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TAKEMATSU, Masaki
Research Institute for Applied Mechanics, Kyushu University

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Masaki TAKEMATSU

The present paper is concerned with the stability problem for a plane parallel flow along a flexible yet still solid boundary. A broken linear velocity profile is used instead of the actual boundary-layer profile. An eigenvalue problem is formulated and solved asymptotically for large values of kR. In a special case in which the boundary is composed of a non-dispersive frictionless medium, stability diagrams are computed. The diagrams describe 'Kelvin-Helmholtz' type of instability as well as some aspects of the modification of 'Tollmien-Schlichting' wave with wall flexibility, the connection between the two thereby being clarified.

§ 1. Introduction

The theoretical problem of the stability of a laminar boundary-layer along a flexible surface was previously investigated by Benjamin[1] and Landahl[2] with a view to examining the Krämer's idea that a flexible surface may serve as a stabilizing device. It was shown that in case the boundary is flexible three different types of instability are possible. These were classified as 'class A' ('Tollmien-Schlichting' type), 'class B' and 'class C' ('Kelvin-Helmholtz' type) instability, and their fundamental properties were examined in detail. However, it remains obscure as yet how these different types of instability are connected with each other on a stability diagram; in fact, the eigen equation derived in their analyses fails to describe 'Kelvin-Helmholtz' type of instability. It may be of theoretical as well as of practical significance to find out a complete stability diagram containing all the possible modes of instability. Straight analytical approach, however, seems to be prohibitive due to the intrinsic difficulty of solving the Orr-Sommerfeld equation to an sufficient degree of approximation for the actual boundary-layer profile.

In the present study, a broken linear velocity profile is assumed as a rough approximation to the actual boundary-layer profile; in this special case, as is well known, exact solutions of the Orr-Sommerfeld equation are available. Because of the simplification, some aspects of the stability characteristics, which critically depend on the profile curvature, may be missed. Nevertheless the simplified model used will make it easy to compute the parts of a stability diagram which are effectively independent of the profile curvature.

§ 2. Formulation of the problem

Consider a steady plane parallel flow along a flexible boundary located at rest at $y = 0$. A definition sketch of the problem is depicted in Fig. 1.

Upon the steady equilibrium flow we superimpose an infinitesimal travelling-wave disturbance having the (x, t)-dependence

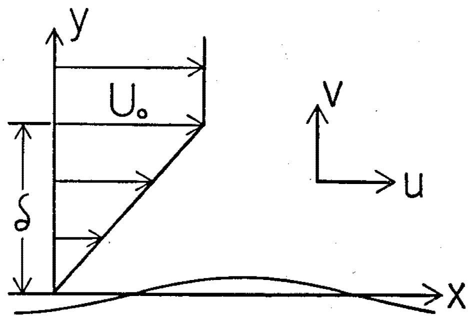

Fig. 1. Definition sketch

$$e(x, t) = \exp\{ik(x-ct)\}, \tag{2.1}$$

where k is the (real) wave number and c the (complex) wave speed. Perturbation velocities (u, v) associated with the disturbance may conveniently be expressed in terms of a stream-function

$$\psi(x, y, t) = \phi(y) \cdot e(x, t),$$

as

$$u = \psi_y = \phi' \cdot e, \qquad v = -\psi_x = -ik\phi \cdot e. \tag{2.2}$$

Here and in what follows, the prime denotes differentiation with respect to y. The amplitude function $\phi(y)$ satisfies the well-known Orr-Sommerfeld equation

$$(U-c)(\phi''-k^2\phi)-U''\phi = \frac{\nu}{ik}(\phi^{\text{iv}}-2k^2\phi''+k^4\phi), \tag{2.3}$$

where ν is the kinematic viscosity of the fluid.

It is convenient to introduce dimensionless variables by reffering all lengths to δ and velocities to U_0. The basic flow under consideration is then specified by

$$U(y) = \begin{cases} 1, & y \geqq 1, \\ y, & 0 \leqq y \leqq 1, \end{cases} \tag{2.4}$$

and (2.3) becomes

$$(1-c)(\phi''-k^2\phi) = (ikR)^{-1}(\phi^{\text{iv}}-2k^2\phi''+k^4\phi), \quad y \geqq 1, \tag{2.5}$$

$$(y-c)(\phi''-k^2\phi) = (ikR)^{-1}(\phi^{\text{iv}}-2k^2\phi''+k^4\phi), \quad 0 \leqq y \leqq 1, \tag{2.6}$$

where R is the Reynolds number $\dfrac{U_0\delta}{\nu}$.

In passing the perturbation pressure p and shear stress τ are expressed as

$$\left.\begin{aligned} p &= -[(U-c)\phi'-\phi U'-(ikR)^{-1}(\phi'''-k^2\phi')] \cdot e, \\ \tau &= R^{-1}(\phi''+k^2\phi) \cdot e. \end{aligned}\right\} \tag{2.7}$$

The boundary conditions to be satisfied at infinity are the usual ones; i.e.

$$\phi(\infty) = \phi'(\infty) = 0. \tag{2.8}$$

The non-slipping and kinematical conditions at the boundary* are

$$\left.\begin{array}{l} U+u = 0 \\ v = \eta_t + U\eta_x \end{array}\right\}, \; y = \eta. \tag{2.9 a, b}$$

Here $\eta(x, t)$ denotes lateral displacements of the flexible boundry and may be expressed as

$$\eta(x, t) = \hat{\eta} \cdot e(x, t), \tag{2.10}$$

with the complex amplitude $\hat{\eta}$.

Let m be the effective mass of the boundary and L a linear operator such that $L\eta$ gives the stress resisting the lateral displacement η. Then, in the absence of inner damping, the dynamical equation to be satisfied by η can be written as

$$M\eta_{tt} + L\eta = F, \tag{2.11}$$

with
$$F = -p + (2v_y + U'\eta_x)R^{-1}.$$

Here M denotes relative mass parameter defined by $M = \frac{m}{\delta\rho}$. In what follows, we take

$$L\eta = Mk^2c_0^2\eta. \tag{2.12}$$

This means that the flexible boundary comprises a non-dispersive medium in which free waves are propagated at constant speed c_0. If use is made of the expressions (2.10) and (2.12), (2.9 a, b) and (2.11) are reduced, after linearization in $\hat{\eta}$, to

$$\left.\begin{array}{l} \hat{\eta} + \phi'(0) = 0, \\ c\hat{\eta} - \phi(0) = 0, \end{array}\right\} \tag{2.13 a, b}$$

$$Mk^2(c_0^2 - c^2)\hat{\eta} = \hat{F}, \tag{2.14}$$

with
$$\hat{F} = -c\phi'(0) - \phi(0) - (ikR)^{-1}\{\phi'''(0) - 3k^2\phi'(0) + k^2\hat{\eta}\}.$$

By eliminating $\hat{\eta}$ from these equations, the following two homogeneous boundary conditions for ϕ are derived;

$$c\phi'(0) + \phi(0) = 0, \tag{2.15}$$

$$\phi'''(0) - (ikR\hat{\beta} + 4)k^2\phi'(0) = 0, \tag{2.16}$$

where

$$\hat{\beta} = M(c_0^2 - c^2).$$

From (2.14) it is readily seen that the case of a rigid boundary is represented

* the boundary is assumed to be inextensible.

by $\hat{\beta} = \infty$.

As, in the present problem, the outer region ($y \geqq 1$) and the boundary-layer region are assumed to obey the respective differential equations (2.5) and (2.6), adequate matching conditions are requisite in addition. The necessary matching conditions may be derived by the requirement that perturbation velocities, stresses and pressure must be continuous across the edge of the boundary-layer ($y=1$). On reference to the expressions (2.7), these can be written in terms of ϕ as

$$\left.\begin{aligned}&[\phi]=[\phi']=[\phi'']=0,\\&[\phi''']=ikR\phi(1-0),\end{aligned}\right\} \tag{2.17}$$

the symbol $[\phi]$ being used for $\phi(1+0)-\phi(1-0)$.
The outer region equation (2.5), together with the conditions at infinity (2.8), yields

$$\phi(y)=A_1e^{-ky}+A_2e^{-\gamma y}, \tag{2.18}$$

where

$$\gamma=\{k^2+ikR(1-c)\}^{1/2},\quad Re(\gamma)>0.$$

Substituting (2.18) to (2.17) and eliminating A_1, A_2, we obtain

$$\phi''(1)+(k+\gamma)\phi'(1)+k\gamma\phi(1)=0, \tag{2.19}$$

$$\{\phi'''(1)-k^2\phi'(1)\}+\gamma\{\phi''(1)-k^2\phi(1)\}+ikR\phi(1)=0. \tag{2.20}$$

The homogeneous differential equation (2.6) for the boundary-layer region and the four homogeneous conditions (2.15), (2.16), (2.19), (2.20) constitute a conventional eigenvalue problem.

Let $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ be a fundamental system of four solutions of (2.6). Then the general solution may be expressed in the form

$$\phi=B_1\phi_1+B_2\phi_2+B_3\phi_3+B_4\phi_4. \tag{2.21}$$

By substituting this expression in the four conditions and equating the resulting algebraic equations in the B's to zero, we have the eigenvalue equation

$$\det.\ (\omega_{ij})=0, \tag{2.22}$$

with the abbreviations

$$\begin{aligned}&\omega_{1j}=\phi_j''(1)+(k+\gamma)\phi_j'(1)+k\gamma\phi_j(1),\\&\omega_{2j}=\phi_j'''(1)-k^2\phi_j(1)+\gamma\{\phi_j''(1)-k^2\phi_j(1)\}+ikR\phi_j(1),\\&\omega_{3j}=c\phi_j'(0)+\phi_j(0),\\&\omega_{4j}=\phi_j'''(0)-(ikR\hat{\beta}+4)k^2\phi_j'(0).\\&\qquad(j=1,2,3,4)\end{aligned} \tag{2.23 a, b, c, d}$$

This gives a relation among eigenvalues k, R and c.

§ 3. Solutions

To evaluate the determinant in (2.22), we need explicit representations of $\phi_{1,2,3,4}$.

As is well known (2.6), which is a special form of the Orr-Sommerfeld equation, can be solved exactly. The two non-viscous solutions ϕ_1, ϕ_2 are given simplly as

$$\phi_1 = \cosh\ (ky), \qquad \phi_2 = \sinh\ (ky). \tag{3.1 a, b}$$

The remaining two viscous solutions ϕ_3, ϕ_4 are, according to Esch[5], conveniently expressible in terms of contour integrals as

$$\phi_{3,4} = \int_{c_3, c_4} \exp\left[\frac{t^3}{3} + izt\right] \cdot \frac{dt}{t^2+\sigma^2}, \tag{3. 2}$$

where

$$z = (kR)^{1/3}[(y-c) - ik^2(kR)^{-1}] = (kR)^{1/3}(y-c) - i\sigma^2,$$
$$\sigma = k(kR)^{-1/3}.$$

The integration contours must go to infinity in the unshaded sectors as shown in Fig. 2, so that $\exp[t^3/3+izt]$ vanishes at each limit. The integral representation (3.2), as it is, is not suitable for numerical purposes. When kR is sufficiently large, however, approximate expressions of immediate use can be successfully derived from (3.2) by a procedure known as the saddle-point method.

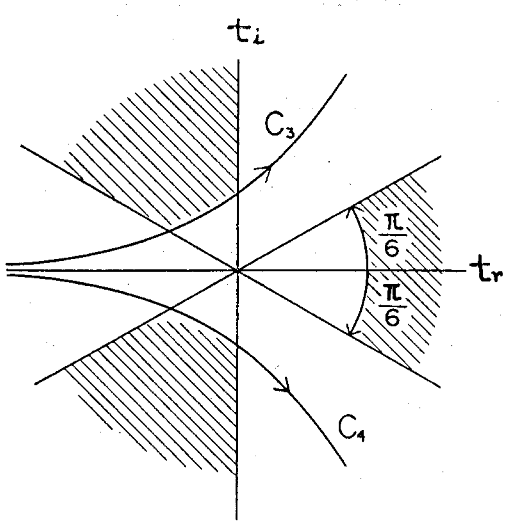

Fig. 2. Integration contours

The results* are

$$\phi_{3,4} \simeq \exp\left[\mp\frac{2}{3}\sqrt{i}\,z^{3/2}\right]\cdot z^{-5/4}\left\{1\mp\frac{101}{48\sqrt{i}}z^{-3/2}+\frac{35905}{3608i}z^{-3}-i\sigma^2 z^{-1}+O(z^{-9/2},\sigma^2 z^{-5/2})\right\}, \tag{3.3 a, b}$$

$$\left(\sqrt{i}=\exp\left[\frac{\pi}{4}i\right],\ -\frac{7\pi}{6}<\arg(z)<\frac{\pi}{6}\right)$$

with irrelevant constant multiples omitted.

The viscous solutions $\phi_{3,4}$ can be expressed in an alternative form. Note that the integrand in (3.2) can be expanded in powers of σ^2, which may be sufficiently small for large kR. Integrating the resulting series term by term yields the following expressions

$$\phi_{3,4}=\chi_{3,4}^{(0)}(\zeta)+\sigma^2\chi_{3,4}^{(1)}(\zeta)+\cdots, \tag{3.4 a, b}$$

with

$$\zeta=(kR)^{1/3}(y-c).$$

In particular, the first terms of the expression are given by

$$\chi_3^{(0)}(\zeta)=\int_\infty^\zeta(\zeta-\zeta')h_1(\zeta')d\zeta',\quad \chi_4^{(0)}(\zeta)=\int_{-\infty}^\zeta(\zeta-\zeta')h_2(\zeta')d\zeta', \tag{3.5 a, b}$$

where the functions h_1 and h_2 are expressed in terms of the Hankel functions of order one-third as

$$h_{1,2}(\zeta)=\zeta^{1/2}H_{1/3}^{(1),(2)}\left[\frac{2}{3}(i\zeta)^{3/2}\right].$$

§ 4. Eigenvalue equations and results

The next step is to substitute the explicit solutions in (2.22) and reduce the eigenvalue equations to manageable forms. For this purpose, we tentatively employ the same approximation procedure as used in the ordinary stability theory for flows with rigid boundaries (cf. Lin[3]). According to the theory, the viscous solutions ϕ_3 and ϕ_4 are conveniently approximated by $\chi_3^{(0)}(\zeta)$ and the leading term of the asymptotic expansion (3.3 b) respectively. Making use of those approximate expressions for $\phi_{3,4}$ together with (3.1 a, b) in (2.22), expanding the determinant and neglecting the terms of $O[(kR)^{-1/2}]$ and $O[e^{-\sqrt{kR}}]$, we have a simplification of the eigenvalue equation for large kR: That is

$$\frac{W_1U'(0)}{W_1-W_0}=\frac{c}{\beta k^2}+\mathscr{F}(\zeta_0),\ (U'(0)=1) \tag{4. 1}$$

where

* In the asymptotic expressions, additional terms, which may be smaller than the error involved, are disregarded.

$$W_0 = kc - k + \frac{1}{2}(1-e^{-2k}),$$

$$W_1 = kc\left\{kc - k + \frac{1}{2}(1+e^{-2k})\right\},$$

$$\zeta_0 = (kR)^{1/3}c,$$

and $\mathscr{F}(\zeta) = \mathscr{F}_r + i\mathscr{F}_i$ is the modified Tietjens function originally introduced by Lin. For small k, the left-hand side is approximately

$$\frac{U'(0)W_1}{W_1 - W_0} \simeq \frac{cU'(0)}{k(1-c)^2}. \tag{4. 2}$$

For the neutral stability case (i.e. $c_i = 0$), (4.1) gives the following pair of relations:

$$\left.\begin{aligned} &\frac{W_1U'(0)}{W_1 - W_0} = \frac{c}{\hat{\beta}k^2} + \mathscr{F}_r(\zeta_0), \\ &\mathscr{F}_i(\zeta) = 0. \end{aligned}\right\} \tag{4. 3}$$

The latter relation implies $\zeta_0 = 2.294$ and $\mathscr{F}_r = 2.292$.

In the limiting case of the rigid boundary ($\hat{\beta} = 0$), (4.3) reduces to

$$\left.\begin{aligned} &\frac{U'(0)c_r}{k(1-c_r)^2} = \mathscr{F}_r(\zeta_0) \text{ (for small } k\text{)}, \\ &\zeta_0 = (kR)^{1/3}c_r = 2.294. \end{aligned}\right\} \tag{4. 4}$$

These are exactly the same relations as derived by Lin[4] for determining the asymptotic behavior of the lower branch of the neutral curve for Blasius flow. Then it is expected that also the modification of the lower branch with wall flexibility may well be described by (4.3).

Letting $R = \infty$ in (4.1), we have an inviscid relation

$$\Delta_0 = (W_0 - W_1) + \Theta W_0 = 0, \tag{4. 5}$$

where

$$\Theta = \hat{\beta}k^2c^{-1}.$$

In particular, for small k, (4.5) simplifies to give

$$k \simeq \frac{(1-c)^2}{M(c_0^2 - c^2)}, \tag{4. 6}$$

predicting instability if $kM < (1-c_0^2)/c_0^2$. This is nothing but the eigen-equation for the 'Kelvin-Helmholtz model' in which the basic flow is assumed to be inviscid and uniform. Thus we see that the instability predicted by (4.5) is essentially 'Kelvin-Helmholtz' instability. It must be emphasized, however, that the viscous eigenvalues given by (4.1) are unrelated to those inviscid eigenvalues given by (4.5); the approximate eigenvalue equation (4.1) is inappropriate to account directly for 'Kelvin-Helmholtz' type of instability. This situation will be seen more clearly in the numerical examples given later.

To determine the effect of finite Reynolds number on 'Kelvin-Helmholtz' waves, we now deduce another appropriate expression for the eigenvalue equation (2.22), retaining higher order terms neglected in (4.1). The proper approach is indicated by the inviscid eigenvalues given by (4.6), which suggest that in general $(kR)^{1/3}c$ as well as $(kR)^{1/3}(1-c)$ are sufficiently large for $kR \gg 1$, so that $\phi_{3,4}$ can be conveniently approximated by the asymptotic series (3.3 a, b). Using (3.3 a, b) for $\phi_{3,4}$ together with the corresponding expressions for their derivatives* in (2.22) and expanding the determinant, the result is

$$\begin{aligned}
&2\Delta_0+(ikR)^{-1/2}\Big[\Big\{2k(1-c)^{-1/2}+\frac{23}{24}(1-c)^{-3/2}-\frac{7i}{24}c^{-3/2}\Big\}\cdot\Delta_0 \\
&-2ic^{-3/2}(W_1-W_0)\,\Theta\Big]+(ikR)^{-1}\Big[\Big\{\frac{71}{24}k(1-c)^{-2}-\frac{7}{24}ikc^{-3/2}(1-c)^{-1/2} \\
&-\frac{161}{1152}ic^{-3/2}(1-c)^{-3/2}+\frac{4813}{2304}(1-c)^{-3}+\frac{455}{2304}c^{-3}\Big\}\cdot\Delta_0 \\
&-\Theta(W_1-W_0)\Big\{2ikc^{-3/2}(1-c)^{-1/2}+\frac{23}{24}ic^{-3/2}(1-c)^{-3/2}-\frac{101}{24}c^{-3}\Big\} \\
&+2k^2\{\Theta(1-c)^{-1}+3W_0c^{-1}-(ck-1)(1-c)^{-1}\}\Big]+O[(kR)^{-3/2}]=0. \qquad (4.\ 7)
\end{aligned}$$

Letting $kR=\infty$ in (4.7), we have again the inviscid relation $\Delta_0=0$. For the neutral case $c_i=0$, if the third term of $O[(kR)^{-1}]$ is discarded in (4.7), it gives only a trivial relation $W_0=W_1=0$. However it is found numerically that, except in that case, the sum of only the first two terms of (4.7) can be a good approximation. We also note that in the limiting case of a rigid boundary, (4.7) gives no meaningful solution unlike another expression (4.1).

Numerical examples: In the present problem, the mechanical properties of a flexible boundary are specified by relative mass parameter M and free wave speed c_0. Stability diagrams are computed by (4.3) and (4.7) for the three cases; $M=10$, 50 and 100, all with $c_0^2=0.5$. The results are presented in Figs. 3, 4 and 5; the labels (A) and (B) are intended to show that the respective curves are determind by (4.3) or (4.7), a_i being the amplification factor defined by $a_i=kc_i$. The curves for rigid wall are also included for comparison. In the calculations of amplified oscillations ($a_i>0$) by (4.3) and (4.4), the same perturbation scheme as used by Schlichting[6] is applied. For (4.7), all numerical works are carried out with the help of an OKITAC-5090H computer. A set of eigenvalues for a negative value of c_i, i.e. $c_i=-0.1$, is also calculated from (4.7) and is listed in Table 1 together with those for $c_i=+0.1$.

From these numerical examples, it is seen that the stability characteristics described by (4.7) are, as is reasonable to expect, largely independent of Reynolds

* These must be determined directly from (3.2) in the same way as for (3.3 a, b)

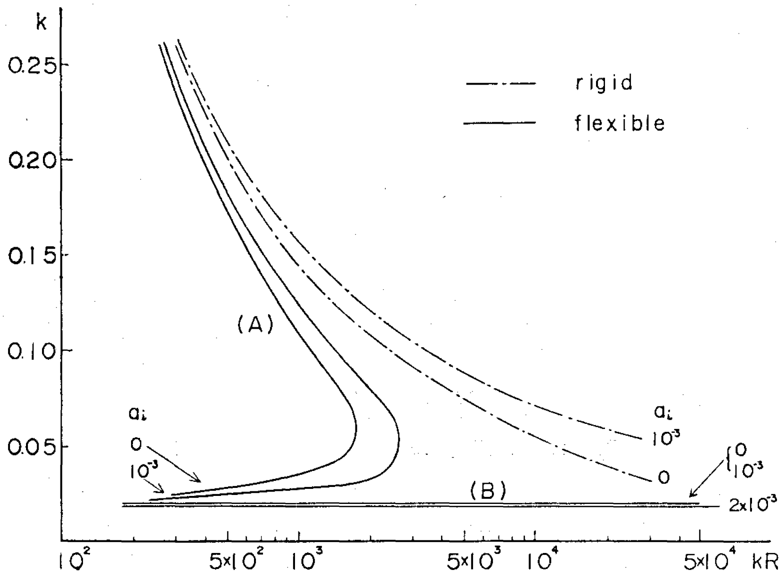

Fig. 3. Stability curves ($M=50$, $c_0^2=0.5$)

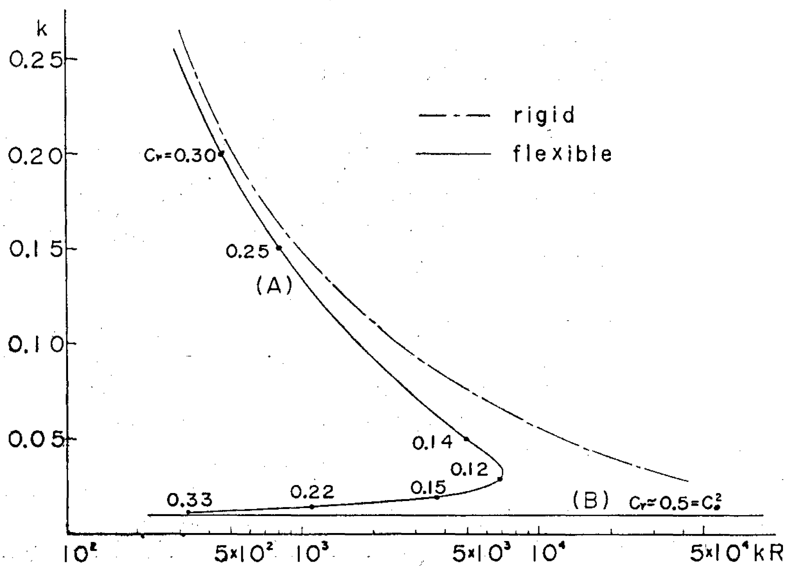

Fig. 4. Neutral stability curves ($M=100$, $c_0^2=0.5$)

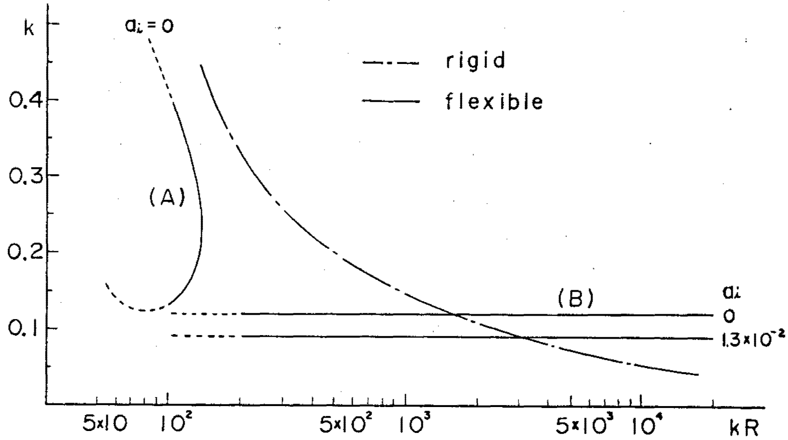

Fig. 5. Stability curves ($M = 10$, $c_0^2 = 0.5$)

Table 1 (M=50, c_0^2=0.5)

kR	2×10^2	5×10^2	2×10^3	2×10^4	∞
k_+ k_-	0.0186 0.0189	0.0187 0.0188	0.0187 0.0187	0.0187 0.0187	0.0187
c_{r+} c_{r-}	0.503 0.549	0.515 0.534	0.521 0.526	0.523 0.524	0.524

The subscripts (+), (−) are indicative of the values corresponding to c_i=+0.1 and −0.1 respectively.

number, and further that the roots of (4.7), as well, occur nearly in complex conjugate pairs just as those of the inviscid relation (4.5) do: for fairly wide range of kR, the stability characteristics of 'Kelvin-Helmholtz' waves are well represented, in all essential respect, by the inviscid equation (4.5). In the figures, the neutral curves labelled (B) are then simply extrapolated from the values given by the inviscid relation.

In contrast to the branches (B), those labelled (A) sensibly depend on Reynolds number. So long as effective stiffness* $\hat{\beta}k^2$ is not too small, the branches (A) have the same trends as the corresponding ones for a rigid boundary. However, as k (and hence effective stiffness) decreases to approach the value at which 'Kelvin-Helmholtz' instability may occur, the branches (A) assume a quite different aspect; after reaching a certain maximum value of kR, they extend in the reverse direction, namely to small kR and approach the other branches (B). The underlying assumptions used in the derivation of (4.1) and

* c.f. (2.14)

(4.7) make it impracticable to extend the calculations to smaller values of kR and examine in detail the connection between the two different kinds of branch. It seems probable, however, that the branches (A) may be finally merged into the branches (B).

We now refer to the eigenvalue equation obtained in the Benjamin's analysis[1]: That is

$$\mathscr{F}(\zeta) = u + iv + \alpha. \tag{4.8}$$

Here α is a 'response coefficient' and is given, in the case of interest, as $\alpha = -\frac{c}{\hat{\beta}k^2}$, u and v being the functions introduced by Lin. Especially, for small k, u can be written in the form

$$u \simeq \frac{cU'(0)}{k(1-c)^2}, \tag{4.9}$$

which is the same as the approximation for the left-hand side of (4.1) (c. f. (4.2)). Then we see that (4.1) is identical to the Benjamin's formula (4.8) except for the term iv which essentially represents the effect of the velocity profile curvature and has no counterpart in the present analysis. (4.1), as stated above, fails to cover 'Kelvin-Helmholtz' type of instability, so with the Benjamin's formula (4.8). In fact, the curve of neutral stability (for Blasius profile) given by (4.8) with $\alpha = -\frac{c}{\hat{\beta}k^2}$ forms such a closed loop* as shown schematically in Fig. 6. Obviously, such a stability dyagram is imcomplete; it must be supplemented, at least, with one more branch extending to infinitely large R. It is to be noted, however, that the shape of the 'lower' branch of the neutral curve is similar to that of our branch labelled (A), and that the c-k relation, corre-

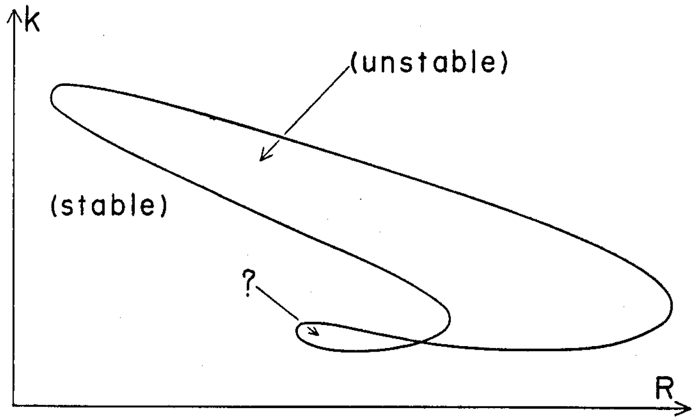

Fig. 6.

* No mention was made of this fact in Benjamin's analysis, nor in Landahl's one either.

sponding to the parts with lowest values of k on the neutral curve, is nearly the same as that predicted by the 'Kelvin-Helmholtz' theory. The present analysis affords some clarification of the behaviour of the 'lower' branch in connection with 'Kelvin-Helmholtz' type of instability. The asymptotic behaviour of the 'upper' branch, which critically depends on the velocity profile curvature, will be examined in a later paper.

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