NUMERICAL TABLES OF SEBAN-BOND SOLUTION FOR EQUATION OF LAMINAR BOUNDARY LAYER ON A CIRCULAR CYLINDER IN AXIAL INCOMPRESSIBLE FLOW

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NUMERICAL TABLES OF SEBAN-BOND SOLUTION FOR EQUATION OF LAMINAR BOUNDARY LAYER ON A CIRCULAR CYLINDER IN AXIAL INCOMPRESSIBLE FLOW

By Jun-ichi OKABE and Gozo KIMURA*

In the simultaneous ordinary differential equations of an infinite number derived by SEBAN and BOND for the laminar boundary layer of an incompressible fluid on the exterior of a circular cylinder with flow parallel to the cylinder axis, the first three equations are solved with the aid of an electronic digital computer. The result is shown in the form of the numerical tables at the end of the paper.

1. Introduction

Sixteen years ago SEBAN and BOND published a very interesting paper, in which they gave a solution for the laminar boundary layer of an incompressible fluid of constant properties on the exterior of a circular cylinder with flow parallel to its axis; the solution is shown graphically as FIGURE 1 of their paper.[1]

Three years later, however, KELLY pointed out inaccuracy contained in their solution.[2] Namely, in connection with another problem, he integrated the differential equations derived by the above-mentioned authors using the REAC, an analog computer that allowed continuous integration with a high degree of accuracy. He observed that the BLASIUS function f_0 had been duplicated quite accurately, while the solutions for f_1 and f_2 had varied somewhat from those given by SEBAN and BOND. (With regard to the definitions of these three functions, see the next section, esp. EQUATION (2.12).) He stated that the difference was probably due to the finite interval size used in the former integration. Since unfortunately, however, results of KELLY's computations were not published except the corrected values of $f_1''(0)$, $f_2''(0)$, $f_1(\infty)$, and $f_2(\infty)$ (see TABLE 1 of section 5), we cannot as yet turn them to full account for practical purposes.

In this circumstance, it is of some significance to take up this problem again with a view to solving the differential equations *as accurately as possible*, in the practical sense of the words, with the aid of an electronic digital computer. The

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1) SEBAN, R. A. and BOND, R., Skin-Friction and Heat-Transfer Characteristics of a Laminar Boundary Layer on a Cylinder in Axial Incompressible Flow, *Journal of the Aeronautical Sciences*, vol. 18 (1951), no. 10, pp. 671–675.

2) KELLY, H. R., A Note on the Laminar Boundary Layer on a Circular Cylinder in Axial Incompressible Flow, *Journal of the Aeronautical Sciences*, vol. 21 (1954), no. 9, *Readers' Forum*, p. 634.

result of our computations is reproduced in the form of the numerical tables at the end of this paper for the convenience of future applications.

2. Problem

For an incompressible fluid of constant properties flowing with uniform free stream velocity U_0 parallel to the axis of a circular cylinder of radius a, the equations of continuity and of the laminar boundary layer can be written respectively as follows:

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial (rv)}{\partial r}=0, \tag{2.1}$$

and

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{2.2}$$

where (x, r) and (u, v) denote the axial, and the radial, coordinate, and the axial, and the radial, velocity component, respectively, ν being the kinematic viscosity of the fluid.

EQUATION (2.1) is satisfied by the introduction of a stream function ψ, such that

$$u=\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{2.3}$$

and

$$v=-\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{2.4}$$

Further making use of the transformations defined by

$$\xi=\frac{4}{a}\sqrt{\frac{\nu x}{U_0}}, \tag{2.5}$$

$$\eta=\sqrt{\frac{U_0}{\nu x}}\frac{r^2-a^2}{4a}, \tag{2.6}$$

and

$$\psi=a\sqrt{\nu U_0 x}\,f(\xi, \eta), \tag{2.7}$$

we can prove without difficulty that EQUATIONS (2.3) and (2.4) become

$$u=\frac{U_0}{2}\frac{\partial f}{\partial \eta}, \tag{2.8}$$

and

$$v=\frac{a}{2r}\sqrt{\frac{\nu U_0}{x}}\left(\eta\frac{\partial f}{\partial \eta}-f\right)-\frac{2\nu}{r}\frac{\partial f}{\partial \xi}, \tag{2.9}$$

respectively, while the remaining one, (2.2), is found to take the form

$$\xi\left(\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}-\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta}\right)$$

$$+\frac{\partial}{\partial\eta}\left\{(1+\xi\eta)\frac{\partial^2 f}{\partial\eta^2}\right\}+f\frac{\partial^2 f}{\partial\eta^2}=0. \tag{2.10}$$

The associated boundary conditions, on the other hand, are

$$\left.\begin{array}{ll} u=v=0 & \text{for} \quad r=a, \\ u=U_0 & \text{for} \quad r=\infty \quad \text{or} \quad x=0. \end{array}\right. \tag{2.11}$$

and

Expanding f in the form

$$f(\xi, \eta)=f_0(\eta)+\xi f_1(\eta)+\xi^2 f_2(\eta)+\cdots, \tag{2.12}$$

substitute it in EQUATION (2.10). Comparing the coefficients of like powers of ξ, we are led to a series of differential equations of an infinite number

$$f_0'''+f_0f_0''=0, \tag{2.13}$$

$$f_1'''+f_0f_1''-f_0'f_1'+2f_0''f_1+\eta f_0'''+f_0''=0, \tag{2.14}$$

$$f_2'''+f_0f_2''-2f_0'f_2'+3f_0''f_2+\eta f_1'''+f_1''+2f_1f_1''-f_1'^2=0, \tag{2.15}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

the boundary conditions for f_0, f_1, f_2, ... are found to be

$$f_0(0)=f_0'(0)=0, \quad f_0'(\infty)=2, \tag{2.16}$$

$$f_1(0)=f_1'(0)=0, \quad f_1'(\infty)=0, \tag{2.17}$$

$$f_2(0)=f_2'(0)=0, \quad f_2'(\infty)=0, \tag{2.18}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where dashes denote differentiation with respect to η. In what follows, just as those three authors did, we shall confine our discussions to the first three unknown functions, f_0, f_1, and f_2, retaining therefore only three equations (2.13), (2.14), and (2.15) together with the same number of boundary conditions (2.16), (2.17), and (2.18).[1)]

3. Solution and flow diagram for f_0

In solving numerically the simultaneous ordinary differential equations (2.13), (2.14), and (2.15), the procedure devised and improved by RUNGE, KUTTA, and GILL (abbreviated in the following as *RKG method*, for convenience) was employed; the computation program in an electronic digital computer was constructed according to the grammar of ALGOL.

In RKG method, the range of the independent variable η is divided into a small segment H, and starting from the initial values $f_i(0)$, $f_i'(0)$, and $f_i''(0)$ ($i=0, 1, 2$), the successive values of $f_i(\eta)$ ($\eta=0+nH$, $n=1, 2, 3,\ldots$) are worked out owing to the well-known formula. In our present case, however, as observed in (2.16) - (2.18), the values of $f_i'(\infty)$, instead of $f_i''(0)$, are prescribed. We

1) As readily noticed, the differential equation (2.13) together with the associated boundary conditions (2.16) are just the same as those for the stream function of the flow along a flat plate placed edgeways to a uniform stream. The name BLASIUS function for f_0 comes from this fact.

should, therefore, take the following way:— First, assuming $f_i''(0)$ appropriately, the value to which $f_i'(\eta)$ converges when η becomes large enough is computed; this might be regarded as $f_i'(\infty)$ practically. By modifying $f_i''(0)$ slightly, a second $f_i'(\infty)$ is produced through the same process. Repeating this procedure a number of times, the limiting value $f_i'(\infty)$ might be brought as close as we wish to the prescribed value. But if we are going to solve these three equations (2.13)-(2.15) all at once, we should have necessarily three degrees of freedom for the choice of $f_i''(0)$ $(i = 0, 1, 2)$ at the same time. Because this makes the situation helplessly complicated, first we treat EQUATION (2.13) solely. After evaluating $f_0''(0)$ through the successive approximation above-mentioned, we next take up the first and the second equations simultaneously permitting only one degree of freedom as ever, $f_1''(0)$. Finally, the third process composed of three equations but containing single undetermined parameter $f_2''(0)$ may be carried out in a similar way.

FIGURE 1 is the flow diagram showing the computation scheme for the function $f_0(\eta)$. The number in the parentheses placed at the beginning of each of the following phrases corresponds to that labelled each step in the figure.

(1) Statement of the data: U is a number not so different from $f_0''(0)$, V being an incremental step of U. D, E, and F are constants for convergency judgments; see (9) - (13) below. Finally, N indicates a constant restricting the number of steps of integration.

(2) Instruction concerning the variation of U: In passing through this station, U is subject to the increment V.

(3) Assignment of the initial values: S denotes the number of steps of integration, counting from the start $X = 0$. From now on, for convenience of printing in an electronic computer we write

$$\eta \equiv X,\ f_0 \equiv A,\ f_0' \equiv A1, \text{ and } f_0'' \equiv A2. \tag{3.1}$$

(4) Printing of the initial values of the variables X, A, $A1$, and $A2$.

(5) Advancement of integration into the next step.

(6) Instruction that if S attains N, or, in other words, the range of integration extends beyond a certain limit, we should escape from the loop and go to (10) directly.

(7) One step of RKG computation method: EQUATION (2.13) may be written in the form

$$\frac{dA}{dX} = A1,\ \frac{dA1}{dX} = A2,$$

and (3.2)

$$\frac{dA2}{dX} \equiv A3 = -A \cdot A2.$$

Now, if the values of A, $A1$, and $A2$ are given at the station $X = SH$ say, then EQUATION (3.2) enables us to find their values at the next station, $X = (S+1)H$.

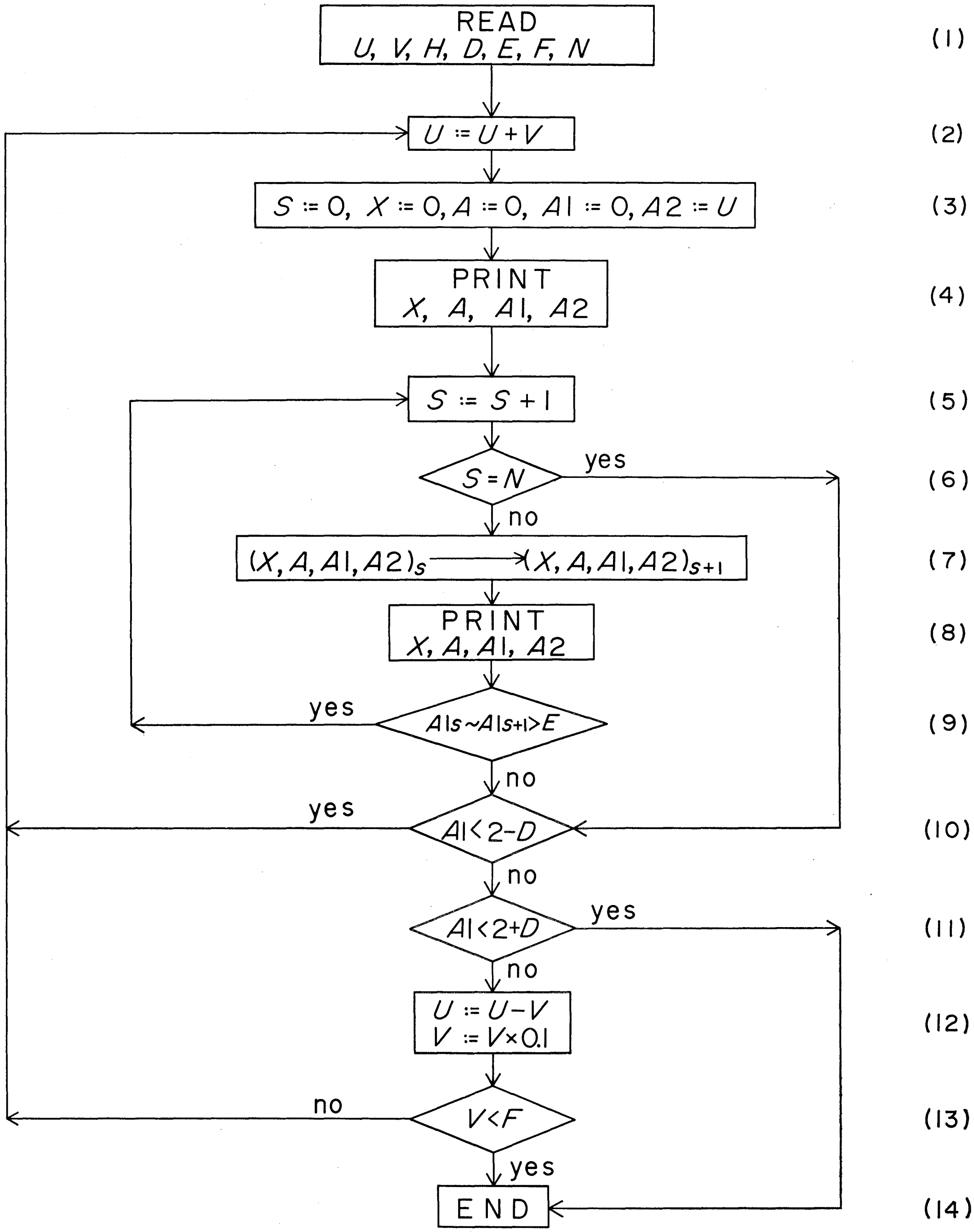

FIGURE 1. Flow diagram for f_0.

(8) Printing of X, A, $A1$, and $A2$ at the new station.

(9) Judgment of convergence of $A1$: By convergence of $A1$ we mean

$$|(A1)_{X=SH}-(A1)_{X=(S+1)H}| \leqq E, \tag{3.3}$$

where $S+1 \leqq N$.[1] If not convergent, we should continue RKG integration.

(10) When $A1$ converges or alternatively the range of integration extends beyond N, criterion should be made whether $A1 < 2-D$. If so, we go back to repeat the computation with the initial value increased by V.

(11) If $2-D \leqq A1 < 2+D$, our computation is over. If $A1 \geqq 2+D$, on the other hand, we should proceed to (12) below.

(12) Return to the *original* U,[2] and multiply V by the factor 0.1.

(13) When V becomes less than F, go to END. Or else return backward and proceed to the computation with a new initial condition.

4. Performance of computation of f_0

With a view to making a rough estimation, first we put

$$H = 0.1,\ N = 100, \text{ and } D = E = F = 10^{-6}, \tag{4.1}$$

and starting tentatively from the initial value

$$f_0''(0) = 1.32821,$$ [3]

we observed after five times of repetition that the process converged in the sense of (3.3) and yielded the values

$$f_0''(0) = 1.328231, \text{ and } f_0'(4.3) = 1.9999996. \tag{4.2}$$

In order to increase accuracy of the computation, next we assigned more stringent values for the constants:[4]

1) The inequality (3.3) might be satisfied as well in a region where $A1$ takes the values maximum or minimum. This possibility can be excluded, however, by plotting the successive values of $A1$ graphically. Incidentally, we assume the same definition for convergency of $B1$ and $C1$ to appear in later sections.

2) The value of U immediately before the last transit through the step (2) in the flow diagram takes place.

3) At the beginning of our calculation U and V are assumed as 1.3282 and 0.00001, respectively. According to the rule (2) of the flow diagram, $f_0''(0)$ is to start from 1.32821.

4) In solving a differential equation by means of RKG method, the value of the dependent variable, g say, at the point $\eta+H$ computed from the data at η coincides with its TAYLOR expansion

$$g(\eta+H) = g(\eta)+Hg'(\eta)+\frac{H^2}{2!}g''(\eta)+\frac{H^3}{3!}g'''(\eta)$$
$$+\frac{H^4}{4!}g^{\mathrm{IV}}(\eta)+\frac{H^5}{5!}g^{\mathrm{V}}(\eta)+\cdots$$

up to the fifth term, so that the truncation error or, in other words, the error due to RKG method itself should be the sum of the series beginning from the sixth term of the right-hand side of the above equality. So long as H is small and $g^{\mathrm{V}}(\eta)$ is not so

$$H = 0.01, \quad N = 600, \qquad\qquad (4.3)$$
$$D = 2.5\times10^{-7},\ E = 10^{-8}, \quad \text{and} \quad F = 10^{-7},$$

and continued the similar computation. After several times of repetition, we arrived finally at the convergent value

$$f_0'(4.58) = 2.0000015\,, \qquad (4.4)$$

the corresponding initial value being

$$f_0''(0) = 1.328231.^{1)} \qquad (4.5)$$

Finally to reproduce the whole result of the computations, we changed the program slightly and printed the values of X, A, $A1$, and $A2$ at intervals of $H = 0.05$ rounding off the figures below the fifth decimal place (see TABLE 2 at the end of the paper). As mentioned already in the introduction of this paper, f_0 is identified with the so-called BLASIUS function or the stream function for the two-dimensional flow along a flat plate. Its behavior in the whole region of the variable η is well-known. It will be remarked that the values shown in our table agree, except the last decimal place (cf. footnote 1 on page 51), with those given by HOWARTH.[2]

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large, therefore, this error may be estimated as order of $H^5/5!$, or 0.83×10^{-12} when $H = 0.01$. Since EQUATION (3.2) is composed of three simultaneous equations, we should expect the truncation error of order

$$3\times500\times0.83\times10^{-12} = 1.25\times10^{-9}$$

at the most, provided we suppose that $A1$ converges after 500 steps, say.

1) Since f_0' or $A1$ thus obtained is found greater than $2+D = 2.00000025$, we should, properly speaking, proceed down to the step (12) of the flow diagram and renew the value of V by multiplying it by 1/10. This process is to be stopped as soon as the new V becomes less than F, the constant (= 0.0000001). In our circumstance, however, to continue the computation further we have to take $V = 0.0000001$, which, owing to accumulation of rounding-off errors arising when numbers are expressed in terms of the binary system, becomes less than F very slightly, and according to the instructions (13) and (14) our computation is over.

Comparing graphically, or otherwise, the two integral curves f_0' produced from the initial values $f_0''(0)=1.328230$ and 1.328231, we can determine without difficulty which one of them would be more appropriate for our purpose of bringing the value of $f_0'(\infty)$ as close as possible to 2. The same remarks hold good as well in determining $f_1''(0)$ and $f_2''(0)$. In connection with the initial value $f_0''(0)$ given by (4.5), see GOLDSTEIN, S., *Modern Developments in Fluid Dynamics*, vol. I (1965), Dover Pub., p. 136, footnote, where it is added that the correct value of $f_0''(0)$ should be 1.32823, and not 1.32824 mentioned in the text (HOSKINS).

Incidentally, let us add a few words about the probable behavior of the function f_0 when the integration is imagined to proceed much further beyond $X = 4.58$. In another occasion, we integrated the equation (3.2) by means of the RKG method starting from the condition (4.5) and with $H = 0.1$, and observed that $A2$ or f_0'' decreased uniformly as X or η increased and that finally at $X = 9.9$ $A2$ arrived at the order 10^{-32}. This example may be interpreted as suggesting the stability (or convergency), and so appropriateness, of our solution explained in the text.

2) HOWARTH, L., On the Solution of the Laminar Boundary Layer Equations, *Proceedings of the Royal Society*, vol. 164 (1938), pp. 547-579, esp. TABLE I, p. 552.

5. f_1 and f_2

With regard to the essential parts of the calculation procedures and the constants of restriction for convergency (D, E, and F), no alterations are made in computing both f_1 and f_2.

The simultaneous differential equations

$$\frac{dA}{dX}=A1\,,\qquad \frac{dB}{dX}=B1\,,$$
$$\frac{dA1}{dX}=A2\,,\qquad \frac{dB1}{dX}=B2\,,\tag{5.1}$$
$$\frac{dA2}{dX}=A3=-A\cdot A2,\quad \frac{dB2}{dX}=B3=-A\cdot B2+A1\cdot B1-2\cdot A2\cdot B-X\cdot A3-A2\,,$$

where we write

$$\eta\equiv X,\ f_0\equiv A,\ f_0'\equiv A1,\ f_0''\equiv A2,\ f_0'''\equiv A3\,,$$
$$f_1\equiv B,\ f_1'\equiv B1,\ f_1''\equiv B2,\ f_1'''\equiv B3\,,\tag{5.2}$$

are solved starting from the conditions

$$A(0)=A1(0)=0,\ A2(0)=1.328231,$$

and

$$B(0)=B1(0)=0\,.\tag{5.3}$$

$B2(0)$, left unspecified so far, should be determined in such a way that the relation at infinity,

$$B1(\infty)=0\,,\tag{5.4}$$

might be satisfied and is found to be 0.694323.[1] Numerical values of B, $B1$, and $B2$ are reproduced in TABLE 3 for various values of X.

Finally f_2 are evaluated by solving the equations

$$\frac{dA}{dX}=A1\,,\qquad \frac{dB}{dX}=B1\,,$$
$$\frac{dA1}{dX}=A2\,,\qquad \frac{dB1}{dX}=B2\,,$$
$$\frac{dA2}{dX}=A3=-A\cdot A2,\quad \frac{dB2}{dX}=B3=-A\cdot B2+A1\cdot B1-2\cdot A2\cdot B-X\cdot A3-A2,$$
$$\frac{dC}{dX}=C1\,,\qquad \frac{dC1}{dX}=C2\,,\tag{5.5}$$
$$\frac{dC2}{dX}=C3=-A\cdot C2+2\cdot A1\cdot C1-3\cdot A2\cdot C-X\cdot B3-B2-2\cdot B\cdot B2+(B1)^2,$$

1) At $X=5.10$ we observe that $B1$ converges in the sense of (3.3) into the value 0.10552577×10^{-5}. The computation is stopped here according to the prohibition that V should not be less than F. See also the footnote 1 on page 55.

with the aid of the initial conditions

$$A(0) = A1(0) = 0,\ A2(0) = 1.328231\,,$$
$$B(0) = B1(0) = 0,\ B2(0) = 0.694323\,, \tag{5.6}$$

and
$$C(0) = C1(0) = 0\,,$$

where we put, in addition to (5.2),

$$f_2 \equiv C,\ f_2' \equiv C1,\ f_2'' \equiv C2, \text{ and } f_2''' \equiv C3. \tag{5.7}$$

Besides,
$$C1(\infty) = 0 \tag{5.8}$$

should be satisfied by choosing $C2\,(0)$ appropriately, and this is found to be -0.1641442.[1] The computated values of C, $C1$, and $C2$ are shown altogether in TABLE 4.

In concluding the paper, it will be interesting to compare the values of $f_1''(0)$, $f_2''(0)$, $f_1(\infty)$, and $f_2(\infty)$ — or, in terms of the notations used in our tables, $B2(0)$, $C2(0)$, $B(\infty)$, and $C(\infty)$, respectively — among the methods by i) SEBAN and BOND, ii) KELLY, and iii) the present authors. We find that the KELLY's result worked out making use of an analog computer was quite accurate — 5 % in the relative error at the most.

TABLE 1.

	$f_1''(0)$	$f_2''(0)$	$f_1(\infty)$	$f_2(\infty)$
SEBAN & BOND	0.704	−0.095	0.051	−0.093
KELLY	0.696	−0.160	−0.019	−0.100
OKABE & KIMURA	0.694	−0.164	−0.018	−0.105

The whole numerical work contained in this paper was performed with the aid of the electronic computer OKITAC 5090H, and our thanks are due to the staff of Computation Center of Kyushu University. We are grateful also to Miss S. HOSHINO for her assistance in preparing the manuscript.

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1) Owing to poor convergency of C-functions (i.e. f_2, f_2', and f_2''), slight modification is found necessary in the computation program for C's. Namely, starting from $C2(0) = -0.164144$, the most accurate value as long as we remain to 6 decimal places, we observe that the convergence defined as $|(C1)_S-(C1)_{S+1}| \leqq E$ (see(3.3)) can be achieved at no point in the whole integration range restricted by $N = 600$ (see (4.2)). (In fact, at the end-point $X = 5.99$, we have $C1 = 0.854247\times10^{-5}$.) Because of difficulty explained in the footnote 1 on page 55, the computation cannot proceed further.

In the next place, therefore, we put from the beginning $C2\,(0) = -0.1641442$, to 7 decimal places ($V = 10^{-7}$), then the rounding-off errors resulting from the transformation into the binary system can be excluded, and we have $V = F$. So the calculation may be carried out further, and the following convergent value is yielded: $C1(5.59) = 0.16724665\times10^{-5}$. Better approximation by means of our scheme is evidently impossible, since we will have $V<F$ without fail.

Numerical tables

Because of convenience of printing in a digital computer, the following notations are employed in the tables, cf. (3.1), (5.2), and (5.7).

$$f_0 \equiv A,\quad f_0' \equiv A1,\quad f_0'' \equiv A2,\quad f_0''' \equiv A3,$$
$$f_1 \equiv B,\quad f_1' \equiv B1,\quad f_1'' \equiv B2,\quad f_1''' \equiv B3,$$
$$f_2 \equiv C,\quad f_2' \equiv C1,\quad f_2'' \equiv C2,\quad f_2''' \equiv C3,$$

and

$$\eta \equiv X.$$

TABLE 2.

$A3 = -A*A2$
$H = 0.01$

X = 0.00	A = 0.00000	A1 = 0.00000	A2 = 1.32823
0.05	0.00166	0.06641	1.32819
0.10	0.00664	0.13282	1.32794
0.15	0.01494	0.19920	1.32724
0.20	0.02656	0.26553	1.32588
0.25	0.04149	0.33177	1.32365
0.30	0.05973	0.39788	1.32032
0.35	0.08128	0.46378	1.31569
0.40	0.10611	0.52942	1.30956
0.45	0.13421	0.59471	1.30173
0.50	0.16557	0.65956	1.29203
0.55	0.20016	0.72388	1.28029
0.60	0.23795	0.78755	1.26636
0.65	0.27890	0.85047	1.25012
0.70	0.32298	0.91252	1.23146
0.75	0.37014	0.97358	1.21032
X = 0.80	A = 0.42032	A1 = 1.03351	A2 = 1.18666
0.85	0.47347	1.09220	1.16045
0.90	0.52952	1.14952	1.13173
0.95	0.58840	1.20533	1.10055
1.00	0.65003	1.25953	1.06701
1.05	0.71432	1.31200	1.03124
1.10	0.78119	1.36262	0.99340
1.15	0.85055	1.41131	0.95371
1.20	0.92229	1.45797	0.91237
1.25	0.99631	1.50252	0.86965
1.30	1.07251	1.54491	0.82582
1.35	1.15077	1.58509	0.78118
1.40	1.23098	1.62302	0.73603
1.45	1.31303	1.65869	0.69068
1.50	1.39681	1.69209	0.64544
1.55	1.48220	1.72324	0.60062
1.60	1.56910	1.75216	0.55651
1.65	1.65738	1.77891	0.51339
1.70	1.74695	1.80352	0.47150
1.75	1.83770	1.82608	0.43109
X = 1.80	A = 1.92953	A1 = 1.84666	A2 = 0.39235
1.85	2.02234	1.86535	0.35544
1.90	2.11603	1.88224	0.32050
1.95	2.21053	1.89743	0.28765
2.00	2.30575	1.91104	0.25694

$X = 2.05$	$A = 2.40161$	$A1 = 1.92316$	$A2 = 0.22841$
2.10	2.49804	1.93392	0.20208
2.15	2.59498	1.94341	0.17792
2.20	2.69236	1.95174	0.15589
2.25	2.79014	1.95903	0.13592
2.30	2.88825	1.96537	0.11793
2.35	2.98666	1.97086	0.10183
2.40	3.08532	1.97558	0.08748
2.45	3.18421	1.97963	0.07479
2.50	3.28328	1.98309	0.06363
2.55	3.38251	1.98602	0.05386
2.60	3.48187	1.98849	0.04537
2.65	3.58135	1.99057	0.03802
2.70	3.68092	1.99231	0.03171
2.75	3.78058	1.99376	0.02631
2.80	3.88029	1.99496	0.02173
2.85	3.98007	1.99594	0.01785
2.90	4.07989	1.99675	0.01459
2.95	4.17974	1.99741	0.01187
3.00	4.27963	1.99795	0.00961
$X = 3.05$	$A = 4.37953$	$A1 = 1.99838$	$A2 = 0.00774$
3.10	4.47946	1.99873	0.00620
3.15	4.57941	1.99900	0.00494
3.20	4.67936	1.99923	0.00392
3.25	4.77933	1.99940	0.00310
3.30	4.87930	1.99954	0.00243
3.35	4.97928	1.99965	0.00190
3.40	5.07926	1.99973	0.00148
3.45	5.17925	1.99979	0.00114
3.50	5.27924	1.99984	0.00088
3.55	5.37924	1.99988	0.00067
3.60	5.47923	1.99991	0.00051
3.65	5.57923	1.99994	0.00039
3.70	5.67923	1.99995	0.00029
3.75	5.77922	1.99997	0.00022
3.80	5.87922	1.99997	0.00017
3.85	5.97922	1.99998	0.00012
3.90	6.07922	1.99999	0.00009
3.95	6.17922	1.99999	0.00007
4.00	6.27922	1.99999	0.00005
$X = 4.05$	$A = 6.37922$	$A1 = 2.00000$	$A2 = 0.00004$
4.10	6.47922	2.00000	0.00003
4.15	6.57922	2.00000	0.00002
4.20	6.67922	2.00000	0.00001
4.25	6.77922	2.00000	0.00001
4.30	6.87922	2.00000	0.00001
4.35	6.97922	2.00000	0.00000
4.40	7.07922	2.00000	0.00000
4.45	7.17922	2.00000	0.00000
4.50	7.27922	2.00000	0.00000
4.55	7.37922	2.00000	0.00000

Note added in proof: Primarily our plan was to reproduce these numerical tables yielded from our computations in phototype. But a few technical difficulties, unexpected at the beginning, prevented us from carrying out our project. In the present form of printing, rewriting f_0, f_1, and f_2 into *A*, *B*, and *C*, respectively would be completely superfluous.

TABLE 3.

$A3 = -A*A2$
$B3 = -A*B2+A1*B1-2*A2*B-X*A3-A2$
$H = 0.01$

$X=$ 0.00	$B=$ 0.00000	$B1=$ 0.00000	$B2=$ 0.69432
0.05	0.00084	0.03306	0.62789
0.10	0.00325	0.06279	0.56139
0.15	0.00706	0.08919	0.49479
0.20	0.01211	0.11227	0.42815
0.25	0.01823	0.13201	0.36159
0.30	0.02526	0.14843	0.29529
0.35	0.03302	0.16155	0.22949
0.40	0.04136	0.17139	0.16451
0.45	0.05011	0.17802	0.10072
0.50	0.05911	0.18149	0.03854
0.55	0.06821	0.18191	−0.02157
0.60	0.07725	0.17938	−0.07909
0.65	0.08610	0.17405	−0.13350
0.70	0.09461	0.16609	−0.18426
0.75	0.10267	0.15569	−0.23082
$X=$ 0.80	$B=$ 0.11014	$B1=$ 0.14309	$B2=$ −0.27267
0.85	0.11694	0.12851	−0.30932
0.90	0.12297	0.11225	−0.34034
0.95	0.12814	0.09458	−0.36536
1.00	0.13241	0.07582	−0.38411
1.05	0.13571	0.05628	−0.39640
1.10	0.13803	0.03629	−0.40217
1.15	0.13934	0.01617	−0.40147
1.20	0.13965	−0.00376	−0.39448
1.25	0.13897	−0.02318	−0.38150
1.30	0.13734	−0.04181	−0.36295
1.35	0.13481	−0.05939	−0.33936
1.40	0.13142	−0.07568	−0.31136
1.45	0.12726	−0.09047	−0.27967
1.50	0.12241	−0.10359	−0.24506
1.55	0.11693	−0.11494	−0.20835
1.60	0.11094	−0.12441	−0.17038
1.65	0.10453	−0.13197	−0.13197
1.70	0.09778	−0.13761	−0.09392
1.75	0.09080	−0.14138	−0.05696
$X=$ 1.80	$B=$ 0.08367	$B1=$ −0.14334	$B2=$ −0.02178
1.85	0.07649	−0.14359	0.01106
1.90	0.06934	−0.14228	0.04106
1.95	0.06229	−0.13954	0.06785
2.00	0.05540	−0.13555	0.09116
2.05	0.04875	−0.13049	0.11083
2.10	0.04237	−0.12453	0.12683
2.15	0.03631	−0.11787	0.13919
2.20	0.03059	−0.11067	0.14804
2.25	0.02525	−0.10312	0.15360
2.30	0.02028	−0.09536	0.15613
2.35	0.01571	−0.08755	0.15595
2.40	0.01153	−0.07981	0.15339
2.45	0.00773	−0.07224	0.14881
2.50	0.00430	−0.06495	0.14257

X	B	$B1$	$B2$
2.55	0.00123	−0.05801	0.13504
2.60	−0.00151	−0.05146	0.12654
2.65	−0.00393	−0.04536	0.11739
2.70	−0.00605	−0.03973	0.10787
2.75	−0.00791	−0.03458	0.09823
2.80	−0.00952	−0.02991	0.08867
2.85	−0.01091	−0.02571	0.07938
2.90	−0.01210	−0.02196	0.07049
2.95	−0.01311	−0.01865	0.06211
3.00	−0.01397	−0.01574	0.05431
3.05	−0.01469	−0.01321	0.04714
3.10	−0.01529	−0.01102	0.04061
3.15	−0.01580	−0.00914	0.03475
3.20	−0.01621	−0.00753	0.02952
3.25	−0.01655	−0.00617	0.02491
3.30	−0.01683	−0.00503	0.02088
3.35	−0.01706	−0.00408	0.01738
3.40	−0.01724	−0.00328	0.01438
3.45	−0.01739	−0.00263	0.01182
3.50	−0.01751	−0.00210	0.00965
3.55	−0.01760	−0.00166	0.00783
3.60	−0.01768	−0.00131	0.00631
3.65	−0.01773	−0.00102	0.00506
3.70	−0.01778	−0.00080	0.00403
3.75	−0.01781	−0.00062	0.00319
3.80	−0.01784	−0.00048	0.00251
3.85	−0.01786	−0.00036	0.00196
3.90	−0.01788	−0.00028	0.00153
3.95	−0.01789	−0.00021	0.00118
4.00	−0.01790	−0.00016	0.00091
4.05	−0.01791	−0.00012	0.00069
4.10	−0.01791	−0.00009	0.00053
4.15	−0.01792	−0.00007	0.00040
4.20	−0.01792	−0.00005	0.00030
4.25	−0.01792	−0.00003	0.00022
4.30	−0.01792	−0.00003	0.00017
4.35	−0.01792	−0.00002	0.00012
4.40	−0.01792	−0.00001	0.00009
4.45	−0.01792	−0.00001	0.00007
4.50	−0.01792	−0.00001	0.00005
4.55	−0.01793	−0.00000	0.00003
4.60	−0.01793	−0.00000	0.00002
4.65	−0.01793	−0.00000	0.00002
4.70	−0.01793	−0.00000	0.00001
4.75	−0.01793	−0.00000	0.00001
4.80	−0.01793	0.00000	0.00001
4.85	−0.01793	0.00000	0.00000
4.90	−0.01793	0.00000	0.00000
4.95	−0.01793	0.00000	0.00000
5.00	−0.01793	0.00000	0.00000
5.05	−0.01793	0.00000	0.00000
5.10	−0.01793	0.00000	0.00000

TABLE 4.

$A3 = -A*A2$
$B3 = -A*B2+A1*B1-2*A2*B-X*A3-A2$
$C3 = -A*C2+2*A1*C1-3*A2*C-X*B3-B2-2*B*B2+B1*B1$
$H = 0.01$

$X=$ 0.00	$C=$ 0.00000	$C1=$ 0.00000	$C2=$ −0.16414
0.05	−0.00022	−0.00902	−0.19554
0.10	−0.00093	−0.01944	−0.22028
0.15	−0.00218	−0.03094	−0.23833
0.20	−0.00403	−0.04316	−0.24967
0.25	−0.00650	−0.05579	−0.25430
0.30	−0.00961	−0.06848	−0.25226
0.35	−0.01335	−0.08091	−0.24364
0.40	−0.01769	−0.09274	−0.22863
0.45	−0.02261	−0.10367	−0.20751
0.50	−0.02804	−0.11340	−0.18066
0.55	−0.03392	−0.12165	−0.14859
0.60	−0.04018	−0.12818	−0.11195
0.65	−0.04671	−0.13278	−0.07152
0.70	−0.05342	−0.13528	−0.02819
0.75	−0.06020	−0.13557	0.01700
$X=$ 0.80	$C=$ −0.06694	$C1=$ −0.13357	$C2=$ 0.06293
0.85	−0.07352	−0.12928	0.10841
0.90	−0.07983	−0.12276	0.15219
0.95	−0.08576	−0.11411	0.19301
1.00	−0.09121	−0.10353	0.22969
1.05	−0.09608	−0.09123	0.26108
1.10	−0.10031	−0.07752	0.28621
1.15	−0.10382	−0.06273	0.30425
1.20	−0.10657	−0.04723	0.31460
1.25	−0.10853	−0.03140	0.31691
1.30	−0.10971	−0.01567	0.31111
1.35	−0.11011	−0.00042	0.29739
1.40	−0.10977	0.01395	0.27625
1.45	−0.10873	0.02709	0.24842
1.50	−0.10708	0.03869	0.21489
1.55	−0.10489	0.04850	0.17685
1.60	−0.10227	0.05632	0.13561
1.65	−0.09930	0.06203	0.09258
1.70	−0.09610	0.06558	0.04921
1.75	−0.09278	0.06697	0.00689
$X=$ 1.80	$C=$ −0.08944	$C1=$ 0.06631	$C2=$ −0.03307
1.85	−0.08618	0.06372	−0.06952
1.90	−0.08309	0.05943	−0.10148
1.95	−0.08026	0.05366	−0.12822
2.00	−0.07775	0.04670	−0.14923
2.05	−0.07560	0.03884	−0.16426
2.10	−0.07387	0.03038	−0.17329
2.15	−0.07257	0.02161	−0.17653
2.20	−0.07171	0.01281	−0.17440
2.25	−0.07129	0.00425	−0.16746
2.30	−0.07128	−0.00387	−0.15641
2.35	−0.07166	−0.01134	−0.14203
2.40	−0.07240	−0.01803	−0.12513
2.45	−0.07345	−0.02382	−0.10655
2.50	−0.07477	−0.02867	−0.08705

X	C	C1	C2
$X=$ 2.55	$C=$ −0.07630	$C1=$ −0.03253	$C2=$ −0.06739
2.60	−0.07800	−0.03541	−0.04819
2.65	−0.07983	−0.03736	−0.03000
2.70	−0.08172	−0.03844	−0.01326
2.75	−0.08366	−0.03872	0.00170
2.80	−0.08558	−0.03830	0.01468
2.85	−0.08748	−0.03728	0.02557
2.90	−0.08930	−0.03578	0.03433
2.95	−0.09105	−0.03389	0.04104
3.00	−0.09269	−0.03171	0.04581
3.05	−0.09422	−0.02933	0.04880
3.10	−0.09562	−0.02685	0.05022
3.15	−0.09690	−0.02434	0.05029
3.20	−0.09805	−0.02184	0.04924
3.25	−0.09909	−0.01943	0.04730
3.30	−0.10000	−0.01712	0.04467
3.35	−0.10080	−0.01497	0.04156
3.40	−0.10150	−0.01297	0.03815
3.45	−0.10210	−0.01115	0.03457
3.50	−0.10262	−0.00952	0.03097
$X=$ 3.55	$C=$ −0.10306	$C1=$ −0.00806	$C2=$ 0.02744
3.60	−0.10343	−0.00677	0.02407
3.65	−0.10373	−0.00565	0.02090
3.70	−0.10399	−0.00468	0.01797
3.75	−0.10420	−0.00384	0.01532
3.80	−0.10438	−0.00314	0.01295
3.85	−0.10452	−0.00255	0.01085
3.90	−0.10463	−0.00205	0.00901
3.95	−0.10473	−0.00164	0.00743
4.00	−0.10480	−0.00130	0.00608
4.05	−0.10486	−0.00103	0.00493
4.10	−0.10490	−0.00081	0.00397
4.15	−0.10494	−0.00063	0.00318
4.20	−0.10497	−0.00049	0.00252
4.25	−0.10499	−0.00037	0.00199
4.30	−0.10501	−0.00029	0.00156
4.35	−0.10502	−0.00022	0.00121
4.40	−0.10503	−0.00016	0.00093
4.45	−0.10503	−0.00012	0.00072
4.50	−0.10504	−0.00009	0.00055
$X=$ 4.55	$C=$ −0.10504	$C1=$ −0.00007	$C2=$ 0.00041
4.60	−0.10505	−0.00005	0.00031
4.65	−0.10505	−0.00004	0.00023
4.70	−0.10505	−0.00003	0.00017
4.75	−0.10505	−0.00002	0.00013
4.80	−0.10505	−0.00001	0.00009
4.85	−0.10505	−0.00001	0.00007
4.90	−0.10505	−0.00001	0.00005
4.95	−0.10505	−0.00000	0.00004
5.00	−0.10505	−0.00000	0.00003
5.05	−0.10505	−0.00000	0.00002
5.10	−0.10505	−0.00000	0.00001
5.15	−0.10505	0.00000	0.00001
5.20	−0.10505	0.00000	0.00001
5.25	−0.10505	0.00000	0.00000

$X = 5.30$	$C = -0.10505$	$C1 = 0.00000$	$C2 = 0.00000$
5.35	−0.10505	0.00000	0.00000
5.40	−0.10505	0.00000	0.00000
5.45	−0.10505	0.00000	0.00000
5.50	−0.10505	0.00000	0.00000
5.55	−0.10505	0.00000	0.00000