential equations with periodic properties" (in Japanese), *Bulletin of Research Institute for Applied Mechanics*, Kyushu University, No. 11, 1958, pp. 21–31.
2) H. YAMADA, "On a method of approximate solution of differential equations" (English abstract), *Reports of Research Institute for Fluid Engineering*, Kyushu University, Vol. 6, No. 2, 1950, pp. 42–46, and "A method of approximate integration of the laminar boundary layer equation, I and II" (English abstract), *ibid.*, Vol. 6, No. 2, 1950, pp. 87–98. The original forms of these papers were written in the Japanese language, and published in the above-mentioned *Reports*, Vol. 3, No. 3, 1947, pp.

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* First, we shall assume that the whole up-and-down motion of the water level within a double-tidal lake may be approximated by the function

$$\zeta = \lambda \cos(\alpha\tau - \mu). \tag{1}$$

Since $\zeta_m = \lambda$ and $\eta_m = \cos\mu$, we have from (5.3)

$$\lambda = \frac{\kappa^2}{\kappa^2 + \kappa'^2} \cos\mu = \frac{\rho^2}{\rho^2 + \rho'^2} \cos\mu. \tag{2}$$

Substitute (2) for ζ in equation (5.2), and equate the both-hand sides of the equation at the junction of the first and the second forms of the right-hand sides, namely at the point $\zeta = 0$, then we readily obtain the relationship

$$\lambda = \rho\sqrt{\sin\mu}. \tag{3}$$

From (2) and (3), we have

$$\lambda = \rho\sqrt{\sqrt{1 + \frac{(\rho^2+\rho'^2)^4}{4\rho^4}} - \frac{(\rho^2+\rho'^2)^2}{2\rho^2}}, \tag{4}$$

and

$$\mu = \sin^{-1}\left\{\sqrt{1 + \frac{(\rho^2+\rho'^2)^4}{4\rho^4}} - \frac{(\rho^2+\rho'^2)^2}{2\rho^2}\right\}; \tag{5}$$

the approximate curve thus obtained is shown also in Figure 10 with a dotted line.

29-35, Vol. 4, No. 3, 1948, pp. 27-42, and Vol. 5, No. 2, 1949, pp. 1-10, respectively.
3) H. YAMADA, "A note on the calculation of non-linear oscillations" (in Japanese), *Reports of Research Institute for Fluid Engineering*, Kyushu University, Vol. 7, No. 2, 1950, pp. 8-16, esp. 10.

(Received October 4, 1968)

Table 1. Solutions of (1.8), $\rho = 0.5$.

$\frac{\pi}{2}\sigma$	A	B	C	D	E
0.0	1.00000	0.50000	0.00000	−0.50000	−1.00000
0.4	0.97709	0.62118	0.18717	−0.26733	−0.72911
0.8	0.90728	0.69487	0.33532	−0.06927	−0.49051
1.2	0.79512	0.63016	0.40018	0.07474	−0.29866
1.6	0.64647	0.50399	0.31493	0.11117	−0.17501
2.0	0.46934	0.34278	0.17751	0.00499	−0.20757
2.4	0.27371	0.15984	0.01253	−0.13900	−0.32078
2.8	0.07135	−0.03152	−0.16363	−0.29812	−0.45687
3.2	−0.12414	−0.21674	−0.33482	−0.45381	−0.59217
3.6	−0.29677	−0.37891	−0.48264	−0.58568	−0.70282
4.0	−0.42516	−0.49458	−0.57971	−0.65337	−0.73818
4.4	−0.45158	−0.48709	−0.53451	−0.58601	−0.64794

Table 2. Approximate solutions.

σ	$\zeta(\rho=1)$	$\zeta(\rho=0.50)$	$\zeta(\rho=0.25)$	σ	$\zeta(\rho=1)$	$\zeta(\rho=0.50)$	$\zeta(\rho=0.25)$
0.0	0.69879	0.27166	0.07647	2.1	−0.76196	−0.33790	−0.11711
				2.2	−0.80636	−0.39581	−0.15485
0.1	0.76196	0.33790	0.11711	2.3	−0.83091	−0.44399	−0.18879
0.2	0.80636	0.39581	0.15485	2.4	−0.83500	−0.48123	−0.21808
0.3	0.83091	0.44399	0.18879	2.5	−0.81853	−0.50662	−0.24199
0.4	0.83500	0.48123	0.21808				
0.5	0.81853	0.50662	0.24199	2.6	−0.78190	−0.51954	−0.25995
				2.7	−0.72602	−0.51966	−0.27151
0.6	0.78190	0.51954	0.25995	2.8	−0.65227	−0.50699	−0.27638
0.7	0.72602	0.51966	0.27151	2.9	−0.56245	−0.48183	−0.27445
0.8	0.65227	0.50699	0.27638	3.0	−0.45878	−0.44481	−0.26576
0.9	0.56245	0.48183	0.27445				
1.0	0.45878	0.44481	0.26576	3.1	−0.34382	−0.39684	−0.25052
				3.2	−0.22039	−0.33910	−0.22912
1.1	0.34382	0.39684	0.25052	3.3	−0.09154	−0.27300	−0.20207
1.2	0.22039	0.33910	0.22912	3.4	0.03957	−0.20019	−0.17005
1.3	0.09154	0.27300	0.20207	3.5	0.16971	−0.12244	−0.13384
1.4	−0.03957	0.20019	0.17005				
1.5	−0.16971	0.12244	0.13384	3.6	0.29567	−0.04168	−0.09434
				3.7	0.41434	0.04011	−0.05251
1.6	−0.29567	0.04168	0.09434	3.8	0.52282	0.12090	−0.00939
1.7	−0.41434	−0.04011	0.05251	3.9	0.61842	0.19873	0.03396
1.8	−0.52282	−0.12090	0.00939	4.0	0.69879	0.27166	0.07647
1.9	−0.61842	−0.19873	−0.03396				
2.0	−0.69879	−0.27166	−0.07647				

Table 3. Exact solution, $\rho = 1$.

σ	ζ_1	ζ_2	ζ_3	$\zeta_1-\zeta_2$	$\zeta_2-\zeta_3$
0.0	0.698789	0.683710	0.683716	0.015079	−0.000006
0.2	0.840594	0.829980	0.829984	0.010614	−0.000004
0.4	0.861479	0.863896	0.863897	−0.002417	−0.000001
0.6	0.758432	0.759908	0.759909	−0.001476	−0.000001
0.8	0.606936	0.608000	0.608001	−0.001064	−0.000001
1.0	0.418745	0.419563	0.419563	−0.000818	0.000000
1.2	0.203728	0.204378	0.204378	−0.000650	0.000000
1.4	−0.027359	−0.026834	−0.026834	−0.000525	0.000000
1.6	−0.262271	−0.261845	−0.261845	−0.000426	0.000000
1.8	−0.486834	−0.486493	−0.486493	−0.000341	0.000000
2.0	−0.683982	−0.683716	−0.683716	−0.000266	0.000000
2.2	−0.830172	−0.829984	−0.829984	−0.000188	0.000000
2.4	−0.863951	−0.863897	−0.863897	−0.000054	0.000000
2.6	−0.759942	−0.759909	−0.759909	−0.000033	0.000000
2.8	−0.608024	−0.608001	−0.608001	−0.000023	0.000000
3.0	−0.419582	−0.419563	−0.419563	−0.000019	0.000000
3.2	−0.204392	−0.204378	−0.204378	−0.000014	0.000000
3.4	0.026822	0.026834	0.026834	−0.000012	0.000000
3.6	0.261835	0.261845	0.261845	−0.000010	0.000000
3.8	0.486485	0.486493	0.486493	−0.000008	0.000000
4.0	0.683710	0.683716	0.683716	−0.000006	0.000000

Table 4. Exact solution, $\rho = 0.5$.

σ	ζ_1	ζ_2	ζ_3	$\zeta_1-\zeta_2$	$\zeta_2-\zeta_3$
0.0	0.27166	0.25061	0.24910	0.02105	0.00151
0.2	0.39794	0.37885	0.37747	0.01909	0.00138
0.4	0.50121	0.48426	0.48304	0.01695	0.00122
0.6	0.56248	0.54899	0.54800	0.01349	0.00099
0.8	0.52156	0.51133	0.51058	0.01023	0.00075
1.0	0.43262	0.42373	0.42309	0.00889	0.00064
1.2	0.31811	0.31014	0.30956	0.00797	0.00058
1.4	0.18624	0.17898	0.17845	0.00726	0.00053
1.6	0.04402	0.03737	0.03688	0.00665	0.00049
1.8	−0.10153	−0.10764	−0.10808	0.00611	0.00044
2.0	−0.24294	−0.24854	−0.24895	0.00560	0.00041
2.2	−0.37188	−0.37697	−0.37734	0.00509	0.00037
2.4	−0.47807	−0.48259	−0.48292	0.00452	0.00033
2.6	−0.54398	−0.54764	−0.54790	0.00366	0.00026
2.8	−0.50753	−0.51030	−0.51051	0.00277	0.00021
3.0	−0.42044	−0.42285	−0.42302	0.00241	0.00017
3.2	−0.30719	−0.30934	−0.30950	0.00215	0.00016
3.4	−0.17630	−0.17826	−0.17840	0.00196	0.00014
3.6	−0.03490	−0.03670	−0.03683	0.00180	0.00013
3.8	0.10990	0.10825	0.10813	0.00165	0.00012
4.0	0.25061	0.24910	0.24899	0.00151	0.00011

Table 5. Exact solution, $\rho = 0.25$.

σ	ζ_1	ζ_2	ζ_3	ζ_4	$\zeta_1-\zeta_2$	$\zeta_2-\zeta_3$	$\zeta_3-\zeta_4$
0.0	0.07647	0.07028	0.06860	0.06815	0.00619	0.00168	0.00045
0.2	0.14972	0.14378	0.14217	0.14174	0.00594	0.00161	0.00043
0.4	0.21555	0.20988	0.20835	0.20793	0.00567	0.00153	0.00042
0.6	0.26866	0.26331	0.26186	0.26147	0.00535	0.00145	0.00039
0.8	0.29858	0.29393	0.29266	0.29232	0.00465	0.00127	0.00034
1.0	0.27376	0.26974	0.26863	0.26833	0.00402	0.00111	0.00030
1.2	0.22394	0.22016	0.21912	0.21884	0.00378	0.00104	0.00028
1.4	0.16092	0.15732	0.15633	0.15606	0.00360	0.00099	0.00027
1.6	0.08939	0.08594	0.08500	0.08474	0.00345	0.00094	0.00026
1.8	0.01329	0.00998	0.00908	0.00883	0.00331	0.00090	0.00025
2.0	−0.06359	−0.06677	−0.06765	−0.06789	0.00318	0.00088	0.00024
2.2	−0.13737	−0.14042	−0.14126	−0.14148	0.00305	0.00084	0.00022
2.4	−0.20377	−0.20668	−0.20747	−0.20769	0.00291	0.00079	0.00022
2.6	−0.25754	−0.26028	−0.26104	−0.26124	0.00274	0.00076	0.00020
2.8	−0.28885	−0.29127	−0.29194	−0.29212	0.00242	0.00067	0.00018
3.0	−0.26530	−0.26742	−0.26800	−0.26816	0.00212	0.00058	0.00016
3.2	−0.21599	−0.21798	−0.21853	−0.21867	0.00199	0.00055	0.00014
3.4	−0.15335	−0.15525	−0.15576	−0.15590	0.00190	0.00051	0.00014
3.6	−0.08215	−0.08396	−0.08446	−0.08459	0.00181	0.00050	0.00013
3.8	−0.00634	−0.00808	−0.00855	−0.00868	0.00174	0.00047	0.00013
4.0	0.07028	0.06860	0.06815	0.06802	0.00168	0.00045	0.00013

Table 6. Exact solution, $\rho = \rho' = 0.5$.

σ	ζ_1	ζ_2	$\zeta_1-\zeta_2$
0.0	0.30902	0.25964	0.04938
0.2	0.34607	0.30713	0.03894
0.4	0.36629	0.33569	0.03060
0.6	0.36202	0.33858	0.02344
0.8	0.31343	0.30067	0.01276
1.0	0.18841	0.18230	0.00611
1.2	0.05268	0.04856	0.00412
1.4	−0.05063	−0.05256	0.00193
1.6	−0.12715	−0.12848	0.00133
1.8	−0.19754	−0.19854	0.00100
2.0	−0.25885	−0.25962	0.00077
2.2	−0.30651	−0.30712	0.00061
2.4	−0.33521	−0.33568	0.00047
2.6	−0.33821	−0.33858	0.00037
2.8	−0.30047	−0.30066	0.00019
3.0	−0.18218	−0.18229	0.00011
3.2	−0.04848	−0.04856	0.00008
3.4	0.05259	0.05256	0.00003
3.6	0.12850	0.12848	0.00002
3.8	0.19856	0.19854	0.00002
4.0	0.25964	0.25962	0.00002