STABILITY OF LAMINAR SHEAR LAYER BETWEEN PARALLEL UNIFORM STREAMS (II)

YAMADA, Hikoji
Research Institute for Applied Mechanics, Kyushu University

https://doi.org/10.5109/7165062

出版情報：Reports of Research Institute for Applied Mechanics. 13 (45), pp.1-15, 1965. 九州大学応用力学研究所
バージョン：
権利関係：

Reports of Research Institute for Applied Mechanics
Vol. XIII, No. 45, 1965

STABILITY OF LAMINAR SHEAR LAYER BETWEEN PARALLEL UNIFORM STREAMS (II)

By Hikoji YAMADA

synopsis In the former half the Orr-Sommerfeld equation of a shear layer between parallel uniform streams has been solved in a certain series of Hermite functions. The non-dimensional lateral distance y is replaced by $\eta=\alpha y$, where α is the wave number of perturbation waves, because neutral oscillation expressed in η extends quite evenly over only neighboring area of x-axis. Thus with first several terms of series the approximation is found to be good enough from $\alpha=0(R=0)$ to $\alpha\doteqdot1(R=\infty)$.

In the latter half the wall effect to the neutral stability when the Reynolds number is quite small has been discussed, for then the disturbance waves are of small wave number and extend up to walls. In this case the shear layer has been replacced by a discontinuous surface of velocity which simplified very much the analysis.

§1 Review of Part I

Several years of repose have passed since the printing of Part I[1] of this work, which we now continue to the end. For the sake of the convenience of description we give in this section a brief summary of Part I, as far as we require in the following.

The non-dimensional velocity $w(y)$ of the shear layer is expressed, by an appropriate choice of coordinates system, in

$$w(y)=\frac{2}{\sqrt{\pi}}\int_0^y e^{-\xi^2}d\xi\,; \tag{1}$$

y is non-dimensional lateral distance from the centre line of the layer. When $w=w'/U$ and $y=y'/\sqrt{4\nu t'}$ (1) represents an exact solution of the plane viscous shear flow, where $2U$ is the relative velocity of twe uniform streams, w' is the dimensional velocity in the shear layer, y' and t' are dimensional distance transverse to the stream and time, and ν is the kinematic viscosity of flowing incompressible fluid.

We superpose on this stream a small disturbance wave train whose non-dimensional stream function is expressed by

$$\psi(x,y\,;t)=\varphi(y)e^{i\alpha(x-ct)}\,; \tag{2}$$

the amplitude $\varphi(y)$ of this disturbance is determined, in the linearised approxima-

(1) H. Yamada: Stability of laminar shear layer between parallel uniform streams (I), Rep. Res. Inst. Appl. Mech. Kyushu Univ., vol. 8 no. **30** (1960), pp. 1–18.

tion, by the Orr-Sommerfeld equation:

$$L(\varphi)=\frac{dy^4}{d^4\varphi}-2\alpha^2\frac{d^2\varphi}{dy^2}+\alpha^4\varphi-i\alpha R\left\{(w-c)\left(\frac{d^2\varphi}{dy^2}-\alpha^2\varphi\right)-\frac{d^2w}{dy^2}\varphi\right\}=0. \quad (3)$$

The characteristic value problem of this equation determines the velocity c of the waves as a function of wave number α, when Reynolds number R is assigned. c is complex in general but for a certain value of α. This special value α and the corresponding real c are fuctions of R, which are called the neutral curves of stability. Our purpose in this paper is the calculation of these neutral curves.

We expand $\varphi(y)$ in $y(-\infty,\infty)$ into an infinite series of Hermite functions:

$$\varphi(y)=\sum_{n=0}^{\infty}a_n\psi_n(y), \quad (4)$$

where Hermite functions are defined as

$$\varphi_n(y)=K_nH_n(y)e^{-y^2/2}\ (K_n=1/\sqrt{2^n n!\sqrt{\pi}}), \quad (5)$$
$$(n=0,1,2,\dots)$$

$H_n(y)$ being the well-known Hermite polynomials[(2)]:

$$H_n(\mathrm{y})=(-1)^n e^{y^2}\left(\frac{d}{dy}\right)^n e^{-y^2} \qquad (n\geq 0). \quad (6)$$

Every terms of $L(\varphi)$ vanish strongly when y tends to $\pm\infty$, and $L(\varphi)=0$ is well replaced by the algebraic simultaneous equations:

$$\beta_m=\int_{-\infty}^{\infty}L(\varphi)\psi_m(y)dy=0, \quad (7)$$
$$(m=0,1,2,\dots)$$

and each β_m are infinite series of the coefficients a_n of (4). Eliminating a'_ns between these series there remains then a characteristic equation, from which we obtain the stability criterion and especially the neutral curve $\alpha=\alpha(R)$. On this neutral curve c is zero by reason of the anti-symmetrical velocity profile.

We have assumed in Part I that $a_n(n\geq 10)$ in one case, and $a_n(n\geq 11)$ in the other case, are so small that they can be put equal to zero. The characteristic equations in these cases are then polynomials of ρ, where $\rho=2\alpha R/\sqrt{\pi}$, such that

$$G_n(\rho)=1+c_1\rho^2+c_2\rho^4+c_3\rho^6+c_4\rho^8+c_5\rho^{10}=0, \quad (8)$$
$$(n=9 \text{ or } 10)$$

c_i's being polynomials of α^2. Assigning several numerical values for α, we find the corresponding ρ's by means of (8), and then the required neutral curve. We have plotted again these curves (for $n=9$ and 10) in Fig.s 1 and 2 of the present paper, in the region of small α where our approximation failed. This failure was felt at $\alpha R\sim 3(\alpha\sim 0.55)$ and larger at smaller values.

Here we undertake the improvement of this defect, as in the followings.

(2) c. f. R. Courant and D. Hilbert: Methods of math. phys., vol. 1 chap. 2 §9; Jahnke-Emde-Lösch: Tafeln höherer Funktionen (1960).

§2 Limiting case $\alpha=0$

Transforming y into η by

$$\eta = \alpha y, \tag{9}$$

(3) becomes an equation on the line η $(-\infty,\infty)$:

$$\mathcal{L}(\varphi) \equiv \frac{d^4\varphi}{d\eta^4} - 2\frac{d^2\varphi}{d\eta^2} + \varphi - iK\left\{(w_1-c)\left(\frac{d^2\varphi}{d\eta^2}-\varphi\right) - \frac{d^2 w_1}{d\eta^2}\varphi\right\} = 0, \tag{10}$$

where

$$K = R\alpha^{-1}, \tag{11}$$

and

$$w_1(\eta) = w\left(\frac{\eta}{\alpha}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\eta/\alpha} e^{-\xi^2}d\xi. \tag{12}$$

In (10) α appears only in $w_1(\eta)$, which, when α tends to zero, tends to a step function:

$$w_1(\eta) = \begin{cases} +1 & (\eta>0), \\ -1 & (\eta<0). \end{cases} \tag{13}$$

Therefore in this limiting case (10) degenerates into

$$\left.\begin{aligned} \ddot{\ddot{\varphi}} - 2\ddot{\varphi} + \varphi - iK(1-c)(\ddot{\varphi}-\varphi) = 0 \quad (\eta>0), \\ \ddot{\ddot{\varphi}} - 2\ddot{\varphi} + \varphi + iK(1+c)(\ddot{\varphi}-\varphi) = 0 \quad (\eta<0), \end{aligned}\right\} \tag{14}$$

and, under the condition that φ vanishes at $\eta=\pm\infty$, has the solution:

$$\varphi = \begin{cases} Ae^{-\eta}+Be^{-k_1\eta} & (\eta>0), \\ Ce^{\eta}+De^{k_2\eta} & (\eta<0), \end{cases} \tag{15}$$

where

$$k_1 = \sqrt{1+iK(1-c)}, \quad k_2 = \sqrt{1-iK(1+c)}. \tag{15'}$$

The condition which joins $\varphi(\eta>0)$ and $\varphi(\eta<0)$ is well-known[(3)], and be written down as

$$\left.\begin{aligned} [\varphi] &= 0, \\ [\dot{\varphi}] &= 0, \\ [\ddot{\varphi}] + 2iK\varphi(0) &= 0, \\ [\dddot{\varphi}] - 2iK\dot{\varphi}(0) &= 0, \end{aligned}\right\} \tag{16}$$

where [ ] means the difference of values of the blacketted quantity at $\eta=+0$ and $\eta=-0$. (15) used in this condition gives

(3) G. P. Drazin: Discontinuous velocity pofiles for the Orr-Sommerfeld equation, J. Fluid Mech., vol. 10 (1961), pp. 571–583; c.f. also H. Yamada: On the stability to long waves of two-dimensional laminar jet, Rep. Res. Inst. Appl. Mech. Kyushu Univ., vol. 12 no. **43** (1964), pp. 1–15.

$$\left.\begin{aligned}&A+B-C-D=0,\\&-A-k_1B-C-k_2D=0,\\&A+k_1^2B-(1-2iK)C-(k_2^2-2iK)D=0,\\&-A-k_1^3B-(1+2iK)C-(k_2^3+2iK)D=0,\end{aligned}\right\}\tag{16'}$$

and eliminating A,B,C,D, the characteristic equation is obtained:

$$(1+c)^2k_1+(1-c)^2k_2=4. \tag{17}$$

The solution of this equation is

$$c=i\gamma(\gamma:\text{real}),\quad K=\frac{4(\sqrt{3}-\gamma)}{(\sqrt{3}\gamma-1)^2}; \tag{18}$$

this result is that of T. Tatsumi and K. Gotoh[4], and of Drazin[3]. Especially for waves of neutral stability $c=0$ and $K=4\sqrt{3}$, *i.e.*

$$\alpha R=4\sqrt{3}\alpha^2\ (\alpha\ll 1), \tag{18'}$$

and using this in (16') we obtain $A=C$, $B=D$, $C=-2D$. We then have the amplitude function in a form, say,

$$\varphi=\begin{cases}2e^{-\eta}-e^{-(2+i\sqrt{3})\eta} & (\eta>0),\\ 2e^{\eta}-e^{(2-i\sqrt{3})\eta} & (\eta<0).\end{cases} \tag{19}$$

This distribution of amplitude is pretty smooth and vanishes exponentially when η tends to $\pm\infty$.

§3 General method of solution

If we observe the problem on the η-axis the amplitude $\varphi(\eta)$ of neutral waves has a well-defined sooth distribution and vanishes exponentially when η tends to $\pm\infty$. This is seen in the case of not-small Reynolds uumber, which has been our subject in Part I. This is also seen in the case of vanishing α (vanishing Reynolds number) of the preceeding section. On the other hand, $\varphi(\eta)$ satisfies a compact differential equation (10) all over the range of α. It had been then better to use η rather than y over the whole range R $(0, \infty)$. In the cases of not-small R, however, η does not differ appreciably from y on the neutral curve and Hermite funcjions in y-variable are as effective as them in η-variable. Contrary is the case of small α, where φ extends in general in a broad range of y-axis. On the neutral curve, α becoming small with R, we shall better use η to contract the range of φ into a convenient neighboring area of $\eta=0$. But this transformation brings with it some changes in the equations and formulas given in Part I. These changes are the replacements of (i) η in the place of y, (ii) 1 in the place of α, (iii) w_1 (η) *i.e.* w (η/α) in the place of $w(y)$, and (iv) K in the place of αR.

Brief summary of the method will be repeated here, with necessary replacements and alterations. Using the expansion:

---

(4) T. Tatsumi and K. Gotoh: The stability of free boundary layers between two uniform streams, J. Fluid Mech., vol. 7 (1960), pp. 433–441.

$$\varphi(\eta)=\sum_{n=0}^{\infty}a_n'\psi_n(\eta) \tag{4'}$$

instead of (4), we define the simultaneous equations:

$$\beta_m'=\int_{-\infty}^{\infty}\mathcal{L}(\varphi)\psi_m(\eta)d\eta=0, \quad (m=0,\ 1,\ 2,\ \dots) \tag{7'}$$

where $\mathcal{L}(\varphi)$ is the expression (10). Then, as in (I-23), we have

$$\beta_m'=\sum_{n=0}^{\infty}\left(\frac{1}{i\rho'}A'_{mn}-B'_{mn}+i\gamma C'_{mn}\right)a'_n \tag{20}$$

and in this, as in (I-24, 25, 26),

$$\rho'=2K/\sqrt{\pi},\quad i\gamma=\sqrt{\pi}c/2; \tag{21}$$

$$\left.\begin{aligned} A'_{mn}=A'_{nm}&=\int_{-\infty}^{\infty}\psi_m(\ddddot{\psi}_n-2\ddot{\psi}_n+\psi_n)d\eta=(A_{mn})_{\alpha=1},\\ C'_{mn}=C'_{nm}&=\int_{-\infty}^{\infty}\psi_m(\ddot{\psi}_n-\psi_n)d\eta=(C_{mn})_{\alpha=1},\\ B'_{mn}&=\int_{-\infty}^{\infty}\psi_m\,\{w_2(\ddot{\psi}_n-\psi_n)-\ddot{w}_2\psi_n\}d\eta, \end{aligned}\right\} \tag{21'}$$

and

$$w_2(\eta)=\int_0^{\eta/\alpha}e^{-\xi^2}d\xi. \tag{22}$$

The constants A'_{mn} and C'_{mn} are easily obtained by the formulas for derivatives of the Hermite functions (I-21, 21'):

$$\ddot{\psi}_n=\frac{\sqrt{(n-1)n}}{2}\psi_{n-2}-\frac{2n+1}{2}\psi_n+\frac{\sqrt{(n+1)(n+2)}}{2}\psi_{n+2}, \tag{23}$$

$$\begin{aligned}\ddddot{\psi}_n=&\frac{\sqrt{(n-3)(n-2)(n-1)n}}{4}\psi_{n-4}-\frac{\sqrt{(n-1)n}(4n-2)}{4}\psi_{n-2}+\frac{6n^2+6n+3}{4}\psi_n\\ &-\frac{\sqrt{(n+1)(n+2)}(4n+6)}{4}\psi_{n+2}+\frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{4}\psi_{n+4},\end{aligned} \tag{23'}$$

and we have

$$\left.\begin{aligned} A'_{n-4,n}=A'_{n,n-4}&=\frac{\sqrt{(n-3)(n-2)(n-1)n}}{4},\\ A'_{n-2,n}=A'_{n,n-2}&=-\frac{\sqrt{(n-1)n}(2n+1)}{2},\\ A'_{n,n}&=\frac{6n^2+14n+11}{4}, \end{aligned}\right\} \tag{24}$$

and

$$C_{n-2,n}=C_{n,n-2}=\frac{\sqrt{(n-1)n}}{2},\quad C_{n,n}=-\frac{2n+3}{2},\tag{24'}$$

all the other A_{mn} and C_{mn} being zero.

Next the evaluation of B'_{mn}. By use of (23) in the third definition of (21') we obtain

$$B'_{mn}=\frac{\sqrt{(n-1)n}}{2}I'_{m,n-2}-\frac{2n+3}{2}I'_{mn}+\frac{\sqrt{(n+1)(n+2)}}{2}I'_{m,n+2}-J'_{mn},\tag{25}$$

and the integrals in this expression:

$$I'_{mn}=\int_{-\infty}^{\infty}\psi_m w_2\psi_n\,d\eta,\quad J'_{mn}=\int_{-\infty}^{\infty}\psi_m \ddot{w}_2\psi_n\,d\eta\tag{25'}$$

are calculated as follows. Using (I-31, 31') *i.e.*

$$\left.\begin{aligned}&\frac{d}{d\eta}(\psi_{m-1}e^{-\eta^2/2})=-\sqrt{2m}\,\psi_m e^{-\eta^2/2}\ (m\geq 1),\\ &\frac{d}{d\eta}(\psi_n e^{\eta^2/2})=\sqrt{2n}\psi_{n-1}e^{\eta^2/2}\qquad (n\geq 0)\end{aligned}\right\}\tag{26}$$

in (25') we have, as in (I-32), the recurrence formula:

$$\left.\begin{aligned}&I'_{mn}=\frac{1}{\sqrt{2m}}(m-1,\ n)+\sqrt{\frac{n}{m}}I'_{m-1,n-1}\quad\binom{m\geq 1}{n\geq 0},\\ &\qquad I'_{0,0}=0,\end{aligned}\right\}\tag{27}$$

where we denote

$$(m,n)=(n,m)=\int_{-\infty}^{\infty}\psi_m\dot{w}_2\psi_n\,d\eta,\tag{28}$$

and by use of (26) and (I-20) *i. e.*

$$\sqrt{2(n+1)}\psi_{n+1}-2\eta\,\psi_n+\sqrt{2n}\,\psi_{n-1}=0\ (n\geq 0),\tag{29}$$

another recurrence formula:

$$\left.\begin{aligned}&(m,n)=\frac{\alpha^2}{1+\alpha^2}\sqrt{\frac{n}{m}}(m-1),\ n-1)-\frac{1}{1+\alpha^2}\sqrt{\frac{n+1}{m}}(m-2,\ n)\quad\binom{m\geq 1}{n\geq 0},\\ &(0,\ 0)=\frac{1}{\sqrt{1+\alpha^2}},\quad (1,\ 0)=(0,\ 1)=0.\end{aligned}\right\}\tag{30}$$

With these formulas then I'_{mn} are found step by step. Finally the evaluation of J'_{mn} accomplishes by use of $\dot{w}_2=-2\alpha^{-3}\eta\,\exp\,(-\eta^2/\alpha^2)$ and (29). Using (28) for $(m+1,\ n)$ we get the expression:

$$-J'_{mn}=\frac{1}{1+\alpha^2}\{\sqrt{2m}(m-1,\ n)+\sqrt{2n}(m,\ n-1)\}\quad\binom{m\geq 0}{n\geq 0};\tag{31}$$

with a table of (m, n) then J'_{mn} are calculated as functions of α.

I'_{mn} and J'_{mn}, and accordingly B'_{mn}, vanish for pairs of suffixes m, n for which $m+n$ are even integers. To the contrary A'_{mn} and C'_{mn} vanish for the pairs m, n for which $m+n$ are odd integers. Then the characteristic determinant equation which results from (20) by elimination of a'_n $(n=0, 1, 2, \cdots)$ has the form:

$$\begin{vmatrix} \frac{A'_{00}}{i\rho'}+i\gamma C'_{00}, & -B'_{01}, & \frac{A'_{02}}{i\rho'}+i\gamma C'_{02}, & -B'_{03}, & \cdots\cdots \\ -B'_{10}, & \frac{A'_{11}}{i\rho'}+i\gamma C'_{11}, & -B'_{12}, & \frac{A'_{13}}{i\rho'}+i\gamma C'_{13}, & \cdots\cdots \\ \frac{A'_{20}}{i\rho'}+i\gamma C'_{20}, & -B'_{21}, & \frac{A'_{22}}{i\rho'}+i\gamma C'_{22}, & -B'_{23} & \cdots\cdots \\ -B'_{30}, & \frac{A'_{31}}{i\rho'}+i\gamma C'_{31}, & -B'_{32}, & \frac{A'_{33}}{i\rho'}+i\gamma C'_{33}, & \cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{vmatrix} = 0. \tag{32}$$

This complex equation for three real variables ρ', γ, and α, α being included in B's, determines ρ' and γ as functions of α, *i.e.* characteristic relations $R(\alpha)$ and $c(\alpha)$ are determined.

§4 Case of neutral stability

As we have seen c, and then γ, vanish in waves of neutral stability. Taking $\gamma=0$ in (32) we are left with one real equation in ρ', which is transformed into the equation:

$$\begin{vmatrix} A'_{00}, & -B'_{01}, & A'_{02}, & -B'_{03}, & \cdots\cdots\cdots\cdots \\ -B'_{10}, & A'_{11}\chi, & -B'_{12}, & A'_{13}\chi, & \cdots\cdots\cdots\cdots \\ A'_{20}, & -B'_{21}, & A'_{22}, & -B'_{23}, & \cdots\cdots\cdots\cdots \\ -B'_{30}, & A'_{31}\chi, & -B'_{32}, & -A'_{33}\chi, & \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{vmatrix} = 0 \tag{32'}$$

for the unknown χ:

$$\chi = (i\rho')^{-2} = -\pi/4K^2. \tag{33}$$

This infinite determinant has been cut off to first eight rows and columns, *i.e.* to the highest suffixes $n'=7$, in the actual calculation of $\chi(\alpha)$.

As stated above B_{mn} are functions of α, which become more and more complex as the suffixes m, n increase. In the calculation of these elements it is simpler to put $\alpha=\tan\theta$ and use θ in place of α. Assigning a value for θ B'_{mn} are constants and (32′) becomes, upon expansion, an algebraic equation χ of fourth degree. Of the stability curve we require here only the part of small α, so that calculations are carried for 6 small values of θ, and the corresponding coefficients of the algebraic equation are as given in Table 1. The negative real roots χ_0 of these equations, which are only one in each of them, and the values K, R, and αR, derived therefrom, are given in Table 2.

Table 1

θ	α	χ^4	χ^3	χ^2	χ	*abs*
0.000	0.000	1	-0.2281	$+0.4819\times10^{-2}$	$+0.8144\times10^{-4}$	-0.8181×10^{-6}
0.100	0.100	1	-0.2271	0.4760	0.8141	0.8024
0.245	0.250	1	-0.2228	0.4602	0.7627	0.6933
0.380	0.399	1	-0.2161	0.4631	0.5573	0.4707
0.464	0.500	1	-0.2137	0.4901	0.3748	0.3241
0.540	0.599	1	-0.2096	0.5123	0.1894	0.1998

Table 2

α	χ_0	ρ'^{-1}	K	R	αR
0.000	-0.01556	0.1247	(6.9282) 7.107	0.000	0.000
0.100	-0.01557	0.1248	7.100	0.7124	0.0715
0.250	-0.01500	0.1225	7.234	1.809	0.4523
0.399	-0.01249	0.1118	7.927	3.166	1.265
0.500	-0.009883	0.09941	8.915	4.461	2.233
0.599	-0.007082	0.08415	10.53	6.313	3.784

K and αR are also indicated as graphs in Fig.s 1 and 2, compared with values caculated by other authers up to now. In these figures we see smooth joining between our present result, indicated as $n'=7$, and our previous results for

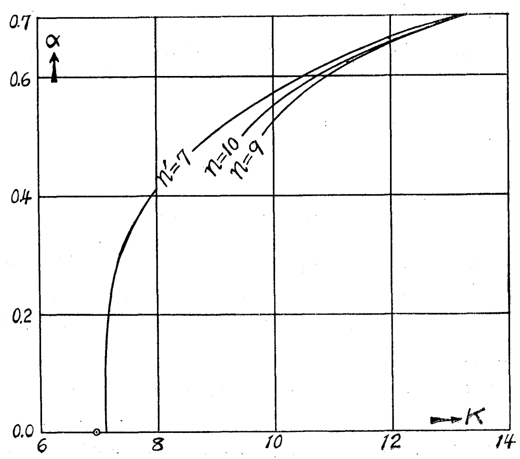

Fig. 1

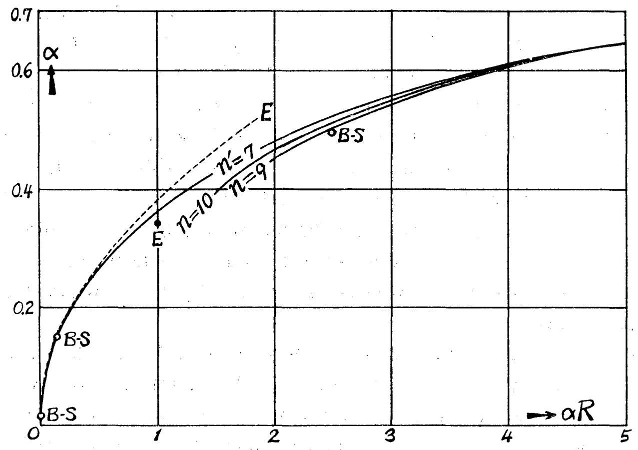

Fig. 2

the case of not-small Reynolds number, indicated as $n=9$ and 10. In Fig. 1 the end point $\alpha=0$ corresponds to $K=7.11$ in our approximation, which is only 2.6% larger than the exact value 6.93 given in the preceeding section. We have thus a complete curve of neutral stability within a few % error. The other curve assigned E is the one which represents the approximation (18′) for very small α, and due originally to *R. E.* Esch[(5)]. We see that this approximation is pretty good when α is smaller than 0.3. The point E is due also to Esch, and points B–S are due to *R.* Betchov and *A.* Szewczyk[(6)]; all calculated by electronic computers. Point E corresponds to our velocity profile, and points B–S are for the velocity profile $w=\tanh y$; owing to the difference between basic velocity profiles B–S points are not to compare with E point and ours. And yet they accord actually with ours all along the entire range of R, the existing small discrepancy being haply due to the profile difference.

§5 Wall Effect (1)

When α is small the domain of existence of the perturbation waves generally extends far into the uniform streams. Consequently when there are side walls confining the stream region, the problem of wall effect on the stability arises. The influence of walls being to constrain or disturb the motion of disturbance it seems very plausible that walls tend to stabilise the waves which are instable without them. In this section we undertake an approximate treatment of this problem.

As we are aiming at the problem of stability of a shear layer in middle part of the channel and not the stability of the boundary layers of confining walls, we can neglect the viscosity effect of the walls on the main stream, assuming

(5) R. E. Esch: The instability of a shear layer between two parallel streams, J. Fluid Mech., vol. 3 (1957), pp. 289–303.

(6) R. Betchov and A. Szewczyk: Stability of a shear layer between parallel streams, The Physics of Fluids, vol. 6 (1963) pp. 1391–1396.

frictionless slipping there. To the disturbance of course the walls act as viscous plates. The actual half breadth of the shear layer, which is 1 in our notation, is assumed to be a small fraction of the half channel width y_1. Then for the disturbance of wave number $\alpha \sim 1$ or larger the channel is broad enough allowing to assume it as boundless. For sufficiently small wave number the wall effect occurs, and for such a case we assume, in the η variable, that the half breadth α is negligible compared with the half channel width $\eta_1 = \alpha y_1$. In other words we assume that the behaviours of the disturbance waves in the neighborhood of walls are the same in both cases of continuous and discontinuous main flows, this assumption being reasonable for waves of sufficiently small wave number. We are thus lead back to (14).

We confine ourselves to the case of anti-symmetrical main flow, whose idealised discontinuous surface lying in the centre line of a channel, side walls situating at $\eta = \pm\eta_1$. $c=0$ for waves of neutral stability also in this case. We have, for the upper and the lower uniform flows, the following fundamental solutions of (14):

$$\eta_1 > \eta > 0: \ e^{\eta},\ e^{-\eta},\ e^{k_1\eta},\ e^{-k_1\eta};$$
$$0 > \eta > \eta_1: \ e^{\eta},\ e^{-\eta},\ e^{k_2\eta},\ e^{-k_2\eta};$$

where

$$k_1 = \sqrt{1+iK}, \quad k_2 = \sqrt{1-iK} = k_1^*, \tag{34}$$

asterisk denoting a complex conjugate. Then, introducing the conditions $\varphi = \dot{\varphi} = 0$ at the walls, we have the amplitude function:

$$\varphi(\eta) = \begin{cases} A\{\cosh(\eta_1-\eta) - \cosh k_1(\eta-\eta_1)\} + B\{k_1\sinh(\eta-\eta_1) - \sinh k_1(\eta-\eta_1)\}, \\ \qquad\qquad (\eta_1 > \eta > 0) \\ C\{\cosh(\eta+\eta_1) - \cosh k_2(\eta+\eta_1)\} + D\{k_2\sinh(\eta+\eta_1) - \sinh k_2(\eta+\eta_1)\}, \\ \qquad\qquad (0 > \eta > -\eta_1) \end{cases} \tag{35}$$

The condition at $\eta=0$, which joins the upper and the lower φ is (16), *i.e.*

$$\left.\begin{aligned} &\varphi(+0) - \varphi(-0) = 0, \quad \dot{\varphi}(+0) - \dot{\varphi}(-0) = 0, \\ &\ddot{\varphi}(+0) + iK\varphi(+0) - \{\ddot{\varphi}(-0) - iK\varphi(-0)\} = 0, \\ &\dddot{\varphi}(+0) - iK\dot{\varphi}(+0) - \{\dddot{\varphi}(-0) + iK\dot{\varphi}(-0)\} = 0, \end{aligned}\right\} \tag{36}$$

and the differential quotients of φ from (35) being imposed, these four equations are linear and homogeneous with regard to A, B, C and D. Eliminating these four then we have a characteristic equation:

$$\begin{vmatrix} F_1 & -F_2 & F_1^* & F_2^* \\ -G_1 & G_2 & G_1^* & G_2^* \\ H_1 & -H_2 & H_1^* & H_2^* \\ -I_1 & I_2 & I_1^* & I_2^* \end{vmatrix} = 0, \tag{37}$$

where

$$\left.\begin{array}{l} F_1=\cosh\eta_1-\cosh(k_1\eta_1), \quad F_2=k_1\sinh\eta_1-\sinh(k_1\eta_1), \\ G_1=\sinh\eta_1-k_1\sinh(k_1\eta_1), \quad G_2=k_1\cosh\eta_1-k_1\cosh(k_1\eta_1), \\ H_1=(1+iK)\cosh\eta_1-(1+2iK)\cosh(k_1\eta_1), \\ \qquad H_2=(1+iK)^{3/2}\sinh\eta_1-(1+2iK)\sinh(k_1\eta_1), \\ I_1=(1-iK)\sinh\eta_1-k_1\sinh(k_1\eta_1), \\ \qquad I_2=(1-iK)k_1\cosh\eta_1-k_1\cosh(k_1\eta_1). \end{array}\right\} \tag{38}$$

(37) can be rewritten into

$$\begin{vmatrix} F_1^{(r)} & F_1^{(i)} & F_2^{(r)} & F_2^{(i)} \\ -G_1^{(i)} & G_1^{(r)} & -G_2^{(i)} & G_2^{(r)} \\ H_1^{(r)} & H_1^{(i)} & H_2^{(r)} & H_2^{(i)} \\ -I_1^{(i)} & I_1^{(r)} & -I_2^{(i)} & I_2^{(r)} \end{vmatrix} = 0, \tag{37'}$$

which we use in the followings. Indexes r and i denote the real and the imaginary parts of the indexed quantity. This equation determines K as a function of η_1, *i.e.* R as a function of α, which is the neutral stability curve.

§6 Wall Effect (2)

To solve (37′) for K is not so easy task that we have made a preliminary estimate of $K(\eta_1)$ for not-small η_1 as follows. When η_1 is large the zeroth approximation for φ is that of $\eta_1=\infty$, which we have found in §2 (19):

$$\left.\begin{array}{l} \varphi^{(0)}=\begin{cases} -2De^{-\eta}+De^{-k_1\eta} & (\eta>0), \\ -2De^{\eta}+De^{k_2\eta} & (\eta<0), \end{cases} \\ K^{(0)}=4\sqrt{3},\ k_1^{(0)}=\sqrt{1+iK^{(0)}}=2+i\sqrt{3},\ k_2^{(0)}=2-i\sqrt{3}. \end{array}\right\} \tag{39}$$

Correcting this expression by the other two fundamental solutions into

$$\varphi=\varphi^{(0)}+Df(\eta)=\varphi^{(0)}+D\begin{cases} ae^{\eta}+be^{k_1^{(0)}\eta} & (\eta>0), \\ ce^{-\eta}+de^{-k_2^{(0)}\eta} & (\eta<0), \end{cases} \tag{40}$$

we let φ satisfy the boundary cconditions $\varphi=\dot{\varphi}=0$ at the side walls $\eta=\pm\eta_1$. At $\eta=\eta_1$ we have

$$\left.\begin{array}{l} -2e^{-\eta_1}+e^{-k_1^{(0)}\eta_1}+ae^{\eta_1}+be^{k_1^{(0)}\eta_1}=0, \\ 2e^{-\eta_1}-k_1^{(0)}e^{-k_1^{(0)}\eta_1}+ae^{\eta_1}+bk_1^{(0)}e^{k_1^{(0)}\eta_1}=0, \end{array}\right\} \tag{41}$$

and then

$$\left.\begin{array}{l} a=2\dfrac{3+i\sqrt{3}}{1+i\sqrt{3}}e^{-2\eta_1}-2\dfrac{2+i\sqrt{3}}{1+i\sqrt{3}}e^{-\left(1+k_1^{(0)}\right)\eta_1}, \\ b=\dfrac{3+i\sqrt{3}}{1+i\sqrt{3}}e^{-2k_1^{(0)}\eta_1}-\dfrac{4}{1+i\sqrt{3}}e^{-\left(1+k_1^{(0)}\right)\eta_1}; \end{array}\right\} \tag{41'}$$

by virtue of symmetry for $\eta<0$

$$c=a^*,\ d=b^* \qquad f(\eta)=-f(-\eta)^*. \tag{42}$$

Next we turn to the joining at $\eta=0$. When $\varphi^{(0)}$ is amended by first order correction, thus replacing K_0 by K and the coefficients $-2D$ and D by A, B for $\eta>0$ and C, D for $\eta<0$, and ignoring inclusion of second order terms, we have

$$\varphi=\begin{cases}Ae^{-\eta}+B(e^{-k_1\eta}+f(\eta)) & (\eta>0),\\ Ce^{\eta}+D(e^{k_2\eta}+f(\eta)) & (\eta<0),\end{cases}\tag{43}$$

for which the joining condition (36) is easily written down:

$$\left.\begin{aligned}&A+(1+f(+0))B-C-(1+f(-0))D=0,\\ &-A+(-k_1+\dot{f}(+0))B-C-(k_2+\dot{f}(-0))D=0,\\ &A+(k_1^2+\ddot{f}(+0))B-(1-2iK)C-\{k_2^2+\ddot{f}(-0)-2iK(1+f(-0))\}D=0,\\ &-A+(-k_1^3+\dddot{f}(+0))B-(1+2iK)C-\{k_2^3+\dddot{f}(-0)+2iK(k_2+\dot{f}(-0))\}D=0.\end{aligned}\right\}\tag{44}$$

Eliminating A, B, C, D between these four equations and writing

$$K=K_0+\epsilon,\quad k_1=k_1^{(0)}+\frac{i}{14}k_2^{(0)}\epsilon,\qquad k_2=k_1^*\tag{45}$$

in the obtained characteristic equation, we find

$$\epsilon=\frac{14}{\sqrt{3}}\{a+a^*+2(b+b^*)+i\sqrt{3}\,(a-a^*+b-b^*)\};\tag{46}$$

inserting then a, b from (41′) this correction reduces to the final expression:

$$\epsilon=\frac{14}{\sqrt{3}}\left\{\begin{aligned}&12e^{-2\eta_1}-e^{-3\eta_1}(18\cos\sqrt{3}\,\eta_1+2\sqrt{3}\sin\sqrt{3}\,\eta_1)\\ &\qquad+e^{-4\eta_1}(9\cos2\sqrt{3}\,\eta_1+\sqrt{3}\sin2\sqrt{3}\,\eta_1)\end{aligned}\right\}.\tag{46′}$$

First approximations of $K(\eta_1)=K_0+\epsilon(\eta_1)$ and $\eta_1K(\eta_1)$ are calculated with this ϵ and given in Table 3, under the heading "1st approx."; $\eta_1K(\eta_1)$ is also inscribed as a curve "1st approx." in Fig. 3. Coefficients of φ to the first approximation are found, solving (44) by use of $K(\eta_1)$ in them, but which we need not in this paper.

As we have approximate values of $K(\eta_1)$ now in hand, we turn to the exact determination of root $K(\eta_1)$ of the determinant equation (37′). Fixing η_1 at several values, K is sought in the neighborhood of the approximate values by the trial-and-

Table 3

η_1	1st approx.		accurate	
	K	$K\eta_1$	K	$K\eta_1$
4.0	6.960	27.84		
3.5	7.031	24.54		
3.0	7.163	21.49		
2.5	7.624	19.06	7.611	19.03
2.2	8.303	18.27	8.252	18.15
2.0	9.091	18.18	8.991	17.98
1.8	10.286	18.51	10.120	18.22
1.5	13.031	19.55	13.058	19.59

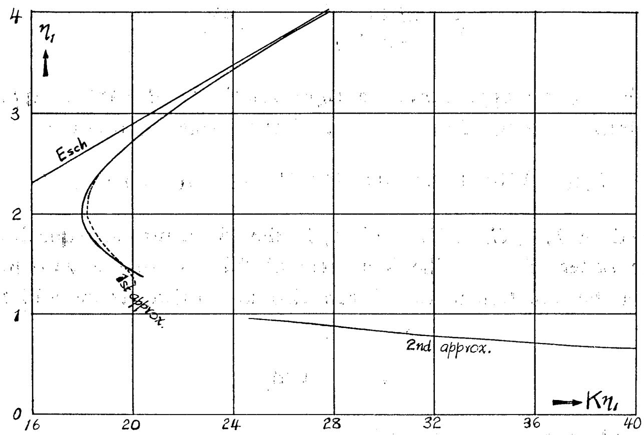

Fig. 3

error method. Result is as shown in Table 3 under the heading "accurate". For $\eta_1 \gtrsim 1.5$ the first approximation proved to be good enough to diminish our labors. Accurate curve $\eta_1 K(\eta_1)$ are also inscribed in Fig. 3.

When η_1 is very small a little consideration shows that K has an expansion of the form:

$$K \infty \frac{\kappa_0}{\eta_1^2} + \kappa_1 + \cdots, \tag{47}$$

where $\kappa_0, \kappa_1, \cdots$ are real constants, and $k_1\eta_1$ has then the expansion:

$$k_1\eta_1 = \sqrt{i\kappa_0} + \frac{1+i\kappa_1}{2\sqrt{i\kappa_0}}\eta_1^2 + \cdots. \tag{48}$$

Using these expansions in (38) we know the orders:

$$F_1, F_2 = O(1), \quad G_1, G_2 = O(\eta_1^{-1}),$$
$$H_1, H_2 = O(\eta_1^{-2}), \quad I_1, I_2 = O(\eta_1^{-3}),$$

and therefore we replace in the determinant equation (37) G, H, I by $\eta_1 G$, $\eta_1^2 H$, $\eta_1^3 I$ and take the limit $\eta_1 \to 0$. (37) then reduces to an equation for the constant κ_0, which after some rewritings takes the expression:

$$\sqrt{\frac{\kappa_0}{2}} = \xi, \qquad \xi(\sinh\xi + \sin\xi)(\cosh\xi - \cos\xi)$$
$$= \frac{7}{2}(\cosh\xi - \cos\xi)^2 + \cosh\xi\cos\xi - 1, \tag{49}$$

from which

$$\xi = 3.705, \ \textit{i.e.}\ \kappa_0 = 27.45 \tag{50}$$

is found. Thus the first approximation in case of small η_1 is

$$K = \frac{27.45}{\eta_1^2}, \quad \eta_1 K = \frac{27.45}{\eta_1}. \tag{51}$$

To find the second approximation right-hand side of (48) is approximated by the first two terms which, after insertion of the value κ_0, becomes

$$k_1\eta_1 = 3.705(1+i) + 0.06748\{(1+\kappa_1) + i(-1+\kappa_1)\}\eta_1^2. \tag{52}$$

This being used in F, $\eta_1 G$, $\eta_1^2 H$ and $\eta_1^3 I$, the determinant equation (37) is expanded to the order of η_1^2. The first term of this expansion vanishes because of (49), and then the coefficient of η_1^2 has also to vanish, from which κ_1 is found to be

$$\kappa_1 = -4.60, \tag{53}$$

and the required second approximation is

$$K = \frac{27.45}{\eta_1^2} - 4.60, \ i.e.\ \eta_1 K = \frac{27.45}{\eta_1} - 4.60\eta_1. \tag{54}$$

The curve representing (54) is inscribed in Fig. 3 as "2nd approx.".

We have thus completed the calculation of $\eta_1 K(\eta_1)$. $\eta_1 K(\eta_1)$ and η_1 divided by y_1 give R and α, therefore we know that $K\eta_1 : \eta_1$ curve is similar to $R : \alpha$ curve *i.e.* neutral stability curve. Application of this curve is shown in Fig. 4, where two cases of side wall distance $y_1 = 5$ and 10 are tried for examples, and $R : \alpha$ curves are inscribed. For $y_1 = 10$ approximation by discontinuous flow seems good. For $y_1 = 5$ it seems appreciably worse. Be that as it may, the appearance of the critical Reynolds number of instability is a characteristic effect of the side walls.

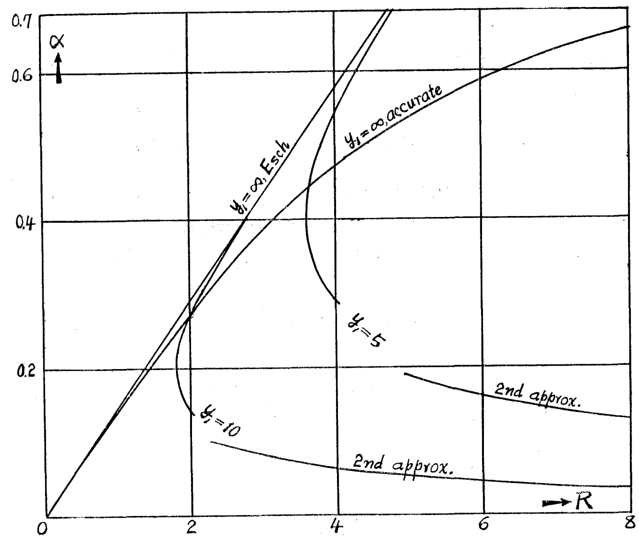

Fig. 4

Acknowledgement

For numerical calculations and preparation of manuscript of this paper assistances are given by Misses M. Kumazawa and K. Kawanishi, to whom we are much obliged.

(Communicated by M. Kurihara. Received January 27, 1965)

Corrigenda to Part I

page	equation	in error	correct
2	(3)	$-w''\varphi$	$+w''\varphi$
7	(20′)	$(n\geq 2)$	$(n\geq 0)$
〃	(21)	ψ_{n-1}	ψ_{n-2}
〃	(21′)	$\sqrt{n(n+1)}$	$\sqrt{(n+1)(n+2)}$