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PRECISE DETERMINATION OF THE SOLITARY WAVE OF EXTREME HEIGHT ON WATER OF A UNIFORM DEPTH

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In view of the facts that a solitary wave which can be produced upon the surface of water of a uniform depth is the limit attained when the wave length of a surface wave grows indefinitely and that therefore it constitutes an idealized model out of which practical knowledges of long waves may be derived, in this paper factors characteristic to the solitary wave of the extreme height, including the wave profile and the distribution of the surface velocity, are given in great detail. As a whole, miscellaneous numerical values already published by various authors are verified. It is established above all that the ratios of wave velocity vs. tidal wave velocity and wave height vs. water depth are 1.2854 and 0.8262 respectively, both of which are supposed to be correct to the 4th decimal places.

1. Introduction

About ten years ago, one of the present authors performed a series of computations concerning surface waves of permanent type; in particular for the solitary wave of the extreme height he found between the wave velocity U and the wave height A the relations

$$U = 1.286\sqrt{gH} \quad \text{and} \quad A = 0.8266\,H.$$

Besides, he gave the numerical tables of the wave profile and of the surface velocity [1].[4] However, the expressions mentioned above are slightly different from those due to McCOWAN [2] who showed

$$U = 1.25\sqrt{gH} \quad \text{and} \quad A = 0.780\,H.$$

Recently LENAU published a new calculation [3], in which he arrived at

$$U = 1.2862\sqrt{gH} \quad \text{and} \quad A = 0.8281\,H,$$

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forms very close to ours. In all these equalities, g and H represent the acceleration due to gravity and the depth of water, respectively.

Taking full advantage of the nowaday situation that the use of electronic digital computers has been remarkably expanded, we have been engaged since some time ago in evaluating the maximum heights of permanent waves for various wave lengths, or, in other words, in determining the so-called breaking index curve from a hydrodynamical point of view. Thus we are now given an opportunity for re-examining and revising the previous computations explained in [**1**]. In the present paper, however, a part of the former procedure is modified, that is to say, VILLAT's formula is to be employed instead of representations in terms of FOURIER series in order to find conjugate harmonic functions. According to our past experience (unpublished), however, the computation technique by means of VILLAT's formula, when applied with the same angular intervals as in FOURIER analysis, through which the same degree of accuracy might be naturally expected, was found to lead us as a matter of fact to the values of U and A sensibly different from (less by 2 % than) those given in the preceding report [**1**]. It should be necessary, therefore, to determine by all means the side which is correct, or *more correct*, speaking more precisely, and further to know the reason of this discrepancy, if possible. On the other hand, as the result of some considerations, the velocity distribution on the water surface for a solitary wave, namely $\tau(\sigma)$ and $\theta(\sigma)$ to appear in the following sections, may be substituted approximately for those of all kinds of waves in which the wave length L satisfies the condition $L/H \gtrsim 10$. The velocity distribution on the surface of the highest solitary wave, therefore, may be used safely as an approximation to that of any wave just breaking, as long as the above condition is fulfilled, and in fact this range covers the greater part of the portion practically useful of the breaking index curve.

We may conclude, therefore, that it should be quite significant to have characteristic values of a solitary wave worked out precisely on the basis of the hydrodynamical theory toward establishing a basis for accurate numerical works to be performed in the future on one hand, and toward computing the breaking index curve in great detail on the other. This is the background of our calculations to be described in the followings. However, only the solitary wave is treated in this paper; the breaking index curve will be discussed separately in near future.

2. Formulation of the problem

The mathematical theory underlying the present computations of a solitary wave is essentially the same as was employed already in the previous paper [**1**], but in what follows it is slightly modified in a manner mentioned elsewhere [**4**]. For the sake of completeness, let us begin our discussions with a brief exposition of our procedure.

Let the x-axis of our physical plane $z = x+iy$ be placed along the bottom of water, and the y-axis directed upward. A permanent solitary wave can be put to rest in reference to a coordinate frame, moving with the velocity of the wave.

Let us suppose in what follows that in this coordinate system the water flows from left to right. Let the center of the wave crest be situated on the y-axis. Denoting the complex potential of this flow by

$$W(z) = \varphi + \mathrm{i}\psi,$$

say, it is well-known that q, the magnitude of the velocity of flow (reduced dimensionless by dividing it with U) at the point z, and θ, the angle between its direction and the horizontal, can be given through the formula

$$\frac{1}{U}\frac{\mathrm{d}W}{\mathrm{d}z} = q\mathrm{e}^{-\mathrm{i}\theta}. \tag{2.1}$$

If, therefore, we should succeed in formulating the function $W(z)$, then our problem would be solved completely.

Since we are not enabled to find out $W(z)$ directly, we first map the W-plane conformally onto the ζ ($= \xi + \mathrm{i}\eta$)-plane by the function

$$\frac{1-\zeta}{1+\zeta} = \cosh\left(\frac{\pi W}{2UH}\right),$$

i.e.

$$\zeta = -\tanh^2\left(\frac{\pi W}{4UH}\right). \tag{2.2}$$

In FIGURE 1 which shows schematically the successive transformations above-mentioned, capital letters and arrows indicate, respectively, the correspondence of representative points situated in each of these three domains and that of the contours enclosing them. Thus the region occupied by the flow in the W-plane, AO_1BCODA, is mapped onto the interior of the unit circle on the ζ-plane in such a way that is specified by the letters and the arrows. COD, the bottom of the flow field, is mapped two-fold on the negative ξ-axis from -1 to 0. Irrespective as to whether we are in the interior or on the boundary, if $-\varphi + \mathrm{i}\psi$ corresponds to $\xi + \mathrm{i}\eta$, then does $\varphi + \mathrm{i}\psi$ to $\xi - \mathrm{i}\eta$ and, in particular, points on the ψ-axis are mapped on the positive part of the ξ-axis from 0 to 1. Now that the wave profile on the z-plane is symmetrical with regard to the y-axis, so are all the streamlines with respect to the same axis too. The complex velocity $q\exp(-\mathrm{i}\theta)$ corresponding to $W = -\varphi + \mathrm{i}\psi$ is found, therefore, to be nothing but the complex conjugate of the same quantity associated with $W = \varphi + \mathrm{i}\psi$. It follows from this fact, through the transformation mentioned above, that on the ζ-plane complex velocities at the points $\xi + \mathrm{i}\eta$ and $\xi - \mathrm{i}\eta$ are conjugate to each other, and that especially on the ξ-axis, irrespective of ξ, whether positive or negative, the complex velocity gives a value, definite and real. Namely the relations,

$$q(\bar{\zeta}) = q(\zeta) \quad \text{and} \quad \theta(\bar{\zeta}) = -\theta(\zeta), \tag{2.3}$$

hold in the region of the ζ-plane consisting of the interior of the unit circle and its boundary. The bar above a letter denotes the complex conjugate of the quantity.

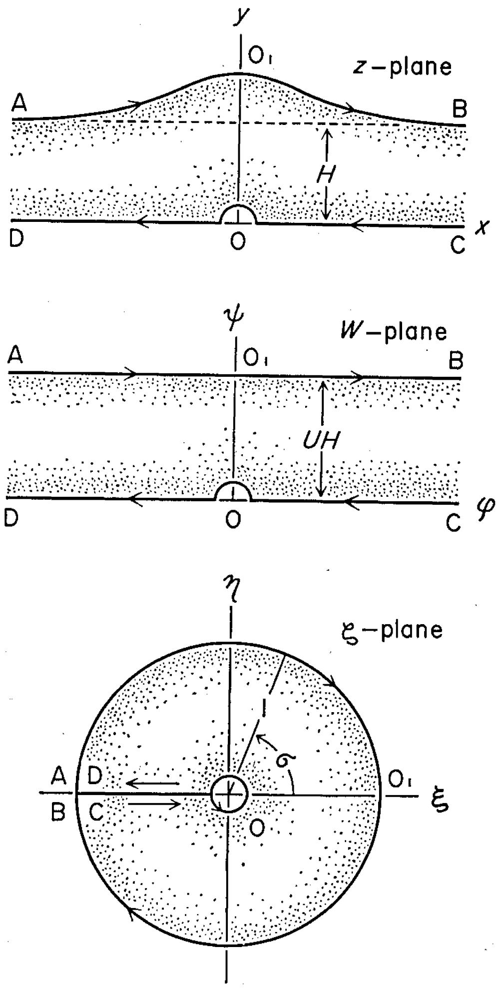

FIGURE 1. Transformations.

Introducing a new function Ω defined by the equation

$$\Omega(z) = \mathrm{i}\ln\left(\frac{1}{U}\frac{\mathrm{d}W}{\mathrm{d}z}\right) = \theta + \mathrm{i}\tau, \tag{2.4}$$

where we put

$$\tau = \ln q, \tag{2.5}$$

let us scrutinize the property of the function $-\mathrm{i}\Omega$ in the unit circle on the ζ-plane. Owing to SCHWARZ's reflection principle analytic continuation may be practised across the cut along the ξ-axis running from -1 to 0, and the function thus yielded is found regular everywhere in the unit circle. Accordingly, Ω itself also is a regular function. On the circumference of the unit circle, on the other hand, Ω may be proved to be regular except the points $\zeta=+1$ and -1 corresponding respectively to an angular crest, as long as it takes place, and to the point at infinity of the z-plane [**5**]. The function Ω, therefore, has the same property as the function ω introduced before by LEVI-CIVITA [**6**], and we may make use of his theory without any alteration: differentiating (2.2) we have

$$\frac{1}{U}\frac{\mathrm{d}W}{\mathrm{d}\zeta} = \mathrm{i}\frac{2H}{\pi}\frac{1}{\sqrt{\zeta}\,(1+\zeta)} \qquad (-\pi < \arg\,\zeta \leqq \pi). \tag{2.6}$$

Rewriting with the aid of (2.4), we are led to

$$\mathrm{d}z = \mathrm{i}\frac{2H}{\pi}\frac{\mathrm{e}^{\mathrm{i}\Omega(\zeta)}}{\sqrt{\zeta}\,(1+\zeta)}\,\mathrm{d}\zeta. \tag{2.7}$$

If we succeed in formulating $\Omega(\zeta)$ in a way or another, then just by integrating (2.7) we can find $\zeta = \zeta(z)$, and simply through substitution we shall arrive at the function $\Omega(z)$ or, in other words, at full understanding of the nature of the field of flow on the physical plane.

Since the circumference of the unit circle $\zeta = \exp\,(\mathrm{i}\sigma)$ $(-\pi < \sigma \leqq \pi)$ corresponds to the surface of water, the Ω-function on the surface, i.e.

$$\Omega(\mathrm{e}^{\mathrm{i}\sigma}) = \theta(\sigma) + \mathrm{i}\tau(\sigma),$$

is generally an analytic function of $\exp\,(\mathrm{i}\sigma)$, a point on the circumference, and moreover needs to satisfy the condition on the water surface. On the physical plane, this is given explicitly as so-called BERNOULLI's theorem which is valid over the surface of water, namely

$$q^2 + 2gU^{-2}y = \text{constant}, \tag{2.8}$$

and by differentiation along the surface (a streamline), we have

$$q\frac{\mathrm{d}q}{\mathrm{d}s} = -\frac{g}{U^2}\frac{\mathrm{d}y}{\mathrm{d}s},$$

where $\mathrm{d}s$ stands for an element of that streamline. If we take $\mathrm{d}s$ in the direction corresponding to σ increasing, then as obvious from FIGURE 1, since $\mathrm{d}s$ is directed reverse to the flow, we should have

$$\mathrm{d}z = -\,\mathrm{d}s\,\mathrm{e}^{\mathrm{i}\theta}, \quad \text{so that} \quad \frac{\mathrm{d}y}{\mathrm{d}s} = -\sin\theta,$$

consequently

$$q\,\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{g}{U^2}\sin\theta. \tag{2.9}$$

In order to transform (2.9) into a condition on the circumference of the unit circle, we have to write $\mathrm{d}s$ in terms of $\mathrm{d}\sigma$ taking account of (2.7). We obtain in this way

$$ds = \frac{H}{\pi} \frac{1}{q \cos \frac{\sigma}{2}} d\sigma. \tag{2.10}$$

Substitution of this relation for ds in (2.9) yields

$$q^2 \frac{dq}{d\sigma} = p \sec \frac{\sigma}{2} \sin \theta, \tag{2.11}$$

where we put $$p = \frac{1}{\pi F^2} \quad \text{and} \quad F = \frac{U}{\sqrt{gH}}. \tag{2.12}$$

To sum up, the central part of our problem may be reduced in constructing, in the domain composed of the interior and the circumference of the unit circle, the function $\Omega(\zeta)$ in such a way that (i) on the circumference the equation (2.11) should be valid, (ii) at $\zeta = -1$ or $\sigma = \pm\pi$, we should have $\theta = \tau = 0$, and (iii) symmetry property,

$$\tau(\bar{\zeta}) = \tau(\zeta) \quad \text{and} \quad \theta(\bar{\zeta}) = -\theta(\zeta),$$

should be valid everywhere. That the solution of this problem exists for an appropriate range of values of p is well-known as the existence theorem.

3. Calculation scheme for the highest wave

In order to work out in practice the wave profile and other things by applying the general theory to the highest solitary wave which we have in mind specifically as our objective, we need preliminarily to take into consideration the facts that at the summit ($\zeta=1$, in other words) the wave has the angle of 120° and that accordingly the velocity should be zero at this point. With a view to providing Ω beforehand with the properties

$$\lim_{\sigma \to \pm 0} \theta(\sigma) = \pm\pi/6 \quad \text{and} \quad \lim_{\sigma \to 0} \tau(\sigma) = -\infty, \tag{3.1}$$

let us divide $\Omega(\zeta)$ into two parts:

$$\Omega(\zeta) = \Omega_0(\zeta) + \Omega_r(\zeta), \tag{3.2}$$

say, in which we put

$$\Omega_0(\zeta) = \frac{i}{3} \ln \frac{1-\zeta}{2}. \tag{3.3}$$

It will be readily noticed that Ω_0 thus introduced satisfies the conditions (ii) and (iii) mentioned at the end of the last section. Besides, on the circumference of the unit circle, we have

$$\Omega_0(e^{i\sigma}) = \theta_0(\sigma) + i\tau_0(\sigma),$$

say, where

$$\theta_0(\sigma)=\begin{cases} +\dfrac{\pi-\sigma}{6} & \text{for} \quad 0<\sigma\leqq\pi, \quad \text{and} \\ -\dfrac{\pi+\sigma}{6} & \text{for} \quad -\pi\leqq\sigma<0, \end{cases}$$

and

$$\tau_0(\sigma)=\frac{1}{3}\ln\left|\sin\frac{\sigma}{2}\right| \quad \text{for} \quad -\pi\leqq\sigma\leqq\pi. \tag{3.4}$$

Properties of these functions agree exactly with those mentioned in (3.1). With respect to the remainder, $\Omega_r(\zeta)$, therefore, we have only to require that the value on the circumference of the unit circle, i.e.

$$\Omega_r(e^{i\sigma}) = \theta_r(\sigma) + i\tau_r(\sigma),$$

say, satisfies the following conditions :

$$\theta_r = 0,\ \tau_r = \text{a finite real constant} \quad \text{at } \sigma = 0, \tag{3.5}$$

and

$$\theta_r = 0,\ \tau_r = 0 \quad \text{at } \sigma = \pm\pi. \tag{3.6}$$

In addition, the function $\Omega_r(\zeta)$ should be regular elsewhere on the circumference and in the interior of the unit circle. What we have to do next is to specify such a function compatible with (i) and (iii) stated before.

Now, as is well-known, a function $\Omega_r(\zeta)$, regular in the unit circle, can be determined through an integral operation apart from a constant, purely real or imaginary, provided that on the circumference of that circle, θ_r, the real part, or τ_r, the imaginary part, is prescribed. Our problem, therefore, may be reduced further to finding $\theta_r(\sigma)$ or $\tau_r(\sigma)$. However, since they are odd and even functions of σ respectively, we have only to evaluate them in the range $0\leqq\sigma\leqq\pi$. Conditions necessary for it are (3.5), (3.6), and also (2.11), the surface condition left untouched so far. With a view to simplifying the subsequent computations, let us introduce a new variable $Q(\sigma)$ defined as

$$Q(\sigma) = (3p)^{-1/3}q(\sigma), \tag{3.7}$$

where we have obviously $q(\sigma) = \exp\{\tau_0(\sigma)+\tau_r(\sigma)\}$.

Replacing $q(\sigma)$ by $Q(\sigma)$ in (2.11), and after integration, we obtain

$$Q^3(\sigma) = \int_0^\sigma \sec\frac{\tilde{\sigma}}{2}\sin\theta(\tilde{\sigma})d\tilde{\sigma}, \tag{3.8}$$

where

$$\theta(\tilde{\sigma}) = \theta_0(\tilde{\sigma})+\theta_r(\tilde{\sigma})\,;$$

in deriving (3.8) we made use of $q(0) = 0$. Connecting (3.8) with (3.7), we should arrive at an alternative form of (2.11). Incidentally, the equality $q(\pi) = 1$ results in

$$\frac{1}{3p}=\int_0^{\pi}\sec\frac{\tilde{\sigma}}{2}\sin\theta(\tilde{\sigma})\,d\tilde{\sigma}, \tag{3.9}$$

through which we are enabled to determine p.

Let us continue transformations. Employing the form readily derived from (3.7), i.e.

$$\tau(\sigma)=\ln q(\sigma)=\frac{1}{3}\ln(3p)+\ln Q(\sigma),$$

we obtain

$$\tau_r(\sigma)=\frac{1}{3}\ln(3p)+t_1(\sigma), \tag{3.10}$$

where we write

$$t_1(\sigma)=\ln Q(\sigma)-\frac{1}{3}\ln\sin\frac{\sigma}{2}\cdot$$

Substitution of (3.8) for $Q(\sigma)$ in the above yields

$$t_1(\sigma)=\frac{1}{3}\ln\int_0^{\sigma}\sec\frac{\tilde{\sigma}}{2}\sin\{\theta_0(\tilde{\sigma})+\theta_r(\tilde{\sigma})\}\,d\tilde{\sigma}-\frac{1}{3}\ln\sin\frac{\sigma}{2}; \tag{3.11}$$

$t_1(0)=0$ will be verified without difficulty. The combined form of (3.10) and (3.11) constitutes our ultimate expression for the water surface condition.

At the present stage, we must take into account one more condition: as evident, this is nothing but an explicit statement that $\theta_r(\sigma)$ and $\tau_r(\sigma)$ are conjugate to each other. If we write it by means of VILLAT's formula, we have

$$\theta_r(\sigma)=\text{constant}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\tau_r(\tilde{\sigma})-\tau_r(\sigma)\}\cot\frac{\tilde{\sigma}-\sigma}{2}\,d\tilde{\sigma}, \tag{3.12}$$

or considering that $\tau_r(\sigma)$ is an even function of σ,

$$\theta_r(\sigma)=\text{constant}-\frac{\sin\sigma}{\pi}\int_0^{\pi}\frac{\tau_r(\tilde{\sigma})-\tau_r(\sigma)}{\cos\tilde{\sigma}-\cos\sigma}\,d\tilde{\sigma}.$$

However, from the fact that θ_r must be zero at $\sigma=0$, it follows that the constant on the right-hand side should vanish identically. Furthermore, rewriting $\tau_r(\sigma)$ in terms of $t_1(\sigma)$ by (3.10), we shall arrive at

$$\theta_r(\sigma)=-\frac{\sin\sigma}{\pi}\int_0^{\pi}\frac{t_1(\tilde{\sigma})-t_1(\sigma)}{\cos\tilde{\sigma}-\cos\sigma}\,d\tilde{\sigma}. \tag{3.13}$$

Through the above formulations, we are led to the conclusion that $\theta_r(\sigma)$ and $\tau_r(\sigma)$ are to be found as the solutions of the simultaneous integral equations (3.11) and (3.13). It is well-known that (3.12) may be expressed in a simple form as a relationship between two FOURIER series representing $\theta_r(\sigma)$ and $\tau_r(\sigma)$. However, in order to attain a satisfactory accuracy by dealing with these series, we need to take a large number of terms. In this paper, we shall choose the alternative of evaluating very accurately the integral appearing in (3.13). Once $\theta_r(\sigma)$ be found out, it can be derived from the definition of $t_1(\sigma)$ (see (3.11)) that

$$t_1(\pi) = \frac{1}{3}\ln\int_0^{\pi} \sec\frac{\tilde{\sigma}}{2}\sin\,\theta(\tilde{\sigma})\,d\tilde{\sigma}.$$

Comparing it with (3.9), we obtain

$$t_1(\pi) = -\frac{1}{3}\ln(3p), \quad \text{or} \quad p = \frac{1}{3}\exp\{-3t_1(\pi)\}. \tag{3.14}$$

Accordingly, from (3.10) τ_r may be calculated by the formula

$$\tau_r(\sigma) = -t_1(\pi)+t_1(\sigma). \tag{3.15}$$

In order to go back from $\theta_r(\sigma)$ and $\tau_r(\sigma)$ thus worked out to the velocity (q, θ), obviously we have only to practise the following computations:

$$q(\sigma) = \exp\{\tau_0(\sigma)+\tau_r(\sigma)\} = \sin^{1/3}\frac{\sigma}{2}\exp\{\tau_r(\sigma)\},$$

and

$$\theta(\sigma) = \theta_0(\sigma)+\theta_r(\sigma) = \frac{\pi-\sigma}{6}+\theta_r(\sigma). \tag{3.16}$$

In concluding this section, it should be noted that out of BERNOULLI's theorem (2.8) and the definition of p (2.12), the maximum wave height A may be determined in the form

$$\frac{A}{H} = \frac{1}{2\pi p}. \tag{3.17}$$

4. Procedure of numerical computations

The iteration method is employed in solving the simultaneous integral equations (3.11) and (3.13). Namely, in the first place by assuming $t_1(\sigma)$ appropriately, $t_1^{(0)}$ say, substitute it for t_1 in (3.13), and evaluate $\theta_r(\sigma)$, $\theta_r^{(1)}$ say, with a method of numerical integration. Next, out of $\theta_r(\sigma)$ thus obtained, calculate numerically $t_1(\sigma)$, $t_1^{(1)}$ say, makig use of (3.11) ; this completes the lst cycle. The similar process may be continued in the 2nd cycle to produce $\theta_r^{(2)}$ and $t_1^{(2)}$, say. Comparing these two kinds of θ_r's and t_1's (in other words, $\theta_r^{(1)}$, $t_1^{(1)}$ and $\theta_r^{(2)}$, $t_1^{(2)}$), provided that both pairs be found coincident down to five places of decimals, then the values of $\theta_r(\sigma)$ and $t_1(\sigma)$ thus obtained are to be regarded as *correct.* On the contrary, if there should be sensible differences, the new $t_1(\sigma)$, i.e. $t_1^{(2)}$, will be brought back again into (3.13) to yield the 3rd $\theta_r(\sigma)$, $\theta_r^{(3)}$ say. Further, repeating the process, we are led to the 3rd $t_1(\sigma)$, $t_1^{(3)}$, and between the 2nd and the 3rd pairs of (θ_r, t_1) the same test of accuracy will be made. The whole procedure will be continued until satisfactory results are arrived at. The process is shown schematically by FIGURE 2.

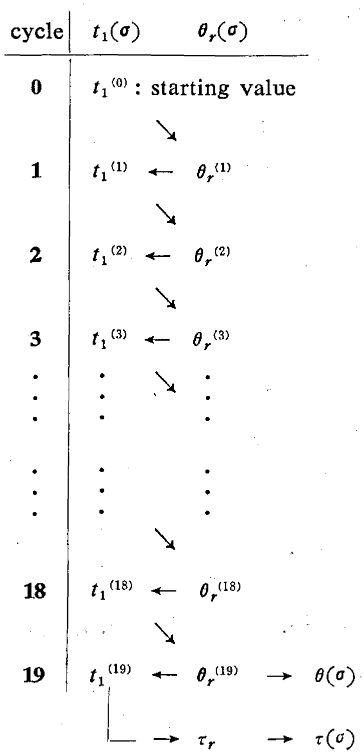

FIGURE 2. Flow diagram.

The numerical integrations contained in (3.11) and (3.13) are carried out with the aid of SIMPSON's rule by evaluating the integrands at intervals of 1° of their arguments ranging from 0° through 180°. However, due to necessity arising from the iteration process mentioned above, contrary to the ordinary SIMPSON's formula, both of the integrand and its integral should be computed with the common step, always at intervals of 1°. For this purpose, we use the following approximation: the integral over the range of 1° may be approximated by the rear-half of the area between the abscissa and the parabola passing through three points situated one after another at intervals of 1°. Or to be more precise, denoting the values of $y(x)$ at the points $x = 0$, h, and $2h$ by a_0, a_1, and a_2 respectively, our approximation formula is written as

$$\int_0^h y(x)\,dx \doteqdot \frac{h}{12}(5a_0 + 8a_1 - a_2). \tag{4.1}$$

However, for the last interval, i.e. from 179° to 180°, the integral is approximated this time by subtracting the rear-half obtained in this way from the value computed through ordinary SIMPSON's formula over the range from 178° to 180°. Comparing with each other the labors involved in the numerical integrations of (3.11) and (3.13), it is evident that the latter is about 180 times as large as the former.

Both of the integrands contained in (3.11) and (3.13) have singular points yielding the indeterminate form 0/0; they are $\tilde{\sigma} = \pi$ for (3.11), and $\tilde{\sigma} = \sigma$ in (3.13). These points may be treated in the following manner. In the first place, evaluating the integrand of (3.11) at the point $\tilde{\sigma} = \pi$ by means of limiting procedure, we have

$$\lim_{\tilde{\sigma}\to\pi-0} \sec\frac{\tilde{\sigma}}{2}\sin\theta(\tilde{\sigma}) = \frac{\cos\theta(\pi)}{-\frac{1}{2}\sin\frac{\pi}{2}}\theta'(\pi) = \frac{1}{3} - 2\theta_r'(\pi), \tag{4.2}$$

since $$\theta_0(\pi) = \theta_r(\pi) = 0, \quad \text{and} \quad \theta_0'(\pi) = -\frac{1}{6},$$

where dashes attached to θ and θ_r denote differentiations with respect to their arguments. Thus all we have to do reduces to computing accurately the value of $\theta_r'(\pi)$. With this end in view, we assume for $\theta_r(\sigma)$ a polynomial of the third degree coincident exactly with its values at four points $\sigma = 177°$, 178°, 179°, and 180°, and out of this curve the value of $\theta_r'(\pi)$ can be estimated as follows.

$$\theta_r'(\pi) = -\{9\theta_r(179°) - 4.5\theta_r(178°) + \theta_r(177°)\}/0.0523599. \tag{4.3}$$

In the next place, let us consider the singular point defined by $\tilde{\sigma} = \sigma$ taking place in (3.13). Since it is supposed from the beginning that $\theta_r(0) = \theta_r(\pi) = 0$, cf. (3.5) and (3.6), the point σ in question may be assumed to be not equal to 0 nor π. The integrand can be transformed through a similar process:

$$\lim_{\tilde{\sigma}\to\sigma} \frac{t_1(\tilde{\sigma}) - t_1(\sigma)}{\cos\tilde{\sigma} - \cos\sigma} = -\frac{t_1'(\sigma)}{\sin\sigma}, \tag{4.4}$$

where $t_1'(\sigma) = dt_1(\sigma)/d\sigma$. As the approximate value of $t_1'(\sigma)$, we employ the central difference having the σ-point as the center. Namely, we obtain approximately

$$t_1'(\sigma) = \{t_1(\sigma°+1°) - t_1(\sigma°-1°)\}/0.0349066. \tag{4.5}$$

After these preliminaries, numerical works of the iteration can be carried out without difficulty. The final values of $\theta_r(\sigma)$ and $t_1(\sigma)$ are given at intervals of 1° of σ. Then from (3.16) we find $q(\sigma)$ and $\theta(\sigma)$; (3.14) yields p; accordingly U and A may be calculated by virtue of (2.12) and (3.17) respectively. And in order to associate $\theta_r(\sigma)$ and $\tau_r(\sigma)$ with properties of a solitary wave, we have to evaluate the function $\Omega_r(\zeta) = \theta_r(\zeta) + i\tau_r(\zeta)$. Writing it as $-i\Omega_r(\zeta) = \tau_r - i\theta_r$, let us make use of SCHWARZ-POISSON's formula with a view to finding out this function. Namely,

$$-\mathrm{i}\Omega_r(\zeta)=\mathrm{i}b+\frac{1}{2\pi}\int_{-\pi}^{\pi}\tau_r(\sigma)\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\,d\sigma,$$

ib being an arbitrary constant, purely imaginary. By replacing $\tau_r(\sigma)$ in the integrand by $t_1(\sigma)$ according to (3.14) and (3.15), and transforming at the same time the both-hand sides back to the expression for $\Omega_r(\zeta)$, we are led to

$$\Omega_r(\zeta)=-b+\frac{\mathrm{i}}{3}\ln(3p)+\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi}t_1(\sigma)\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\,d\sigma.$$

Now, considering the physical condition that at the center of the bottom, $z=0$ i.e. $\zeta=0$, it should be that $\theta=0$ or $\mathrm{re}\,\Omega(0)=0$, since it is obvious from (3.3) that $\mathrm{re}\,\Omega_0(0)=0$, we must have accordingly $\mathrm{re}\,\Omega_r(0)=0$, and so the constant b in the above equality should vanish identically. Thus

$$\Omega_r(\zeta)=\mathrm{i}\left\{\frac{1}{3}\ln(3p)+\frac{1}{2\pi}\int_{-\pi}^{\pi}t_1(\sigma)\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\,d\sigma\right\}. \tag{4.6}$$

After we have succeeded in working out $\Omega_r(\zeta)$ in this way, we are enabled to find the value of $\Omega(\zeta)$ at any point within the unit circle with the aid of (3.2). Introducing this function thus determined into the right-hand side of (2.7), after integration we can find ζ as a function of z. By substitution for ζ in $\Omega(\zeta)$, we can determine $\Omega(z)$, or the nature of the flow field.

We are now in a position to discuss the wave profile $x(\sigma)$, $y(\sigma)$ of the solitary wave of the maximum height. Once the profile is determined, the distribution of velocity on the surface of that wave may be readily derived from $q(\sigma)$ and $\theta(\sigma)$. Starting from (2.7), through substitution of $\exp(\mathrm{i}\sigma)$ for ζ, we may write

$$dz(\sigma)=-\frac{H}{\pi}\frac{e^{i\theta(\sigma)}}{q(\sigma)\cos\frac{\sigma}{2}}\,d\sigma.$$

The wave profile, therefore, can be expressed in the forms

$$\frac{x_0(\sigma)}{H}=-\frac{1}{\pi}\int_0^{\sigma}\frac{\cos\theta(\sigma)}{q(\sigma)\cos\frac{\sigma}{2}}\,d\sigma,$$

and

$$\frac{y_0(\sigma)}{H}=-\frac{1}{\pi}\int_0^{\sigma}\frac{\sin\theta(\sigma)}{q(\sigma)\cos\frac{\sigma}{2}}\,d\sigma, \tag{4.7}$$

where x_0 and y_0 are connected with x and y by the relations

$$x_0=x,\quad \text{and}\quad y_0=y-(A+H). \tag{4.8}$$

This translation of the origin is simply due to convenience of tabulation of our final results.

The numerical integrations in (4.7) are performed using ordinary SIMPSON'S rule, and results are produced from $\sigma = 0^\circ$ to 180° at intervals of 2°. Two remarks are necessary for these integrations. First, we have to note that the value of x_0/H increases without bounds at $\sigma = 180^\circ$, and so we adopt the convention that the computation of x_0/H is stopped as soon as $\sigma = 178^\circ$ is reached. Second, as obvious from the first equation of (3.16), $q(\sigma)$ behaves like $(\sigma/2)^{1/3}\, e^{-t_1(\pi)}$ in the neighborhood of $\sigma = 0$. So that $\sin\theta(\sigma)/q(\sigma)\cos(\sigma/2)$ and $\cos\theta(\sigma)/q(\sigma)\cos(\sigma/2)$ may be approximated by $\sin 30^\circ(2/\sigma)^{1/3}\, e^{t_1(\pi)}$ or $2^{-2/3}\, e^{t_1(\pi)}\,\sigma^{-1/3}$, and by $\cos 30^\circ (2/\sigma)^{1/3}\, e^{t_1(\pi)}$ or $3^{1/2}\, 2^{-2/3}\, e^{t_1(\pi)}\sigma^{-1/3}$, respectively. Over a small distance after starting from the point $\sigma = 0$, therefore, we may put approximately

$$\frac{\sin\theta}{q\cos\frac{\sigma}{2}} = 2^{-2/3}\, e^{t_1(\pi)}\sigma^{-1/3}\,(1+s_1\sigma+s_2\sigma^2+s_3\sigma^3+s_4\sigma^4),$$

and $$\frac{\cos\theta}{q\cos\frac{\sigma}{2}} = 3^{1/2}\, 2^{-2/3}\, e^{t_1(\pi)}\sigma^{-1/3}(1+c_1\sigma+c_2\sigma^2+c_3\sigma^3+c_4\sigma^4); \tag{4.9}$$

the numerical coefficients s_i and c_i ($i = 1, 2, 3,$ and 4) are to be determined from the known values of the left-hand sides at the points $\sigma = 1^\circ$, 2°, 3°, and 4°.[1] Since, therefore, the integrals on the right-hand sides of (4.7) can be given in the forms

$$\int_0^\sigma \frac{\sin\theta}{q\cos\frac{\sigma}{2}}\, d\sigma = 2^{-2/3}\, 3\, e^{t_1(\pi)}\,\sigma^{2/3}\Big(\frac{1}{2}+\frac{s_1}{5}\sigma+\frac{s_2}{8}\sigma^2+\frac{s_3}{11}\sigma^3+\frac{s_4}{14}\sigma^4\Big),$$

and $$\int_0^\sigma \frac{\cos\theta}{q\cos\frac{\sigma}{2}}\, d\sigma = 2^{-2/3}\, 3^{3/2}\, e^{t_1(\pi)}\,\sigma^{2/3}\Big(\frac{1}{2}+\frac{c_1}{5}\sigma+\frac{c_2}{8}\sigma^2+\frac{c_3}{11}\sigma^3+\frac{c_4}{14}\sigma^4\Big), \tag{4.10}$$

we can evaluate the integral for the values $\sigma = 2^\circ$ and 4°. On the other hand, for σ greater than 4°, SIMPSON's rule is put to use. In this way, a point situated on the surface of water is given in terms of σ, the parameter, and the magnitude and the direction of the velocity prevailing at that point are given by $q(\sigma)$ and $\theta(\sigma)$ respectively.

Finally, as a test of accuracy of numerical computations, we estimate the quantity

$$\frac{q^2(\sigma)}{2\pi p} + \frac{y_0(\sigma)}{H} = \Delta(\sigma), \tag{4.11}$$

say, at each station of σ used in computations, the second term on the left-hand side being the calculated value of the second expression in (4.7), while on the other hand the first term might be readily understood by a glance at (3.8). Namely, differentiating once with respect to σ, the both hand sides of that equa-

1) In practice, the numerical coefficients s's and c's were determined by solving the simultaneous equations through the ordinary procedure of elementary algebra; no routines frequently employed for an electronic computer, such as *sweep-out method*, were not used.

tion, dividing with Q, again integrating with σ, and finally substituting for Q the right-hand side of (3.7), we are led to

$$\frac{q^2(\sigma)}{2\pi p} = \frac{1}{\pi}\int_0^{\sigma} \frac{\sin\theta(\sigma)}{q(\sigma)\cos\frac{\sigma}{2}}\, d\sigma = -\frac{y_0(\sigma)}{H}. \tag{4.12}$$

So that, theoretically speaking, the quantity defined as $\Delta(\sigma)$ should vanish identically. Its deviation from zero is nothing but the error due to difference of two computation procedures applied to the same quantity, and may be regarded as a measure indicating the accuracy of numerical works involved.

5. Results of numerical computations

In reference [**1**], by giving relevant numerical coefficients, both of $\theta_r(\sigma)$ and $\tau_r(\sigma)$ were expressed in terms of FOURIER series.[1] In the present paper, we started our iteration procedure from the values of $t_1(\sigma)$ computed out of the series of $\tau_r(\sigma)$ at intervals of 1°. The essential part of our numerical works consists in repeating the cyclic computation $t_1(\sigma)\rightarrow \theta_r(\sigma)\rightarrow t_1(\sigma)\rightarrow \cdots$ until at last we arrive at a state in which further continuation can give rise to no sensible change of values. Although it is generally convenient in such a case to leave the convergence criterion to an electronic computater and, if the condition is proved satisfied, to make a step forward into the next automatically, we took the alternative, leaving room for all-round decision for ourselves in any chance necessary in the course of the computation, since convergence has to be fulfilled at as many as 180 points in the range of σ (from 1° to 180°) for each of $t_1(\sigma)$ and $\theta_r(\sigma)$.

The machine time to complete one cycle calculation $t_1^{(n-1)}\rightarrow \theta_r^{(n)}\rightarrow t_1^{(n)}$ was about 13 minutes in the average, so that at the initial stage convergence was inspected when the 5th and the 10th cycles have been finished. Afterwards, however, we used to proceed examining the results after every 2 or 3 cycles. Having confirmed that $\theta_r^{(18)}$ and $t_1^{(18)}$ were different from $\theta_r^{(17)}$ and $t_1^{(17)}$ respectively only by a quantity less than 10^{-5} for all values of σ without fail, we performed the computation of the 19th cycle, which we took as the final result. At first, because the computation method of $\theta_r'(\pi)$ were somewhat too rough, we found after 11 cycles of iteration still sensible values of $\Delta(\sigma)$ in the neighborhood of $\sigma=\pi$. The computation scheme of $\theta_r'(\pi)$, therefore, was modified into more accurate form mentioned in (4.3) in order that we may be able to continue sebsequent cycles more effectively.

In TABLE 1 values of q, θ, $-x_0/H$, $-y_0/H$, and Δ are given altogether for various values of σ from 0° to 180°. On the other hand, in FIGURE 3 the profile is shown of the solitary wave of the maximum amplitude, besides at the lower part of the same figure the fine structure of the profile in the immediate neighborhood of its top is reproduced in a magnified scale. It will be remarked that

1) See eq. (15) of reference [**1**].

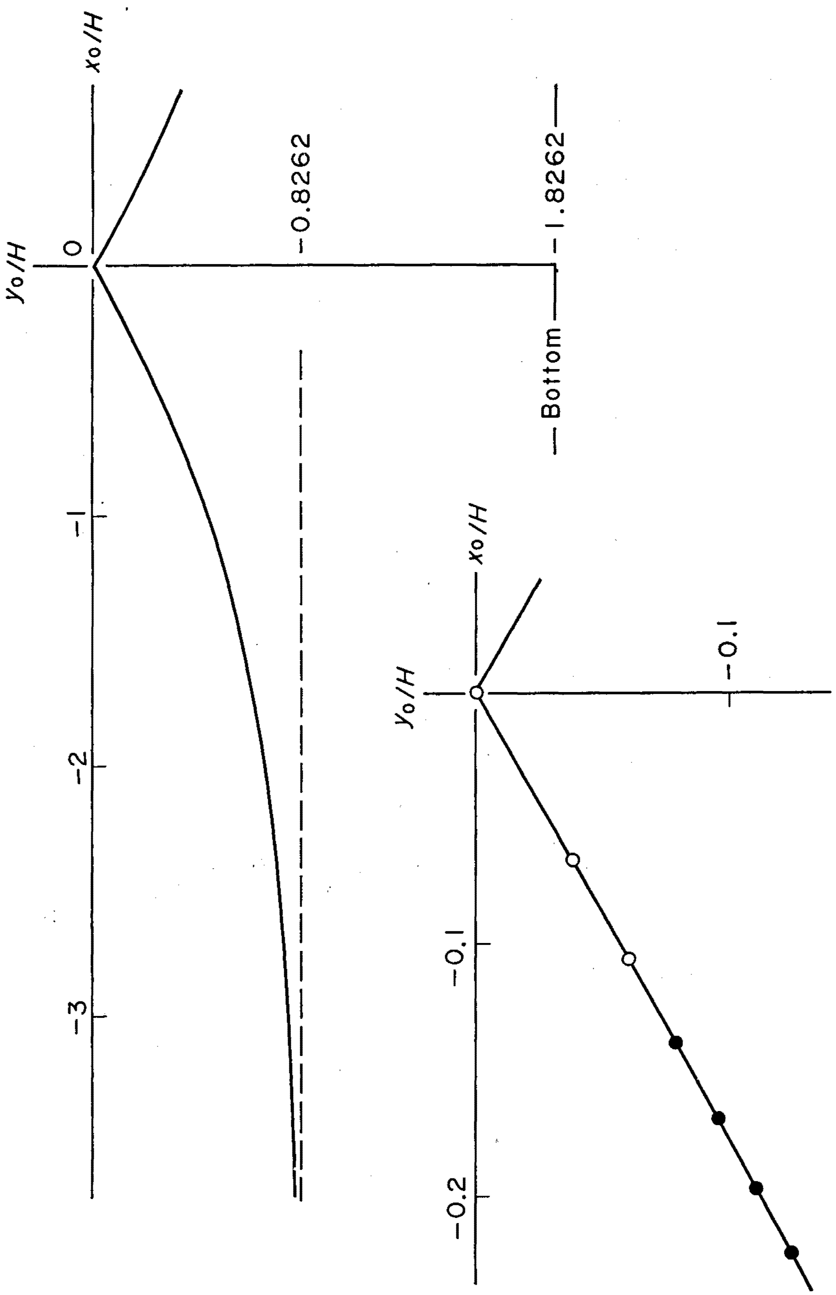

FIGURE 3. Wave profile.

the part of the curve computed from the approximate expansions (4.11) and (4.12) for $\sigma = 0°$, $2°$ and $4°$ (the open circles) is connected quite smoothly with the portion constructed through SIMPSON's formula for $\sigma \geqq 6°$ (the closed circles). Results embodied as TABLE 1 and FIGURE 3 were yielded by 19 cycles, and it is observed everywhere in the range of the variable that $\Delta(\sigma) < 0.0001$. Finally, the characteristic value, p, is given as

$$p = 0.19265, \qquad (5.1)$$

from which the followings are derived:

$$U = 1.2854\sqrt{gH} \quad \text{and} \quad A = 0.8262\,H. \qquad (5.2)$$

The relations summarized in (5.1) and (5.2) may be regarded not only as a verification of computations published already by one of the present authors [1] and by LENAU [2], but also as the formulas having improved accuracy, correct possibly to the 4th decimal places, compared with the previous calculations above-mentioned.

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References

1. YAMADA, H.: On the highest solitary wave. Rep. Res. Inst. Appl. Mech., Kyushu Univ. **5**, No. 18, 53-67 (1957).
2. McCOWAN, J.: On the solitary wave. Phil. Mag. (5) **32**, 45-58 (1891); On the highest wave of permanent type. Phil. Mag. (5) **38**, 351-358 (1894).
3. LENAU, C. W.: The solitary wave of maximum amplitude. J. Fluid Mech. **26**, 309-320 (1966).
4. YAMADA, H.: Permanent gravity waves on water of uniform depth. Rep. Res. Inst. Appl. Mech., Kyushu Univ. **6**, No. 23, 127-139 (1958).
5. LEWY, H.: On steady free surface flow in a gravity field. Comm. Pure Appl. Math. **5**, 413-414 (1952).
6. LEVI-CIVITA, T.: Détermination rigoureuse des ondes permanentes d'ampleur finie. Math. Ann. **93**, 264-314 (1925).

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TABLE 1.

$\sigma°$	$q(\sigma)$	$\theta(\sigma)$	$-x_0(\sigma)/H$	$-y_0(\sigma)/H$	$\Delta(\sigma)\times 10^6$
0	0.00000	0.52360	0.00000	0.00000	0
2	0.21543	0.51425	0.06705	0.03834	4
4	27082	50740	10675	06059	4
6	30941	50122	14026	07908	3
8	33993	49543	17036	09546	3
10	36556	48992	19818	11040	3
12	0.38785	0.48463	0.22435	0.12427	3
14	40767	47951	24926	13730	3
16	42561	47453	27315	14965	3
18	44204	46967	29621	16142	3
20	45723	46491	31858	17271	3
22	0.47139	0.46024	0.34036	0.18357	3
24	48466	45565	36163	19405	3
26	49718	45112	38247	20421	3
28	50904	44665	40293	21406	3
30	52031	44224	42306	22365	3
32	0.53107	0.43787	0.44291	0.23299	3
34	54136	43355	46250	24211	3
36	55124	42926	48187	25103	3
38	56074	42501	50105	25976	3
40	56990	42078	52006	26832	3
42	0.57875	0.41659	0.53893	0.27671	3
44	58732	41241	55768	28496	3
46	59562	40826	57633	29308	3
48	60367	40413	59489	30106	3
50	61151	40001	61338	30892	3
52	0.61913	0.39590	0.63182	0.31668	3
54	62657	39180	65023	32433	3
56	63382	38772	66862	33188	3
58	64091	38363	68700	33934	3
60	64784	37956	70538	34672	3
62	0.65462	0.37548	0.72379	0.35402	3
64	66127	37140	74223	36124	3
66	66779	36732	76072	36840	3
68	67418	36324	77927	37549	3
70	68047	35915	79789	38253	3
72	0.68665	0.35506	0.81660	0.38951	3
74	69273	35095	83540	39644	3
76	69872	34684	85432	40332	3
78	70462	34271	87336	41016	3
80	71043	33857	89255	41696	3
82	0.71617	0.33441	0.91188	0.42372	3
84	72184	33023	93139	43045	3
86	72744	32603	95108	43716	3
88	73297	32181	97096	44383	3
90	73844	31757	99106	45049	3
92	0.74386	0.31329	1.01140	0.45712	3
94	74923	30900	03198	46374	3
96	75455	30467	05284	47035	3
98	75982	30030	07398	47695	3
100	76505	29590	09543	48354	3

TABLE 1. *continued*

$\sigma°$	$q(\sigma)$	$\theta(\sigma)$	$-x_0(\sigma)/H$	$-y_0(\sigma)/H$	$\Delta(\sigma)\times10^6$
102	0.77025	0.29147	1.11721	0.49012	3
104	77540	28700	13935	49671	3
106	78053	28248	16186	50330	3
108	78563	27792	18479	50990	3
110	79070	27331	20815	51651	3
112	0.79575	0.26865	1.23198	0.52313	3
114	80079	26393	25631	52976	3
116	80580	25916	28118	53642	3
118	81080	25433	30663	54310	3
120	81580	24943	33270	54981	3
122	0.82078	0.24447	1.35944	0.55655	3
124	82577	23943	38690	56333	3
126	83075	23432	41514	57015	3
128	83573	22912	44423	57701	3
130	84073	22384	47424	58392	3
132	0.84573	0.21846	1.50525	0.59089	3
134	85075	21299	53734	59793	3
136	85578	20740	57062	60503	3
138	86084	20171	60521	61220	3
140	86593	19589	64124	61946	3
142	0.87105	0.18995	1.67887	0.62681	3
144	87621	18386	71826	63426	3
146	88142	17762	75963	64182	3
148	88668	17121	80322	64950	3
150	89200	16463	84933	65732	3
152	0.89738	0.15784	1.89831	0.66528	3
154	90285	15084	1.95058	67341	3
156	90841	14361	2.00668	68173	3
158	91408	13610	2.06728	69026	3
160	91986	12829	2.13324	69903	3
162	0.92579	0.12015	2.20569	0.70807	3
164	93190	11161	28617	71744	3
166	93820	10262	37681	72718	2
168	94476	09308	48077	73737	2
170	95162	08287	60289	74813	2
172	0.95890	0.07181	2.75129	0.75962	1
174	0.96673	05962	2.94120	77208	−1
176	0.97541	04574	3.20674	78601	−9
178	0.98557	02894	3.65718	80253	−70
180	1.00000	00000	∞	82620	−70