PSEUDO-PLASTIC SUBSTANCE AS GENERALIZED STOKESIAN FLUID

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By Kazuo KAWATATE

Synopsis

Here is proposed a method for describing pseudo-plastic substance, where it is represented as a generalized Stokesian fluid possessed of yield value.

The method of calculations to determine the characteristic values of a substance will be explained on various cases with a rod, a band, a rotating disk, a rotating cylinder and a capillary applied as viscometer.

1. Constitutive equations

In considering problems of primary dimension, what is known as Bingham plastic substance presents typical plastic phenomenon. There is not observed any flow in this Bingham plastic substance, as plain from Fig. 1, in a stress region lower than the yield value τ_0. It is only in a higher stress region that the flow begins to appear. Putting D for strain rate, τ for shearing stress and m for constant mobility of the matter in quest, we can express as follows,

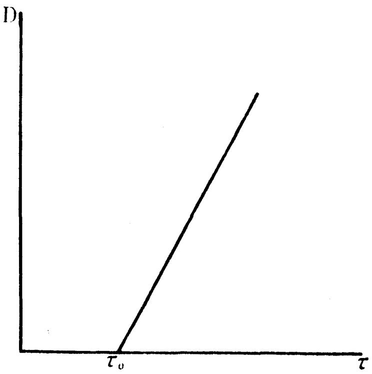

Fig. 1.

$$\left.\begin{array}{ll} \text{for} \quad \tau_0 \geqq \tau, & D=0 \\ \text{for} \quad \tau_0 < \tau, & D=m(\tau-\tau_0) \end{array}\right\}. \tag{1}$$

Oldroyd is the first to extend this one-dimensional plastic body in an ideal form into three-dimensional one [1]. He has solved various problems by denoting yield value with τ_0, deviatoric stress tensor with $p_{\cdot j}^{i}$, strain rate tensor with $d_{\cdot j}^{i}$ and plastic viscosity with η_p and by using

$$\left.\begin{array}{ll} \text{for} \quad \tau_0 \geqq \sqrt{\frac{1}{2} p_{\cdot n}^{m} p_{\cdot m}^{n}}, & d_{\cdot j}^{i}=0 \\ \text{for} \quad \tau_0 < \sqrt{\frac{1}{2} p_{\cdot n}^{m} p_{\cdot m}^{n}}, & \left(1-\frac{\tau_0}{\sqrt{\frac{1}{2} p_{\cdot n}^{m} p_{\cdot m}^{n}}}\right) p_{\cdot j}^{i} = 2\eta_p d_{\cdot j}^{i} \end{array}\right\}. \tag{2}$$

[1] Oldroyd, J. G., Proc. Camb. Phil. Soc., 43 (1947), 100.

Both equations (1) and (2) which are generally called the constitutive or rheological equations are representing the characteristic of Bingham solid in this case. The important part in the above relations is played by yield value τ_0 and mobility m or plastic viscosity η_p.

There are various ways made known so far in describing one-dimensional pseudo-plastic substance, one of which will be given here as an example [2],

$$\left.\begin{array}{lll} \text{for} & \tau_0 \geqq \tau, & D=0 \\ \text{for} & \tau_0 < \tau, & D=m(\tau-\tau_0)+A(\tau-\tau_0)^B \end{array}\right\}, \tag{3}$$

where A and B are assumed as the material constants to be determined. Formulation of these relations in three dimensional terms is a much-desired matter in many fields such as lubrication, calendering or screw extractions in chemical engineering, and mixing and kneading of soup or butter. As it is with the case of an ideal plasticity, the first question arising in the description of pseudo-plastic substance is the condition of yield and the next is the relations among stress, strain or strain rate in that yield region.

In the mathematical theory of plasticity, what are called by the names of Mises and Tresca are well known. Especially the condition of Mises is frequently used which tells the yield begins when the elastic strain energy of deformation has reached some critical value. Since the mechanical properties of an isotropic body are independent of the rotation of the reference axes, the condition under which plastic flow starts appearing must be expressed in terms of stress invariants provided that the psedo-plastic substance is treated as isotropic. Letting certain Cartesian coordinate system be (X, Y, Z) and considering the simple problem in which only p_{XY} is not zero and all the other stress components become zero, we have the result that only the stress invariant of second order remains non-zero. This fact also seems to justify well enough the frequent adoption of Mises' condition that has a form the invariant of second order is constant. Hence the condition of Mises will be adopted here as the criterion of the region of plastic flow [3].

We proceed next to discuss the behavior of pseudo-plastic materials in the plastic region. At this stage, we are reminded of the formulation of the so-called Stokesian fluids [4] [5] [6]. Letting strain rate tensor be $d_{.j}^{i}$, stress tensor $s_{.j}^{i}$, pressure p and shearing stress tensor $p_{.j}^{i}$, we have

$$s_{.j}^{i}=-p\delta_{.j}^{i}+p_{.j}^{i}.$$

The continuum that satisfies the following four conditions is called "Stokesian fluid."

1. $p_{.j}^{i}=f_{.j}^{i}(d_{.n}^{m})$. (4)

[2] Murakami, K., Chem. High Polymer 9 (1952), 469.
[3] Prager, W. and Hodge, P. G., Theory of Perfect Plastic Solids, 1951.
[4] Green, A. E. and Adkins, J. E., Large Elastic Deformations, 1960, 337.
[5] Eringen, A. C., Nonlinear Theory of Continuous Media, 1962, 160.
[6] Fredrickson, A. G., Principles and Applications of Rheology, 1964, 68,

2. If $d_{\cdot n}^{m} \to 0$, then $p_{\cdot j}^{i} \to 0$. (5)
3. $p_{\cdot j}^{i}$ is explicitly independent of the position.
4. Functions $f_{\cdot j}^{i}$ are independent with respect to the orientation of axes.

Letting the Hamilton-Cayley theorem in the matrix theory play the major role, it will be proved to be expressed as follows,

$$p_{\cdot j}^{i}=\alpha_0\delta_{\cdot j}^{i}+\alpha_1 d_{\cdot j}^{i}+\alpha_2 d_{\cdot m}^{i} d_{\cdot j}^{m}, \tag{6}$$

$$\alpha_k=\alpha_k(\mathrm{I}_d,\ \mathrm{II}_d,\ \mathrm{III}_d) \quad k=0,\ 1,\ 2, \tag{7}$$

$$\alpha_0\ (0,\ 0,\ 0)=0, \tag{8}$$

where

$$\left.\begin{aligned} \mathrm{I}_d&=d_{\cdot m}^{m}\\ \mathrm{II}_d&=\frac{1}{2}(\mathrm{I}_d^2-d_{\cdot n}^{m}d_{\cdot m}^{n})\\ \mathrm{III}_d&=\det.|d_{\cdot n}^{m}|=\frac{1}{3}(-\mathrm{I}_p^3+3\mathrm{I}_d\mathrm{II}_d+d_{\cdot m}^{l}d_{\cdot n}^{m}d_{\cdot l}^{n}) \end{aligned}\right\}. \tag{9}$$

This may be looked upon as the formulation of the isotropic non-Newtonian flow from purely fluid viewpoint. Such a logic provides us with a key to describe our pseudo-plastic substance, which was initiated by Cauchy long ago [4] and, later on, extended in various ways, having been used for the establishment of the rheological equation of diversified visco-elastic substances such as Rivlin-Ericksen fluid in which the rate of strain rate is considered beside the strain rate [7], Segawa's fluid where the strain and the strain rate are considered [8], and Koh-Eringen's fluid where the thermoviscoelastic material is accounted for [9].

Now puting anew p for mechanical mean pressure, $p_{\cdot j}^{i}$ for deviatoric stress tensor and the same signs for the rest, we can assume from the analogy with the one dimensional body as follows,

$$\left.\begin{aligned} &\text{for} \quad \tau_0 \geqq \sqrt{\frac{1}{2}p_{\cdot n}^{m}p_{\cdot m}^{n}}, \quad d_{\cdot j}^{i}=0\\ &\text{for} \quad \tau_0 < \sqrt{\frac{1}{2}p_{\cdot n}^{m}p_{\cdot m}^{n}}, \quad \left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p_{\cdot n}^{m}p_{\cdot m}^{n}}}\right)p_{\cdot j}^{i}=f_{\cdot j}^{i}(d_{\cdot n}^{m}) \end{aligned}\right\}. \tag{10}$$

Then quite similarly to the case of Stokesian fluid, we obtain

$$\left.\begin{aligned} &\text{for} \quad \tau_0 \geqq \sqrt{\frac{1}{2}p_{\cdot n}^{m}p_{\cdot m}^{n}}, \quad d_{\cdot j}^{i}=0\\ &\text{for} \quad \tau_0 < \sqrt{\frac{1}{2}p_{\cdot n}^{m}p_{\cdot m}^{n}}, \quad \left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p_{\cdot n}^{m}p_{\cdot m}^{n}}}\right)p_{\cdot j}^{i}=\alpha_0\delta_{\cdot j}^{i}+\alpha_1 d_{\cdot j}^{i}+\alpha_2 d_{\cdot m}^{i}d_{\cdot j}^{m} \end{aligned}\right\}. \tag{11}$$

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[4] Green, A. E. and Adkins, J. E., op. cit..
[7] Rivlin, R. S., and Ericksen, J. L., J. Ratl. Mech. Anal. 4 (1955), 323.
[8] Segawa, W., J. Soc. Materials Sci., Japan, 13 (1964), 350.
[9] Koh, S. L. and Eringen, A. C., Int. J. Engng. Sci., 1 (1963), 199.

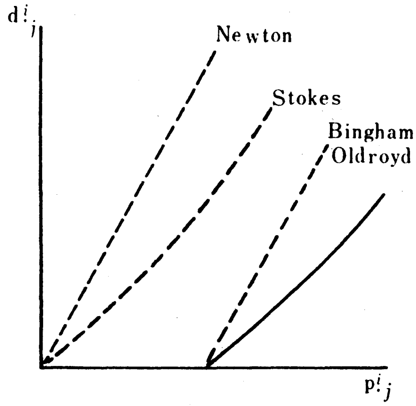

Fig. 2.

That is, in equation (11) the pseudo-plastic substance is defined as Stokesian fluid endowed with the conception of yield value. There α_k are given by equation (7). Incidentally, if we can attain one to one correspondence between $\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}}\right)p^{i}_{\cdot j}$ and $d^{i}_{\cdot j}$ within the range of the yield region $\tau_0<\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}$, in other words, if the phenomenon of hysteresis such as structural destruction is neglected or if consideration is given exclusively to loading instead of unloading, we can solve equation (10) with respect to $d^{i}_{\cdot j}$

$$d^{i}_{\cdot j}=h^{i}_{\cdot j}(z^{k}_{\cdot l}),\quad z^{k}_{\cdot l}=\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}}\right)p^{k}_{\cdot l}. \tag{12}$$

Following the similar logic, we can induce

$$d^{i}_{\cdot j}=\beta_0\delta^{i}_{\cdot j}+\beta_1\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}}\right)p^{i}_{\cdot j}+\beta_2\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}}\right)^2 p^{i}_{\cdot k}p^{k}_{\cdot j}, \tag{13}$$

where

$$\beta_k=\beta_k(\mathrm{I}_z,\ \mathrm{II}_z,\ \mathrm{III}_z)=\beta_k(\mathrm{I}_p,\ \mathrm{II}_p,\ \mathrm{III}_p)\quad k=0,\ 1,\ 2, \tag{14}$$

$$\left.\begin{aligned}&\mathrm{I}_p=p^{m}_{\cdot m}\\ &\mathrm{II}_p=\frac{1}{2}(\mathrm{I}_p^2-p^{m}_{\cdot n}p^{n}_{\cdot m})\\ &\mathrm{III}_p=\det.|p^{m}_{\cdot n}|=\frac{1}{3}(-\mathrm{I}_p^3+3\mathrm{I}_p\mathrm{II}_p+p^{l}_{\cdot m}p^{m}_{\cdot n}p^{n}_{\cdot l})\end{aligned}\right\}. \tag{15}$$

If we treat incompressible material henceforward, then we obtain

$$\mathrm{I}_d=d^{m}_{\cdot m}=0. \tag{16}$$

Again, since $p^{i}_{\cdot j}$ is a deviatoric stress, it is resulted

$$\mathrm{I}_p=p^{m}_{\cdot m}=0. \tag{17}$$

Accordingly, from equations (11) and (13) we obtain as follows,

$$\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}}\right)p^{i}_{\cdot j}=\alpha_1 d^{i}_{\cdot j}+\alpha_2\left(d^{i}_{\cdot m}d^{m}_{\cdot j}-\frac{1}{3}d^{m}_{\cdot n}d^{n}_{\cdot m}\delta^{i}_{\cdot j}\right) \tag{18}$$

$$d^{i}_{\cdot j}=\beta_1\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}}\right)p^{i}_{\cdot j}+\beta_2\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}p^{m}_{\cdot n}p^{n}_{\cdot m}}}\right)^2\left(p^{i}_{\cdot k}p^{k}_{\cdot j}-\frac{1}{3}p^{m}_{\cdot n}p^{n}_{\cdot m}\delta^{i}_{\cdot j}\right) \tag{19}$$

And equations (7) and (14) become

$$\left.\begin{aligned}\alpha_k&=\alpha_k(\mathrm{II}_d,\ \mathrm{III}_d)\\ \beta_k&=\beta_k(\mathrm{II}_p,\ \mathrm{III}_p),\end{aligned}\right\},\quad k=1,\ 2. \tag{20}$$

In order to investigate, at this point, the one-to-one corresponding conditions between the strain-rate invariants (II_d, III_d) and the stress invariants (II_p, III_p), we will put as follows,

$$\left.\begin{aligned}&D_{\mathrm{I}}=d^{\ m}_{\cdot n}d^{\ n}_{\cdot m}\ . && D_{\mathrm{II}}=d^{\ l}_{\cdot m}d^{\ m}_{\cdot n}d^{\ n}_{\cdot l}\\ &P_{\mathrm{I}}=p^{\ m}_{\cdot n}p^{\ n}_{\cdot m}\ . && P_{\mathrm{II}}=p^{\ l}_{\cdot m}p^{\ m}_{\cdot n}p^{\ n}_{\cdot l}\\ &Z_{\mathrm{I}}=z^{\ m}_{\cdot n}z^{\ n}_{\cdot m}=\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}P_{\mathrm{I}}}}\right)^2P_{\mathrm{I}}, && Z_{\mathrm{II}}=\left(1-\frac{\tau_0}{\sqrt{\frac{1}{2}P_{\mathrm{I}}}}\right)^3P_{\mathrm{II}}\end{aligned}\right\}. \tag{21}$$

The last two equations may be written as follows,

$$\left.\begin{aligned}\sqrt{\frac{P_{\mathrm{I}}}{2}}&=\sqrt{\frac{Z_{\mathrm{I}}}{2}}+\tau_0\\ P_{\mathrm{II}}&=\left(1+\frac{\tau_0}{\sqrt{\frac{1}{2}Z_{\mathrm{I}}}}\right)^3Z_{\mathrm{II}}\end{aligned}\right\}. \tag{22}$$

Using equations (16) and (17) on the other hand, we can easily carry out calculations as follows from equations (9) and (15),

$$\left.\begin{aligned}D_{\mathrm{I}}&=-2\mathrm{II}_d,\quad D_{\mathrm{II}}=3\mathrm{III}_d\\ P_{\mathrm{I}}&=-2\mathrm{II}_p,\quad P_{\mathrm{II}}=3\mathrm{III}_p\end{aligned}\right\}, \tag{23}$$

all we have to do is, after all, to investigate the corresponding condition between (D_{I}, D_{II}) and (Z_{I}, Z_{II}). If equation (12) and Hamilton-Cayley theorem are considered jointly as follows,

$$d^{\ k}_{\cdot m}d^{\ m}_{\cdot n}d^{\ n}_{\cdot l}=\mathrm{I}_d d^{\ k}_{\cdot m}d^{\ m}_{\cdot l}-\mathrm{II}_d d^{\ k}_{\cdot l}+\mathrm{III}_d\delta^{\ k}_{\cdot l}\,, \tag{24}$$

$$d^{\ k}_{\cdot m}d^{\ m}_{\cdot n}d^{\ n}_{\cdot l}=\frac{1}{2}D_{\mathrm{I}}d^{\ k}_{\cdot l}+\frac{1}{3}D_{\mathrm{II}}\delta^{\ k}_{\cdot l}\qquad(\text{for } \mathrm{I}_d=0)\,, \tag{25}$$

it is obtained from equation (18) by putting $\alpha_k=\alpha_k(D_{\mathrm{I}},\ D_{\mathrm{II}})$ [5],

$$\left.\begin{aligned}Z_{\mathrm{I}}&=\alpha_1^2D_{\mathrm{I}}+2\alpha_1\alpha_2D_{\mathrm{II}}+\frac{1}{6}\alpha_2^2D_{\mathrm{I}}^2\\ Z_{\mathrm{II}}&=\alpha_1^3D_{\mathrm{II}}+\frac{1}{2}\alpha_1^2\alpha_2D_{\mathrm{I}}^{\ 2}+\frac{1}{2}\alpha_1\alpha_2^2D_{\mathrm{I}}D_{\mathrm{II}}+\frac{1}{3}\alpha_2^3\left(D_{\mathrm{II}}^2-\frac{1}{12}D_{\mathrm{I}}^3\right)\end{aligned}\right\}. \tag{26}$$

Similarly from equation (19) it is obtained by putting $\beta_k=\beta_k(P_{\mathrm{I}},\ P_{\mathrm{II}})$ as follows,

$$\left.\begin{aligned}D_{\mathrm{I}}&=\beta_1^2Z_{\mathrm{I}}+2\beta_1\beta_2Z_{\mathrm{II}}+\frac{1}{6}\beta_2^2Z_{\mathrm{I}}^2\\ D_{\mathrm{II}}&=\beta_1^3Z_{\mathrm{II}}+\frac{1}{2}\beta_1^2\beta_2Z_{\mathrm{I}}^2+\frac{1}{2}\beta_1\beta_2^2Z_{\mathrm{I}}Z_{\mathrm{II}}+\frac{1}{3}\beta_2^3\left(Z_{\mathrm{II}}^2-\frac{1}{12}Z_{\mathrm{I}}^3\right)\end{aligned}\right\}. \tag{27}$$

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[5] Eringen, A. C., op. cit., 297.

To obtain a unique solution with respect to D_{I} and D_{II} for equation (26), the condition that the following functional determinent must not vanish, i. e.,

$$\frac{\partial Z_{\mathrm{I}}}{\partial D_{\mathrm{I}}}\cdot\frac{\partial Z_{\mathrm{II}}}{\partial D_{\mathrm{II}}}-\frac{\partial Z_{\mathrm{I}}}{\partial D_{\mathrm{II}}}\cdot\frac{\partial Z_{\mathrm{II}}}{\partial D_{\mathrm{I}}}\neq 0, \tag{28}$$

will answer the purpose in accordance with the implicit function theorem. The same logic will be valid with equation (27). In the above-mentioned epuations, $\frac{1}{2}\alpha_1=\frac{1}{2}\alpha_1(D_{\mathrm{I}}, D_{\mathrm{II}})$ expresses the apparent plastic viscosity while $\alpha_2=\alpha_2(D_{\mathrm{I}}, D_{\mathrm{II}})$ the effect of apparent normal stress [6], but here we will take up only the apparent plastic viscosity for consideration,

$$\left.\begin{aligned}&\alpha_1=\alpha(D_{\mathrm{I}}, D_{\mathrm{II}})\\&\alpha_2=0\end{aligned}\right\}. \tag{29}$$

Namely, we will consider such a simple case as

$$\left.\begin{aligned}&Z_{\mathrm{I}}=\alpha^2 D_{\mathrm{I}}\\&Z_{\mathrm{II}}=\alpha^3 D_{\mathrm{II}}\end{aligned}\right\}. \tag{30}$$

Then equation (28) becomes

$$\alpha+2\frac{\partial\alpha}{\partial D_{\mathrm{I}}}D_{\mathrm{I}}+3\frac{\partial\alpha}{\partial D_{\mathrm{II}}}D_{\mathrm{II}}\neq 0, \tag{31}$$

or considering as $\alpha=\alpha(\sqrt{D_{\mathrm{I}}}, \sqrt[3]{D_{\mathrm{II}}})$, we have

$$\alpha+\frac{\partial\alpha}{\partial\sqrt{D_{\mathrm{I}}}}\sqrt{D_{\mathrm{I}}}+\frac{\partial\alpha}{\partial\sqrt[3]{D_{\mathrm{II}}}}\sqrt[3]{D_{\mathrm{II}}}\neq 0. \tag{32}$$

If α representing the characteristic of sample satisfies this condition, equation (30) can be solved uniquely with respect to D_{I} and D_{II}. And we may put as $\alpha=\alpha(D_{\mathrm{I}}, D_{\mathrm{II}})=\alpha(Z_{\mathrm{I}}, Z_{\mathrm{II}})$. Substituting this into equation (27), we can induce the following relations,

$$\left.\begin{aligned}&\frac{Z_{\mathrm{I}}}{\alpha^2}=\beta_1^2 Z_{\mathrm{I}}+2\beta_1\beta_2 Z_{\mathrm{II}}+\frac{1}{6}\beta_2^2 Z_{\mathrm{I}}^2\\&\frac{Z_{\mathrm{II}}}{\alpha^3}=\beta_1^3 Z_{\mathrm{II}}+\frac{1}{2}\beta_1^2\beta_2 Z_{\mathrm{I}}^2+\frac{1}{2}\beta_1\beta_2^2 Z_{\mathrm{I}}Z_{\mathrm{II}}+\frac{1}{3}\beta_2^3(Z_{\mathrm{II}}^2-\frac{1}{12}Z_{\mathrm{I}}^3)\end{aligned}\right\}. \tag{33}$$

When this is solved with respect to (β_1, β_2), there exist at least three groups of roots. Through an extremely rudimentary trial calculation, we can evaluate a group of real roots in a simple form as follows,

$$\left.\begin{aligned}&\beta_1=\beta(Z_{\mathrm{I}}, Z_{\mathrm{II}})=\frac{1}{\alpha(Z_{\mathrm{I}}, Z_{\mathrm{II}})}\\&\beta_2=0\end{aligned}\right\}. \tag{34}$$

By the way, we will obtain an equation by expressing a condition where the dissipative energy of deformation in the plastic region is positive as follows,

[6] Fredrickson, A. G., op. cit., 75.

$$p_{.n}^{m}\, d_{.m}^{n} > 0\,. \tag{35}$$

This becomes by equation (11)

$$\alpha_0 \mathrm{I}_d + \alpha_1 D_{\mathrm{II}} + \alpha_2 D_{\mathrm{III}} > 0\,, \tag{36}$$

and by equation (13)

$$\beta_0 \mathrm{I}_p + \beta_1 \left(1 - \frac{\tau_0}{\sqrt{\frac{1}{2}P_{\mathrm{II}}}}\right) P_{\mathrm{II}} + \beta_2 \left(1 - \frac{\tau_0}{\sqrt{\frac{1}{2}P_{\mathrm{II}}}}\right)^2 P_{\mathrm{III}} > 0\,. \tag{37}$$

When the conditions of $\mathrm{I}_d=0$, $\mathrm{I}_p=0$, $\alpha_1=\alpha$, $\alpha_2=0$, $\beta_1=\beta$, $\beta_2=0$, $D_{\mathrm{II}}>0$ and $P_{\mathrm{II}}>0$ are added further, we obtain ultimately

$$\left.\begin{array}{l} \alpha > 0 \\ \beta > 0 \end{array}\right\}, \tag{38}$$

which naturally proves that the signs of stress and strain rate are equivalent. In the following sections, method of estimation of apparent plastic viscoty α or apparent mobility β will be described by the aid of various viscometers. The fundamental equations there are

$$\text{continuous equation (incompressible)},\quad d_{.m}^{m}=0\,, \tag{39}$$

$$\text{kinematical relations},\quad d_{.j}^{i}=\frac{1}{2}\left(v^i|_j+v_j|^i\right), \tag{40}$$

$$\left.\begin{array}{ll} \text{equilibrium equations}, & (-p\delta_{.j}^{m}+p_{.j}^{m})|_m=0 \\ & p=\frac{1}{3}s_{.m}^{m}\,,\quad j=1,2,3, \end{array}\right\}, \tag{41}$$

$$\text{rheological equations, for } \tau_0 \geqq \sqrt{\tfrac{1}{2}P_{\mathrm{II}}}\,,\quad d_{.j}^{i}=0\,, \tag{42}$$

$$\left.\begin{array}{ll} \text{for } \tau_0 < \sqrt{\tfrac{1}{2}P_{\mathrm{II}}}\,, & \left(1-\dfrac{\tau_0}{\sqrt{\frac{1}{2}P_{\mathrm{II}}}}\right) p_{.j}^{i}=\alpha d_{.j}^{i} \\ \text{or} & d_{.j}^{i}=\beta\left(1-\dfrac{\tau_0}{\sqrt{\frac{1}{2}P_{\mathrm{II}}}}\right) p_{.j}^{i} \end{array}\right\}, \tag{43}$$

provided that

$$\left.\begin{array}{l} \alpha=\alpha(D_{\mathrm{II}},\, D_{\mathrm{III}})=\alpha(\sqrt{D_{\mathrm{II}}},\, \sqrt[3]{D_{\mathrm{III}}}) \\ \beta=\beta(P_{\mathrm{II}},\, P_{\mathrm{III}})=\beta(\sqrt{P_{\mathrm{II}}},\, \sqrt[3]{P_{\mathrm{III}}}) \end{array}\right\}. \tag{44}$$

For coordinate system Eulerian system is adopted, where " $|_i$ " denotes covariant differentiation and v^i velocity vector. Incidentally, the body forces are neglected in the equations of equilibrium.

2. Rod viscometer

It will be assumed that a round rod with radius a is passing through a hollow cylinder of length l and inner diameter $2b$ with driving force F and at steady velocity V. Putting $h = b-a$ and in case of $h/a \ll 1$, $a/l \ll 1$, we may express velocity component using cylindrical coodinate as follows,

$$\left.\begin{aligned} v_r &= v_\theta = 0 \\ v_z &= v(r) \end{aligned}\right\}. \tag{1}$$

The boundary conditions to be added here will be

$$\left.\begin{aligned} v(a) &= V \\ v(b) &= 0 \end{aligned}\right\}. \tag{2}$$

The physical component of strain rate will become

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{dv}{dr} \\ 0 & 0 & 0 \\ \frac{dv}{dr} & 0 & 0 \end{pmatrix}, \tag{3}$$

where the condition of incompressibility $I_d = 0$ is satisfied. It is also understood that only p_{rz} is not zero among the deviatoric stress components and the rest becomes extinct. We have, therefore, $I_p=0$, $P_{\mathrm{II}}=2p_{rz}^2$, $P_{\mathrm{III}}=0$, $\sqrt{\frac{1}{2}P_{\mathrm{II}}} = |p_{rz}|$ and the rheological equation will be

$$\left.\begin{aligned} &\text{for} \quad \tau_0 \geqq |p_{rz}|, \quad \frac{dv}{dr} = 0 \\ &\text{for} \quad \tau_0 < |p_{rz}|, \quad \frac{1}{2}\frac{dv}{dr} = \left(1 - \frac{\tau_0}{|p_{rz}|}\right)\beta p_{rz} \end{aligned}\right\}. \tag{4}$$

The equilidrium equation is

$$\frac{dp_{rz}}{dr} + \frac{p_{rz}}{r} = 0. \tag{5}$$

Solving this equation, we get $p_{rz} = \text{const.}/r$. Since its balance to external force F is considered to be

$$F + \int_0^{2\pi} p_{rz} lr d\theta = 0$$

we can express as follows,

$$p_{rz} = -\frac{F}{2\pi l} \cdot \frac{1}{r}. \tag{6}$$

Putting here as

$$r_* = \frac{F}{2\pi l \tau_0}, \tag{7}$$

eqnation (4) will become

$$\left.\begin{aligned} &\text{for } r_* \leqq r, \quad \frac{dv}{dr}=0 \\ &\text{for } r_* > r, \quad \frac{dv}{dr}=2\beta\tau_0\left(-\frac{r_*}{r}+1\right) \end{aligned}\right\}, \tag{8}$$

and considering jointly with the boundary condition (2), we obtain

$$\left.\begin{aligned} &\text{in case } r_* \leqq a, \quad v(r)=0 \\ &\text{in case } a<r_*\leqq b, \text{ in the range of } r_*\leqq r\leqq b, \ v(r)=0 \\ &\qquad \text{in the range of } a\leqq r<r_*, \ v(r)=\int_r^{r_*} 2\beta\tau_0\left(\frac{r_*}{r}-1\right)dr \\ &\text{in case } b<r_*, \ v(r)=\int_r^b 2\beta\tau_0\left(\frac{r_*}{r}-1\right)dr \end{aligned}\right\}. \tag{9}$$

If we turn to the point of $r=a$, the result will be

$$\left.\begin{aligned} &\text{i) } r_*\leqq a, && V=0 \\ &\text{ii) } a<r_*\leqq b, && V=\int_a^{r_*} 2\beta\tau_0\left(\frac{r_*}{r}-1\right)dr \\ &\text{iii) } b<r_*, && V=\int_a^b 2\beta\tau_0\left(\frac{r_*}{r}-1\right)dr \end{aligned}\right\}. \tag{10}$$

Let us consider here as a special case Bingham plasticity. That is, putting $\beta=\frac{1}{2\eta_p}$ in the above equation and reverting r_* to F by using equation (7), we have

$$\left.\begin{aligned} &\text{i) } F\leqq 2\pi la\tau_0, && V=0 \\ &\text{ii) } 2\pi la\tau_0<F\leqq 2\pi lb\tau_0, && V=\frac{1}{\eta_p}\left\{\frac{F}{2\pi l}\ln\frac{F}{2\pi la\tau_0}-\left(\frac{F}{2\pi l}-\tau_0 a\right)\right\} \\ &\text{iii) } 2\pi lb\tau_0<F, && V=\frac{1}{\eta_p}\left\{\frac{F}{2\pi l}\ln\frac{b}{a}-\tau_0(b-a)\right\} \end{aligned}\right\}, \tag{11}$$

which essentially reaches the equation obtained by Oldroyd [1].

Let us proceed to the evaluation of mobility β in case when the result of experiment is given by $V=V(F)$. Since

$$\beta=\beta(\sqrt{D_{\mathrm{II}}},\ \sqrt[3]{D_{\mathrm{III}}})=\beta(\sqrt{D_{\mathrm{II}}},\ 0)=\beta(|p_{rz}|)=\beta\left(\frac{r_*}{r}\tau_0\right), \tag{12}$$

we can readily demonstrate the following

$$\frac{\partial\beta}{\partial r_*}=-\frac{r}{r_*}\cdot\frac{\partial\beta}{\partial r}. \tag{13}$$

From equation (10), therefore, we get the following for the case of $a<r_*\leqq b$,

$$\frac{dV}{dr_*}=\int_a^{r_*} 2\frac{\partial\beta}{\partial r_*}\left(\frac{r_*}{r}-1\right)\tau_0 dr+\int_a^{r_*} 2\beta\frac{1}{r}\tau_0 dr$$

[1] Oldroyd, J. G., op. cit., 396.

$$= -2\tau_0 \int_a^{r_*} \frac{\partial \beta}{\partial r}\left(1-\frac{r}{r_*}\right) dr + 2\tau_0 \int_a^{r_*} \beta \frac{dr}{r}$$

$$= -2\tau_0 \left\{\left[\beta\left(1-\frac{r}{r_*}\right)\right]_a^{r_*} + \int_a^{r_*} \beta \frac{dr}{r_*}\right\} + 2\tau_0 \int_a^{r_*} \beta \frac{dr}{r}$$

$$= 2\tau_0 \beta\left[\frac{r_*}{a}\tau_0\right]\left(1-\frac{a}{r_*}\right) + \frac{1}{r_*} V$$

$$\therefore \beta\left[\frac{r_*}{a}\tau_0\right] = \frac{1}{2\tau_0(r_*-a)}\left(r_* \frac{dV}{dr_*} - V\right).$$

Rewriting it by using equation (7), we get for the case of $\tau_0 < \frac{F}{2\pi la} \leq \frac{b}{a}\tau_0$,

$$\beta\left[\frac{F}{2\pi la}\right] = \frac{1}{2\left(\frac{F}{2\pi l} - \tau_0 a\right)}\left(F\frac{dV}{dF} - V\right). \tag{14}$$

We obtain similarly as follows,

when $\frac{b}{a}\tau_0 < \frac{F}{2\pi la}$,

$$\beta\left[\frac{F}{2\pi la}\right] = \beta\left[\frac{F}{2\pi lb}\right] \frac{\frac{F}{2\pi l} - \tau_0 b}{\frac{F}{2\pi l} - \tau_0 a} + \frac{1}{2\left(\frac{F}{2\pi l} - \tau_0 a\right)}\left(F\frac{dV}{dF} - V\right). \tag{15}$$

It will be assumed, for instance, that $V=V(F)$ has been given by equation (11). Then it will be resulted that

i) in the case of $\tau_0 < \frac{F}{2\pi la} \leq \frac{b}{a}\tau_0$, from equation (14)

$$\beta\left[\frac{F}{2\pi la}\right] = \frac{1}{2\eta_p},$$

ii) in the case of $\frac{b}{a}\tau_0 < \frac{F}{2\pi la} \leq \left(\frac{b}{a}\right)^2 \tau_0$ or $\tau_0 < \frac{F}{2\pi lb} \leq \frac{b}{a}\tau_0$, from equation (15)

$$\beta\left[\frac{F}{2\pi la}\right] = \frac{1}{2\eta_p},$$

iii) similarly for every F, $\beta = \frac{1}{2\eta_p}$.

3. Plate viscometer

The problem to be considered here is about the two-dimensional case where a plate is surrounded by the walls over the length l on its upside and downside with the interval h filled with sample. If $h/l \ll 1$, the velocity will be given by

$$\left.\begin{aligned} v_x &= u(z) \\ v_y &= v_z = 0 \end{aligned}\right\}. \tag{1}$$

The bonndary condition then is

$$\left.\begin{array}{l} u(0)=0 \\ u(h)=V \end{array}\right\}, \tag{2}$$

where x and y axes are taken in the plane of the wall and z axis is taken perpendicular to them, and the plate is assumed to be pulled along x axis with steady velocity V. Among the strain rate components there is only one that does not become zero as follows,

$$d_{zx} = \frac{1}{2}\frac{du}{dz}. \tag{3}$$

Needless to say, incompressible condition $\mathrm{I}_d=0$ has been satisfied there. The constitutive equation becomes

$$\left.\begin{array}{ll} \text{for} \quad \tau_0 \geqq |p_{zx}|, & \dfrac{du}{dz}=0 \\ \text{for} \quad \tau_0 < |p_{zx}|, & \dfrac{du}{dz}=2\left(1-\dfrac{\tau_0}{|p_{zx}|}\right)\beta p_{zx} \end{array}\right\}. \tag{4}$$

On the other hand, from the follwing equilibrium equation

$$\frac{\partial p_{zx}}{\partial z}=0, \tag{5}$$

we obtain

$$p_{zx} = \frac{f}{2l},$$

where it is provided that f is the force required per unit length along y direction. Equation (4) will be therefore,

$$\left.\begin{array}{ll} \text{for} \quad f \leqq 2\tau_0 l, & \dfrac{du}{dz}=0 \\ \text{for} \quad f > 2\tau_0 l, & \dfrac{du}{dz}=2\beta\left(\dfrac{f}{2l}-\tau_0\right) \end{array}\right\}, \tag{6}$$

and considering boundary condition (2), we have

$$\text{for} \quad f \leqq 2\tau_0 l, \quad u(z)=0 \qquad \therefore V=0,$$

$$\text{for} \quad f > 2\tau_0 l, \quad u(z)=2\beta\left[\frac{f}{2l}\right]\left(\frac{f}{2l}-\tau_0\right)z \qquad \therefore V=2h\left(\frac{f}{2l}-\tau_0\right)\beta\left[\frac{f}{2l}\right].$$

Thus using this viscometer, we can induce β easily, the stress being formally independent of the position.

4. Rotating disc viscometer

Let us assume that there is a disc of radius R on a fixed plate at the distance h which is filled with sample. The velocity component will be approximated by

$$\left.\begin{array}{l} v_r=v_z=0 \\ v_\theta=r\omega(z) \end{array}\right\}. \tag{1}$$

The boundary coudition is

$$\left.\begin{array}{l} \omega(0)=0 \\ \omega(h)=\Omega \end{array}\right\}. \tag{2}$$

The only one remaining otherwise than zero among the components of strain rate is

$$d_{\theta z}=\frac{1}{2}r\frac{d\omega}{dz}, \tag{3}$$

and the constitutive equation becomes

$$\left.\begin{array}{ll} \text{for} \quad \tau_0 \geqq |p_{\theta z}|, & \dfrac{d\omega}{dz}=0 \\ \text{for} \quad \tau_0 < |p_{\theta z}|, & \left(1-\dfrac{\tau_0}{|p_{\theta z}|}\right)p_{\theta z}=\dfrac{1}{2}\alpha r\dfrac{d\omega}{dz} \end{array}\right\}. \tag{4}$$

The equilibrium equations are

$$\left.\begin{array}{l} \dfrac{\partial p}{\partial r}=\rho r\omega^2 \\ \dfrac{\partial p_{\theta z}}{\partial z}=0 \end{array}\right\}. \tag{5}$$

It is understood, however, that we do not consider the equilibrium of pressure gradient and centrifugal force since it has nothing to do with the decision of α in this case, we obtain as follows from the second equation,

$$p_{\theta z}=f(r). \tag{6}$$

Then the above will give us together with equations (4) and (2)

$$\frac{d\omega}{dz}=\text{const.}=\frac{\Omega}{h}. \tag{7}$$

Equation (4), therefore, will be written as

$$\left.\begin{array}{ll} \text{for} \quad \tau_0 \geqq p_{\theta z}, & \omega(z)=0 \\ \text{for} \quad \tau_0 < p_{\theta z}, & p_{\theta z}=\tau_0+\dfrac{1}{2}\alpha r\dfrac{\Omega}{h} \end{array}\right\}. \tag{8}$$

Letting driving torque be T, we get

$$T=\int_0^R p_{\theta z}2\pi r^2 dr=\frac{2\pi}{3}R^3\tau_0+\pi\cdot\frac{\Omega}{h}\int_0^R \alpha r^3 dr. \tag{9}$$

Considering Bingham solid ($\alpha=2\eta_p$), we have

$$\text{for} \quad T \leqq \frac{2\pi}{3}R^3\tau_0, \quad \Omega=0 \tag{10}$$

$$\text{for} \quad T > \frac{2\pi}{3}R^3\tau_0, \quad T = \frac{2\pi}{3}R^3\tau_0 + \eta_p \frac{\pi}{2}\frac{\Omega}{h}R^4 \Bigg\}$$

where $D-\tau$ curve and $\Omega-T$ curve assume the similar form. This is resulted from the assumption of $\omega=\omega(z)$, and it corresponds to the case of plate viscometer in which $D-\tau$ diagram and $V-f$ diagram assumed the similar form although non-Bingham solid was analysed there.

Now getting back to equation (9), we will evaluate $\alpha = \alpha\left[r\frac{\Omega}{h}\right]$ from $\Omega-T$ diagram. Since we have

$$\frac{\partial\alpha}{\partial\Omega} = \frac{r}{\Omega}\frac{\partial\alpha}{\partial r}, \tag{11}$$

it will be resulted

$$\begin{aligned}\frac{dT}{d\Omega} &= \frac{\pi}{h}\int_0^R \alpha r^3 dr + \frac{\pi\Omega}{h}\int_0^R \frac{\partial\alpha}{\partial r}\frac{r^4}{\Omega}dr \\ &= \frac{1}{\Omega}\left(T-\frac{2\pi}{3}R^3\tau_0\right) + \frac{\pi}{h}\left\{\left[\alpha r^4\right]_0^R - 4\int_0^R \alpha r^3 dr\right\} \\ &= -\frac{3}{\Omega}\left(T-\frac{2\pi}{3}R^3\tau_0\right) + \frac{\pi}{h}R^4\alpha\left[\frac{R\Omega}{h}\right].\end{aligned}$$

Then we finally have

$$\begin{aligned}\alpha\left[\frac{R\Omega}{h}\right] &= \frac{h}{\pi R^4}\left\{\frac{dT}{d\Omega} + \frac{3}{\Omega}\left(T-\frac{2\pi}{3}R^3\tau_0\right)\right\} \\ &= \frac{h}{\pi R^4\Omega}\left\{\Omega\frac{dT}{d\Omega} + 3\left(T-\frac{2\pi}{3}R^3\tau_0\right)\right\}.\end{aligned} \tag{12}$$

5. Rotating cylinder viscometer

Let us assume that the diameter of a fixed inner cylinder is $2a$ and the diameter of rotating outer cylinder $2b$ and that the interval is filled with sample over the length l. The velocity can be reasonably expressed, excepting the terminal parts, as follows,

$$\left.\begin{aligned} v_r &= v_z = 0 \\ v_\theta &= r\omega(r)\end{aligned}\right\}. \tag{1}$$

The boundary condition will be assumed to be given by

$$\left.\begin{aligned}\omega(a) &= 0 \\ \omega(b) &= \Omega\end{aligned}\right\}. \tag{2}$$

The component of strain rate that does not become zero is only

$$d_{r\theta} = \frac{1}{2}r\frac{d\omega}{dr}. \tag{3}$$

Rheological equation will be

$$\left.\begin{aligned} &\text{for}\quad \tau_0 \geqq |p_{r\theta}|, \quad \frac{d\omega}{dr}=0 \\ &\text{for}\quad \tau_0 < |p_{r\theta}|, \quad \frac{1}{2}r\frac{d\omega}{dr}=\left(1-\frac{\tau_0}{|p_{r\theta}|}\right)\beta p_{r\theta} \end{aligned}\right\}. \tag{4}$$

If the equilibrium of pressure gradient and centrifugal force is also left out of consideration here, the equilibrium equation will become

$$\frac{dp_{r\theta}}{dr}+\frac{2p_{r\theta}}{r}=0\,. \tag{5}$$

We obrain, therefore, the following equation,

$$p_{r\theta}=\frac{\text{const.}}{r^2}$$

Letting driving torque be T, we get

$$T=\int_0^{2\pi} p_{r\theta} l r^2 d\theta\,. \tag{6}$$

Determining the indefinite constant from the above, we obtain

$$p_{r\theta}=\frac{T}{2\pi l}\cdot\frac{1}{r^2}\,. \tag{7}$$

Putting here as

$$r_*^2=\frac{T}{2\pi l\tau_0}, \tag{8}$$

we can re-write equation (4) as follows,

$$\left.\begin{aligned} &\text{for}\quad r_*^2 \leqq r^2, \quad \frac{d\omega}{dr}=0 \\ &\text{for}\quad r^2 < r_*^2, \quad r\frac{d\omega}{dr}=2\beta\tau_0\left(\frac{r_*^2}{r^2}-1\right) \end{aligned}\right\}. \tag{9}$$

There will be obtained by considering the boundary condition (2) further,

$$\left.\begin{aligned} &\text{for}\quad r_*^2 \leqq a^2, \quad \omega(r)=0 \\ &\text{for}\quad a^2 < r_*^2 \leqq b^2, \quad \text{if}\quad r_*^2 \leqq r^2 \leqq b^2, \quad \omega(r)=\Omega \\ &\qquad\qquad\qquad\qquad \text{if}\quad a^2 \leqq r^2 < r_*^2, \quad \omega(r)=\int_a^r 2\beta\tau_0\left(\frac{r_*^2}{r^2}-1\right)\frac{1}{r}\,dr \\ &\text{for}\quad b^2 < r_*^2, \quad \omega(r)=\int_a^r 2\beta\tau_0\left(\frac{r_*^2}{r^2}-1\right)\frac{1}{r}\,dr \end{aligned}\right\}. \tag{10}$$

Considering the position where $r=b$, we get the following result,

$$\left.\begin{aligned} &\text{i)}\ r_*^2 \leqq a^2, \qquad \Omega=0 \\ &\text{ii)}\ a^2 < r_*^2 \leqq b^2, \quad \Omega=\int_a^{r_*} 2\beta\tau_0\left(\frac{r_*^2}{r^2}-1\right)\frac{1}{r}\,dr \\ &\text{iii)}\ b^2 < r_*^2, \qquad \Omega=\int_a^{b} 2\beta\tau_0\left(\frac{r_*^2}{r^2}-1\right)\frac{1}{r}\,dr \end{aligned}\right\}. \tag{11}$$

Especially, putting as $2\beta=\frac{1}{\eta_p}$, we have the equation obtained by Reiner-Rivlin.

$$\left.\begin{array}{lll} \text{i) } T\leqq 2\pi la^2\tau_0, & \Omega=0 \\ \text{ii) } 2\pi la^2\tau_0<T\leqq 2\pi lb^2\tau_0, & \Omega=\frac{T}{4\pi\eta_p la^2}-\frac{\tau_0}{2\eta_p}-\frac{\tau_0}{2\eta_p}ln\frac{T}{2\pi la^2\tau_0} \\ \text{iii) } 2\pi lb^2\tau_0<T, & \Omega=\frac{T}{4\pi\eta_a l}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)-\frac{\tau_0}{\eta_p}ln\frac{b}{a} \end{array}\right\}. \tag{12}$$

Let us assume β next from the relation between Ω and T obtained by experiment. Since we have

$$\beta=\beta(\sqrt{P_{\mathrm{I}}},\,0)\equiv\beta(|p_{r\theta}|)=\beta\left[\frac{r_*^2}{r^2}\tau_0\right],$$

it is easy to demonstrate

$$\frac{\partial\beta}{\partial r_*}=-\frac{r}{r_*}\frac{\partial\beta}{\partial r}. \tag{13}$$

Accordingly, after repeating the process almost similar to the case of a rod viscometer, we obtain

in the case $a<r_*\leqq b$,

$$\beta\left[\frac{r_*^2}{a^2}\tau_0\right]=\frac{1}{2\tau_0\left(\frac{r_*^2}{a^2}-1\right)}r_*\frac{d\Omega}{dr_*}.$$

Rewriting the above by using equation (8) and

$$\frac{r_*}{2}\frac{d\Omega}{dr_*}=T\frac{d\Omega}{dT},$$

we get

for $\tau_0<\frac{T}{2\pi la^2}\leqq\frac{b^2}{a^2}\tau_0$,

$$\beta\left[\frac{T}{2\pi la^2}\right]=\frac{1}{\frac{T}{2\pi la^2}-\tau_0}T\frac{d\Omega}{dT}. \tag{14}$$

Similarly

for $\frac{b^2}{a^2}\tau_0<\frac{T}{2\pi la^2}$,

$$\beta\left[\frac{T}{2\pi la^2}\right]=\beta\left[\frac{T}{2\pi lb^2}\right]\frac{\frac{T}{2\pi lb^2}-\tau_0}{\frac{T}{2\pi la^2}-\tau_0}+\frac{1}{\frac{T}{2\pi la^2}-\tau_0}T\frac{d\Omega}{dT}. \tag{15}$$

Assuming that the relation between Ω and T has been for instance given by equation (12) and using equations (14) and (15), we can of course get $\beta=\frac{1}{2\eta_p}$.

6. Capillary viscometer

It is assumed that a sample passes through a tube $2R$ in diameter and l in length. In case of $l \gg R$, the velocity will be given by

$$\left.\begin{aligned} v_r = v_\theta = 0 \\ v_z = w(r) \end{aligned}\right\}, \tag{1}$$

except the parts adjacent to the inlet and the outlet of the tube. The boundary condition then is

$$w(R) = 0 . \tag{2}$$

The only one that does not become zero among strain rate is

$$d_{rz} = \frac{1}{2}\frac{dw}{dz} . \tag{3}$$

The non-vanishing deviatoric stress component is, therefore, only p_{rz}, and its relation with the strain rate will be given by

$$\left.\begin{aligned} &\text{for} \quad \tau_0 \geqq |p_{rz}|, \quad \frac{dw}{dr} = 0 \\ &\text{for} \quad \tau_0 < |p_{rz}|, \quad \frac{1}{2}\frac{dw}{dr} = \beta\left(1 - \frac{\tau_0}{|p_{rz}|}\right)p_{rz} \end{aligned}\right\} . \tag{4}$$

The equilibrium equations will be

$$\left.\begin{aligned} \frac{\partial p}{\partial r} = 0 \\ -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rp_{rz}) = 0 \end{aligned}\right\}, \tag{5}$$

and it is seen from the first equation that p is a function only of z,

$$p = p(z) . \tag{6}$$

On the other hand, since p_{rz} is a function only of r from equation (4), it leads to a conclusion, when viewed jointly with the second equation of equation (5), that it is

$$\frac{\partial p}{\partial z} = -k \text{ (constant)}, \; k > 0 . \tag{7}$$

Here, the negative sign is attached, because z axis is taken in the direction to which pressure lowers,

$$k = \frac{p(0) - p(l)}{l} . \tag{8}$$

Again from the second equation of equation (5), therefore, we can obtain

$$p_{rz} = -\frac{kr}{2} .$$

Putting here as

$$r_* = \frac{2\tau_0}{k}, \tag{9}$$

equation (4) will be

$$\left.\begin{aligned} &\text{for } r \leqq r_*, \quad \frac{dw}{dr} = 0 \\ &\text{for } r_* < r, \quad -\frac{dw}{dr} = 2\beta\tau_0\left(\frac{r}{r_*} - 1\right) \end{aligned}\right\}. \tag{10}$$

Letting the flow rate be Q, we obtain as follows from the boundary condition and equation (10),

$$\begin{aligned} &\text{for } R \leqq r_*, \quad Q = 0, \\ &\text{for } r_* < R, \quad Q = \int_0^R w(r) 2\pi r dr \\ &\qquad = \Big[w(r)\pi r^2 \Big]_0^R - \int_0^R \frac{dw}{dr} \pi r^2 dr \\ &\qquad = \int_{r_*}^R 2\beta\tau_0\left(\frac{r}{r_*} - 1\right)\pi r^2 dr \\ &\qquad = 2\pi\tau_0 \int_{r_*}^R \beta\left(\frac{r}{r_*} - 1\right) r^2 dr . \end{aligned} \tag{11}$$

Especially, we obtain Buckingham-Reiner equation by considering Bingham solid [6],

$$\begin{aligned} &\text{for } R \leqq \frac{2\tau_0}{k}, \quad Q = 0, \\ &\text{for } R > \frac{2\tau_0}{k}, \quad Q = \frac{\pi\tau_0}{\eta_p}\left(\frac{R^4}{8\tau_0}k - \frac{R^3}{3} + \frac{2}{3} \cdot \frac{\tau_0^3}{k^3}\right). \end{aligned} \tag{12}$$

Incidentally, we will estimate β from the relation between Q and k that is evaluated by experiment. The process is the same as that followed in the case of rod and rotating cylinder viscometer. Availing the fact that

$$\beta = \beta(\sqrt{P_{\mathrm{I}}}, 0) \equiv \beta(|p_{rz}|) = \beta\left[\frac{r}{r_*}\tau_0\right], \tag{13}$$

we can demonstrate

$$\frac{\partial\beta}{\partial r_*} = -\frac{r}{r_*}\frac{\partial\beta}{\partial r}. \tag{14}$$

Using partial integration further, we can obtain for $r_* < R$,

$$\beta\left[\frac{R}{r_*}\tau_0\right] = \frac{1}{2\pi\tau_0 R^3\left(\frac{R}{r_*} - 1\right)}\left(-r_*\frac{\partial Q}{\partial r_*} + 3Q\right). \tag{15}$$

Again rewriting equation (15) by equation (9) and

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[6] Fredrickson, A. G., op. cit., 184.

$$-r_* \frac{\partial Q}{\partial r_*} = k \frac{\partial Q}{\partial k}, \tag{16}$$

we can show in case of $R > \frac{2\tau_0}{k}$,

$$\beta\left[\frac{k}{2} R\right] = \frac{1}{2\pi R^3\left(\frac{kR}{2} - \tau_0\right)} \left(k \frac{\partial Q}{\partial k} + 3Q\right). \tag{17}$$

If the relation between Q and k is assumed to be given by equation (12), it is quite easy to confirm that we can obtain $\beta = \frac{1}{2\eta_p}$.

Incidentally, replacing in equation (17) as

$$\left.\begin{array}{l} 2\beta\left[\frac{kR}{2}\right]\left(\frac{kR}{2} - \tau_0\right) \longrightarrow D \\ \frac{4Q}{\pi R^3} \longrightarrow H \\ \frac{kR}{2} \longrightarrow S_m \end{array}\right\}. \tag{18}$$

we can get a equation obtained by Rabinowitsch and Mooney for one-dimensional non-Newtonian flow, which is quite natural considering the procedure of the calculation carried out [6],

$$D = \frac{1}{4}\left(S_m \frac{dH}{dS_m} + 3H\right). \tag{19}$$

7. Supplement

The author has so far carried out calculations by using various viscometers. It is needless to say that the main objective is in the method of describing pseudoplastic body.

Though the characteristic values to be dealt here are generally the functions of external atomosphere, such as temperature and pressure, those external conditions have been regarded as constant for simplicity. Not only that, to render treatment easier the author has adopted various assumptions, i. e., that stress is the function only of strain rate, that yield condition is made to be given by that of Mises, and that calculation is based on incompressibility, all of which are valid only when they have been testified by experiments. In comparing these with the practical problems, we have to give full allowance to them.

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[6] Fredrickson, A. G, op. cit., 20.