ON THE TABLES OF u^<27/17> AND u^<-27/7>

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ON THE TABLES OF $u^{27/7}$ AND $u^{-23/7}$ [1)]

By Jun-ichi OKABE

Summary

In spite of our inadequate knowledge of the nature of shear flow turbulence, there often arises requirement of calculating by any means the turbulent boundary layer of a body immersed in a uniform stream. According to our experiences, the method due to MILLIKAN (1932) has been found very convenient because of its comparatively less labor for the computations; cf. **section 1.**

That is to say, for REYNOLDS numbers less than 3×10^6, the thickness of the turbulent boundary layer in a steady condition is given by the formula

$$\delta = 0.3799 R_L{}^{-1/5} u_1{}^{-23/7}\left(\int_0^x u_1{}^{27/7}\,\mathrm{d}x\right)^{4/5}$$

for a two-dimensional obstacle, and also by

$$\delta = 0.3799 R_L{}^{-1/5} u_1{}^{-23/7} r^{-1}\left(\int_0^x u_1{}^{27/7} r^{5/4}\mathrm{d}x\right)^{4/5}$$

for an axisymmetric body having a blunt nose, where

$\delta(x) \equiv \Delta/L$ (Δ: boundary layer thickness, L: characteristic length of the body),

$u_1(x) \equiv U_1/U$ (U_1: velocity of flow just outside the boundary layer, U: characteristic velocity of flow),

$x = X/L$ (X: distance along the surface of the body from the forward stagnation in two-dimensional stream, or along the meridian when flow is axially symmetric),

$r(x) \equiv R/L$ (R: distance of the surface from the axis of revolution),

and $R_L \equiv UL/\nu$ (REYNOLDS number).

Finally, by means of the criterion due to HOWARTH (1935), the separation point of the boundary layer can be determined by the equation

$$\Gamma = -0.06\,,$$

where Γ is a non-dimensional parameter and is defined by

$$\Gamma \equiv R_L{}^{1/4}\vartheta^{5/4}u_1{}^{-3/4}\mathrm{d}u_1/\mathrm{d}x\,,$$

ϑ being the non-dimensional momentum thickness, or we may put

$$\Gamma \cong R_L{}^{1/4}\left(\frac{7}{72}\delta\right)^{5/4} u_1{}^{-3/4}\frac{\mathrm{d}u_1}{\mathrm{d}x}\,.$$

1) Condensed English version of the four papers originally written in Japanese and published by Research Committee for Hydrology, of which the present writer is a member, in *Bulletin of Research Institute for Applied Mechanics*, Kyushu University, under the title '$u^{27/7}$, $u^{-23/7}$ no Suhyo (Tables of $u^{27/7}$ and $u^{-23/7}$)'. These four papers will be referred to later in more detail.

Several years ago, Research Committee for Hydrology in Research Institute for Applied Mechanics, Kyushu University, published the tables which contained the numerical values of $u^{27/7}$ and $u^{-23/7}$ together with their first differences, for the values of u ranging from 0.500 to 1.500 at intervals of 0.001, with a view to making it possible to carry out with slight labor the numerical integrations involved in the above expressions (in what follows these tables will be referred to as 'the Tables' for short); cf. **section 2.**

In the following two sections, accuracy of the various procedures of calculating turbulent boundary layers is tested for a linearly retarding (**section 3**), and a linearly accelerating (**section 4**), flows, $u_1 = 1-x$ and $(x+1)/2$, respectively. By comparing with the solutions (the momentum thickness, the local skin friction, and Γ) due to HOWARTH (1935) in the case of linear retardation, it is noticed that MILLIKAN's method gives the result quite comparable except only for the local skin friction in the immediate neighborhood of separation point. For the flow of linear acceleration, on the other hand, his formula is found to yield the values of the momentum thickness, the local skin friction, and Γ intermediate between those obtained through HOWARTH's, and SQUIRE-YOUNG's (1937), methods. Namely in all the computations of these three quantities, the curves constructed making use of MILLIKAN's approximation start from the curves of HOWARTH near the origin ($x=0$) and approach to those of SQUIRE and YOUNG as x increases. It is expected, therefore, from both of these two examinations, that also in general cases the method of MILLIKAN in which necessary computations can be performed very easily with the aid of the Tables would have almost the same degree of approximation as the remaining two, in each of which we should solve a non-linear differential equation of the first order numerically.

In **section 5**, assuming five kinds of the two-dimensional velocity distributions

$$u_1 = (1-x)^n ,$$

where $$n = 2,\ 3/2,\ 1,\ 1/2,\ \text{and}\ 1/3 ,$$

along a smooth surface, the thickness of the boundary layers is worked out by numerical integrations (SIMPSON's rule) in order to check off the Tables in part. It is verified that the approximate result thus produced agrees almost completely with the values obtained exactly from the formula, provided Δx, the step of the non-dimensional coordinate along the plate, is taken sufficiently small (0.005). In three cases when the velocity decreases and increases linearly along the plate ($u_1 = (1-x)$, $(x+1)/2$, and $(x+2)/2$), computations are repeated for a number of choices of Δx, or what amounts to the same thing, of Δu_1, the step of the non-dimensional velocity: $|\Delta u_1| = 0.005$, 0.01, 0.02, 0.04, 0.08, 0.12, and 0.24. It is proved that the accuracy of the numerical integrations may be said nearly equivalent to each other in so far as $|\Delta u_1| < 0.04$ and is maintained practically satisfactorily until $|\Delta u_1|$ reaches the value 0.24. For velocity distributions other than a linear function of x, approximating them by means of a number of straight lines and applying the above estimation to each of these segments, it is expected that we should arrive at some choice of $|\Delta u_1|$ or Δx not so far from being adequate.

In **section 6**, employing the parameter Γ, the positions x_s of the separation points are determined for each of the velocity distributions $(1-x)^n$

from the condition $\Gamma(x_s) = -0.06$. It is noticed that the values of $u_1(x_s)$ show sensible uniformity among these five separation points, where $u_1(x_s)$ denotes the velocity of the main flow just outside the boundary layer at the point of separation. From the similarity in boundary layer separations of two-dimensional flows, in other words, the assumption that the separation point would be independent of REYNOLDS number, it is concluded that in a group of steady flow patterns defined by the equation of the form $x = \alpha\varphi(u_1)$, where φ and α denote an arbitrary function and a parameter variable within that group, respectively, separation from the wall occurs at the point $u_1 =$ constant. If this functional condition is found to be satisfied at least approximately between x and u_1, then by knowing, theoretically or experimentally, one of the points of separation among a series of systematic modification of the flow, there exists a possibility of predicting roughly the remaining points of separation without labor of calculating the boundary layers for these flows.

Finally in **section 7**, disturbances in boundary layer thickness and in surface traction originated from sinusoidal, steady variations superposed upon uniform velocity distribution is studied. After the numerical computations with the aid of the Tables, the followings are mentioned which, in spite of restricted variety of the investigated examples, would form a part of more or less general conclusions:

1) Disturbance generated in δ, the thickness of the boundary layer, develops with the same wave length as the undulation contained in the velocity.

2) As the amplitude grows up of the superposed sinusoidal term of the velocity, degree of variation produced on the part of the boundary layer thickness increases much more markedly.

3) Denoting the boundary layer thickness corresponding to the basic uniform flow by δ_0, change of δ is more pronounced on the side $\delta > \delta_0$ than on $\delta < \delta_0$. In other words, the disturbed boundary layer appears thicker, in the mean, than the original condition.

4) Variation of δ propagates along the plate amplifying in the manner roughly proportional to $\delta_0(x)$.

5) Of course, the phase and the wave length of the undulation produced in δ become different as the corresponding ones of the velocity distributions are varied. But still the amplitude of δ as a function of x is subjected to no essential modifications.

6) Circumstances are more complicated for the local skin friction. However, disturbance is found larger on the positive side (increasing the non-dimensional local skin friction over the value of the uniform stream) than on the negative side, and if a large number of wave lengths are contained in the whole length of the plate, then the total skin friction will increase most likely compared with the value in the uniform flow.

In the **appendix**, the same problem as section 7 is discussed for the case of laminar boundary layers. By virtue of TANI's approximate formula (1941), conclusions are derived which are quite similar to the case of turbulent boundary layers.

1. Introduction

Recently, in many branches of hydraulic engineering, there arise increasing requirements for calculating the turbulent boundary layer of a body immersed in a uniform stream. A number of computation procedures were published for boundary layers (in this note, unless otherwise explicitly stated, as the *boundary layer* we mean the one, of a liquid or a gas of constant density, formed on a two-dimensional smooth surface, completely turbulent from the beginning, and at the same time quasi-steady, namely steady on the time-average). Because of our inadequate knowledge of the nature of shear flow turbulence, however, so far as the present writer is aware, no method which may be accepted in all cases as sufficiently accurate has not been found out.

In the next place, labor involved in computations should be taken into consideration. Usually numerical works complicated and tedious, more or less, are necessary in the course of calculations of boundary layers. However, now that the assumptions underlying the theories are not completely verified, we cannot help doubting whether it would be worth-while without fail to carry out for all the problems with which we are dealing numerical computations in great detail.

Besides even if all our doubts should be dissolved and the method of exact treatment should be established some day, since there can be practical demands in some cases for evaluating at the minimum expenses of labor and time properties of boundary layers roughly or selectively, we might expect with certainty that a rapid estimation, although accompanied by some degree of error inevitably, would not lose all its significance in the future.

2. MILLIKAN's method

With a view to reducing computational labor in MILLIKAN's formula,[1] which is, according to our past experiences, one of the most simplified and at the same time the most convenient methods for calculating boundary layers, Research Committee for Hydrology in Research Institute for Applied Mechanics, Kyushu University, published several years ago the tables including $u^{27/7}$ and $u^{-23/7}$ together with their first differences for the values of u ranging from 0.500 to 1.500 at intervals of 0.001.[2]

Namely, by applying MILLIKAN's method to two-dimensional boundary layers, which was invented for axisymmetric flows originally, we can derive the formula

$$\delta(x) = 0.3799 R_L^{-1/5} u_1^{-23/7} \left(\int_0^x u_1^{27/7} dx \right)^{4/5}, \tag{2.1}$$

where δ and u_1 are the boundary layer thickness and the velocity of the main stream just outside the boundary layer, both reduced non-dimensional by dividing

1) MILLIKAN, C. B., The Boundary Layer and Skin Friction for a Figure of Revolution, *Transactions of the American Society of Mechanical Engineers, Applied Mechanics Section*, vol. 54, no. 2 (1932), pp. 29-39.

2) RESEARCH COMMITTEE FOR HYDROLOGY, Tables of $u^{27/7}$ and $u^{-23/7}$ ($u = 0.500 \sim 1.500$), (in Japanese), *Bulletin of Research Institute for Applied Mechanics* (abbreviated as *Bulletin* hereafter), no. 5 (1954), pp. 1-16. In what follows, these tables will be referred to as 'the Tables' for short.

them with the characteristic length L (for instance, the overall length of the submerged body) and the characteristic velocity U[1] (for instance, the general flow) respectively, X, or x in non-dimensional form constructed in the same way, being the distance along the wall of the obstacle measured from the forward stagnation, where the boundary layer is assumed to begin with vanishing thickness. Further, R_L is the REYNOLDS number defined by UL/ν, ν being the kinematic viscosity of the fluid.[2] Furthermore, when δ has been found, τ_0, the intensity of skin friction, may be evaluated in the form

$$\tau_0/\rho U^2 = \kappa R_L^{-1/4}\delta^{-1/4}u_1^{7/4}, \tag{2.2}$$

where ρ is the density of the fluid which is assumed to be constant throughout this note while κ denotes a coefficient roughly equal to 0.0232.[3] Equations (2.1) and (2.2) can be both obtained by assuming that the same relationships as in the flat-plate-boundary-layer (the so-called 1/7th power law of velocity profile[4]) would be valid approximately also at each point on the surface of a two-dimensional obstacle, in other words, that the whole boundary layer on the latter might be reproduced substantially by the boundary layer (or, to be more precise, the train of boundary layers) on an infinite number of flat plates of infinitesimal length placed in close formation, the velocity just outside each of the boundary layers (or the undisturbed flow, so to speak, corresponding to each infinitesimal plate) being imagined to vary according to the velocity distribution of the main flow around the real obstacle. In this picture the deformation of velocity profile in the boundary layer taking place as a matter of fact as we go downstream along the surface is completely ignored, and it is dangerous therefore to apply the method to the flow accompanied by a large pressure gradient parallel to the stream.

Concerning the separation of boundary layers, BURI and HOWARTH remarked

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1) In the original papers, the symbol U_0 was used in place of U.

2) For an axisymmetric body with a blunt nose, the result corresponding to (2.1) may be obtained without difficulty and it reads

$$\delta(x) = 0.3799R_L^{-1/5}u_1^{-23/7}r^{-1}\left(\int_0^x u_1^{27/7}r^{5/4}\mathrm{d}x\right)^{4/5},$$

where $r(x)$ is the non-dimensional distance of the surface from the axis of revolution, cf. FEDIAEVSKY, C. C., The Boundary Layer and the Drag of a Body of Revolution at Large Reynolds Number, *Journal of the Aeronautical Sciences*, vol. 3, no. 1 (1935), pp. 17–20. It should be noted that the error of order (δ/r) has been neglected in deriving this formula. Concerning the effect of transverse curvature, see the following: LANDWEBER, L., Effect of Transverse Curvature on Frictional Resistance, *Report* 689 (1949), The David W. Taylor Model Basin.

3) For flow in a smooth pipe the coefficient 0.0225 is deduced from BLASIUS' empirical formula. In the present case, however, the value of κ is modified in such a way that the law of frictional resistance $C_f = 0.074R_L^{-1/5}$ is derived instead of $0.072R_L^{-1/5}$; see GOLDSTEIN, S., *Modern Developments in Fluid Dynamics*, vol. II (1938), **163**, pp. 361–362.

4) It is verified experimentally for moderate REYNOLDS numbers less than 3×10^6, cf. GOLDSTEIN, S., *op. cit.*, **163**, p. 362. According to SCHLICHTING, H., *Boundary Layer Theory* (translated by KESTIN, J.) (1960), XXI, **a**, p. 537, equation (21.11), the range of REYNOLDS numbers in which the 1/7th power law holds is

$$5\times10^5 < R_L < 10^7.$$

that it occurs at the point x_s where the non-dimensional parameter

$$\Gamma = R_L^{1/4}\vartheta^{5/4}u_1^{-3/4}\mathrm{d}u_1/\mathrm{d}x \tag{2.3}$$

takes the value -0.06,[1] where ϑ denotes the momentum thickness, reduced non-dimensional as before by means of L. Now if we substitute for ϑ in this expression the value obtained by connecting equation (2.1) with

$$\vartheta = 7\delta/72\,, \qquad (2.4)^{2)}$$

the relation verified approximately for the flow along a flat plate (the 1/7th power law), then we are readily enabled to compute the value of Γ for various values of x and in particular to determine the point x_s where the critical equation

$$\Gamma(x_s) = -0.06 \tag{2.5}$$

is satisfied. It should be noticed, of course, that there exists a fundamental contradiction in this procedure. That is to say, the value of ϑ and accordingly that of Γ worked out in this way are both based upon the uniform flow, whereas, on the other hand, the separation is caused directly from the deformation of velocity profile owing to adverse pressure gradient prevailing along the surface of the body.[3]

1) GOLDSTEIN, S., *op. cit.*, **166**, FIG. 118, p. 375, and **194**, pp. 436-439.

2) GOLDSTEIN, S., *op. cit.*, **163**, equation (63), p. 361.

3) Then how much error should be expected in the approximate value of ϑ obtained in this manner? In order to answer this question, the present writer refers only briefly to his paper 'On the Frictional Resistance of Ships', *Reports of Research Institute for Fluid Engineering*, vol. 4, no. 2 (1947), pp. 1-20, where the behaviors of two-dimensional boundary layers formed on the surface of lenticular shapes of an infinite draft and with the ratio (length L/breadth B) varying among 4, 5, 6, 7, and 8 are calculated in detail. The body is supposed to move in the direction of its center line with a uniform speed U in a fluid extending to an infinite distance in all directions, the REYNOLDS number defined by UL/ν being assumed 4.57×10^6. A result of the calculation is shown in the following table.

L/B \ ϑ \ x	4		5		6		7		8	
	ϑ_H	ϑ_M	ϑ_H	ϑ_M	ϑ_H	ϑ_M	ϑ_H	ϑ_M	ϑ_H	ϑ_M
0.60	0.000 816	0.000 814	0.000 868	0.000 865	0 000 903	0.000 901	0.000 929	0.000 929	0.000 952	0.000 950
0.70	0.001 098	0.001 088	0.001 135	0.001 125	0.001 157	0.001 149	0.001 173	0.001 168	0.001 187	0.001 179
0.80	0.001 522	0.001 488	0.001 519	0.001 491	0.001 499	0.001 484	0.001 500	0.001 480	0.001 493	0.001 474

x is the non-dimensional distance measured from the forward stagnation along the surface of the obstacle; ϑ_H and ϑ_M denote both the non-dimensional momentum thicknesses. ϑ_M is obtained by the method explained in the text, while ϑ_H stands for the results from HOWARTH's method, less crude but at the same time much more laborious than the former, cf. GOLDSTEIN, S., *op. cit.*, **194**, pp. 436-439, or HOWARTH, L., The Theoretical Determination of the Lift Coefficient for a Thin Elliptic Cylinder, *Proceedings of the Royal Society*, A., vol. 149 (1935), pp. 558-586, esp. pp. 574-579. We may notice clearly that although there exists slight difference between these two kinds of ϑ, validity of our approximation has been verified so far as this example is concerned, and that, stating more specifically, ϑ_H is always greater than ϑ_M. It may be concluded, therefore,

L/B	$x_{s,H}$	$x_{s,M}$
4	0.910	0.913
5	0.918	0.921
6	0 928	0.932
7	0.939	0.942
8	0.949	0.951

(*continued on the following page*)

But so far as our own experiences are concerned, Γ decreases with negative sign very rapidly without exception as the separation point is approached. Accordingly despite of the fact that the accuracy of Γ itself computed in this way is quite low, it might be expected that the error involved in x_s could not be so large, and that for the same reason ambiguity accompanying the critical value on the right-hand side of equation (2.5) would not constitute essential obstacle in determining separation points practically.[1)]

3. Comparison with another method (retarded flow)

We are thus enabled to evaluate δ, the thickness of the boundary layer, $\tau_0/\rho U^2$, the intensity of skin friction, and Γ, the parameter characterizing separation, through the formulas (2.1), (2.2), and (2.3), respectively. Then, how high accuracy does this solution possess? To answer this question let us investigate in more detail in sections 3 and 4, boundary layers generated from main flows retarded and accelerated, respectively.[2)] Only, since the velocity profile (the so-called 1/7th power law) underlying MILLIKAN's method is already quite flat, acceleration will not probably yield any sensible effect upon the profile. On the contrary, however, we should expect much larger error owing to retardation of the flow, because there is great room left for the defect of the profile. So let us first consider a retarded flow.[3)]

As a material of comparison, we shall take up the following example. The leading edge of a plate with length L is encountered by a stream of velocity U flowing parallel to the plate. As we go down along the plate, the flow is retarded uniformly until on the trailing edge the velocity in the main stream attains the value zero. By taking the coordinate X along the surface from the leading edge, writing $U_1(X)$ for the flow just outside the boundary layer, and then deriving from them the non-dimensional quantities defined respectively by

$$x \equiv X/L \quad \text{and} \quad u_1 \equiv U_1/U\,, \tag{3.1}$$

the following expression is found between u_1 and x:

$$u_1 = 1 - x\,. \tag{3.2}$$

Substituting this equation for u_1 in (2.1), we may readily perform the integration and are led to the result

$$\delta(x) = 0.3799 R_L^{-1/5}(1-x)^{-23/7}(7/34)^{4/5}\{1-(1-x)^{34/7}\}^{4/5}\,, \tag{3.3}$$

that if we make use of ϑ_H, the separation would be found at a point a little upstream compared with the case when ϑ_M is substituted for ϑ. In the above table, $x_{s,H}$ and $x_{s,M}$ denote the separation points calculated with ϑ_H and ϑ_M, respectively.

1) According to PRANDTL, separation occurs for the values of Γ ranging from -0.05 to -0.09, cf. GOLDSTEIN, S., *op. cit.*, **194**, p. 438.

2) For a uniform flow, by putting $u_1 = 1$ we can verify easily that (2.1) is reduced to the well-known formula for the flow along a flat plate, cf. GOLDSTEIN, S., *op. cit.*, **163**, equation (65), p. 362.

3) RESEARCH COMMITTEE FOR HYDROLOGY, Tables of $u^{27/7}$ and $u^{-23/7}$ (supplementary note), *Bulletin*, no. 7 (1955), pp. 39–42.

assuming that the boundary layer starts with vanishing thickness, namely

$$\delta(0) = 0 . \tag{3.4}$$

For various values of x we are now in a position to compute the values of δ from (3.3), ϑ through (2.4), τ_0 and Γ by virtue of (2.2) and (2.3), respectively.

On the other hand, according to HOWARTH,[1] another variable defined by the relation

$$\chi \equiv R_L^{1/4}\vartheta^{5/4} \tag{3.5}$$

is found to satisfy the differential equation

$$\frac{d\chi}{dx} + \frac{5}{4}\frac{\chi}{u_1}\frac{du_1}{dx}(2+H) = \frac{5}{4}\frac{\zeta}{u_1^{1/4}}, \tag{3.6}$$

where H and ζ are the known functions of Γ;[2] the relation between Γ and χ is readily found to be

$$\Gamma = \chi u_1^{-3/4} du_1/dx . \tag{3.7}$$

For the case when

$$R_L = 10^5 , \tag{3.8}$$

starting with the initial condition

$$\chi(0) = 0 , \tag{3.9}$$

successive values of χ may be obtained by solving equation (3.6) numerically. In terms of χ, values of ϑ, Γ, and τ_0 are evaluated through (3.5), (3.7), and by the equality

$$\frac{\tau_0}{\rho U^2} = \zeta u_1^{7/4}\frac{\vartheta}{\chi} , \tag{3.10}$$

respectively. In FIGURES 1, 2, and 3, corresponding to the velocity distribution (3.2) and the REYNOLDS number (3.8), ϑ, $\tau_0/\rho U^2$ together with Γ worked out through the procedures due to MILLIKAN and HOWARTH are shown for comparison.[3] Provided we could assume that the conclusions drawn from this example would be valid generally (in other words, for general velocity distributions and within a certain range of REYNOLDS numbers including 10^5), we might mention as follows—

1) In a retarded region, MILLIKAN's formula based upon the so-called 1/7th power law of the velocity profile is found to yield results not so different from those by HOWARTH's less crude but more laborious method in the whole range of the flow, except for the values of $\tau_0/\rho U^2$ near the separation. In practical computations, therefore, of estimating τ_0 or ϑ, the latter in which we should solve the non-linear differential equation of the first order numerically may be substituted

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1) HOWARTH, L., *op. cit.*, cf. footnote 3) on p. 42.

2) GOLDSTEIN, S., *op. cit.*, FIG.'s 117 and 118, p. 375; FIG.'s 10 and 11 in HOWARTH's original paper.

3) In solving equation (3.6), the method due to RUNGE and KUTTA was employed.

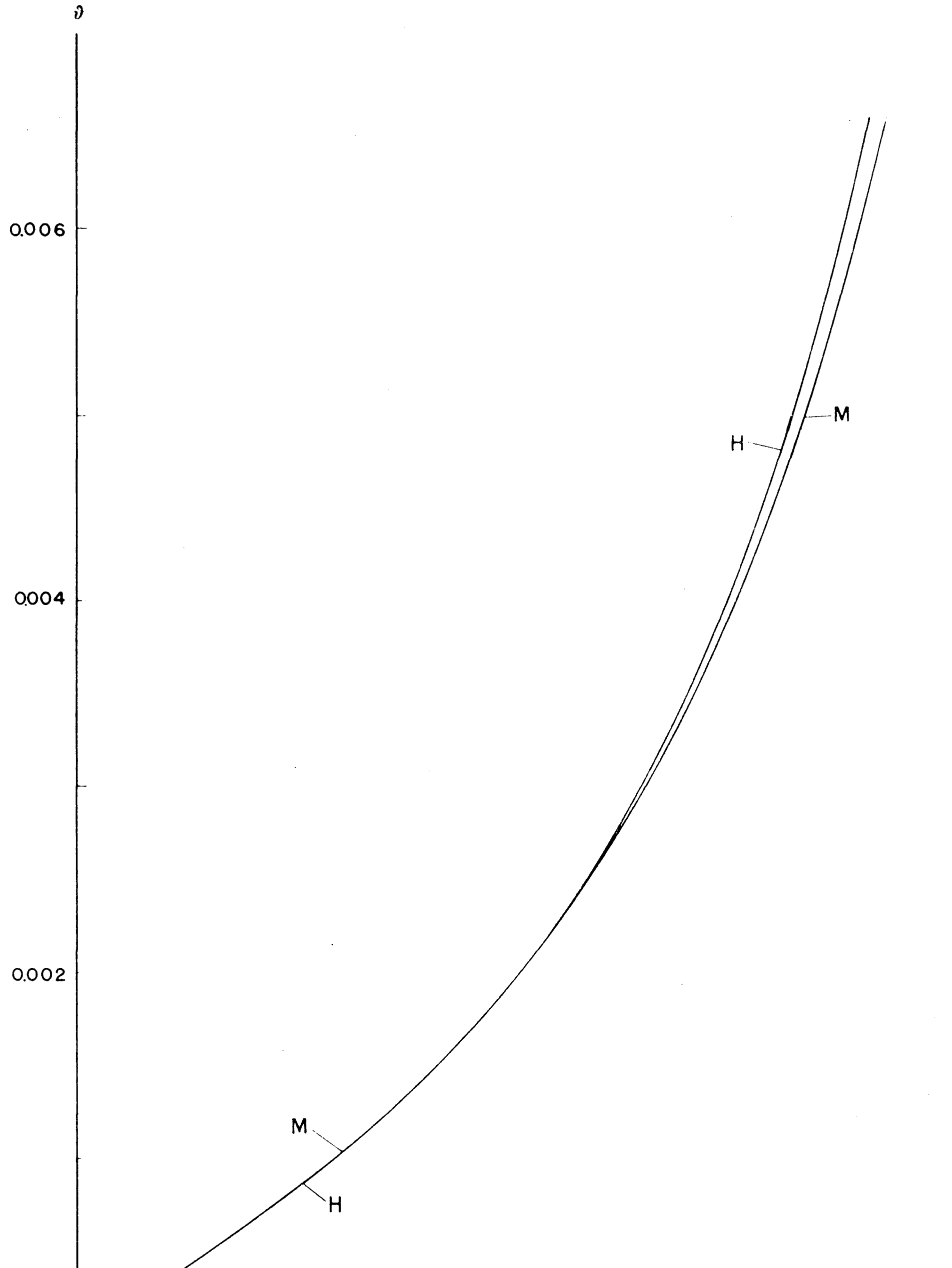

FIGURE 1. Momentum thickness ϑ.
M : MILLIKAN's method, H : HOWARTH's method.

by the former which can be carried out with slight labor by the aid of our Tables.

2) On the other hand in the immediate neighborhood of the separation point, τ_0 from MILLIKAN's formula shows large discrepancy, as naturally expected from the underlying assumptions, from the value obtained by HOWARTH's method.

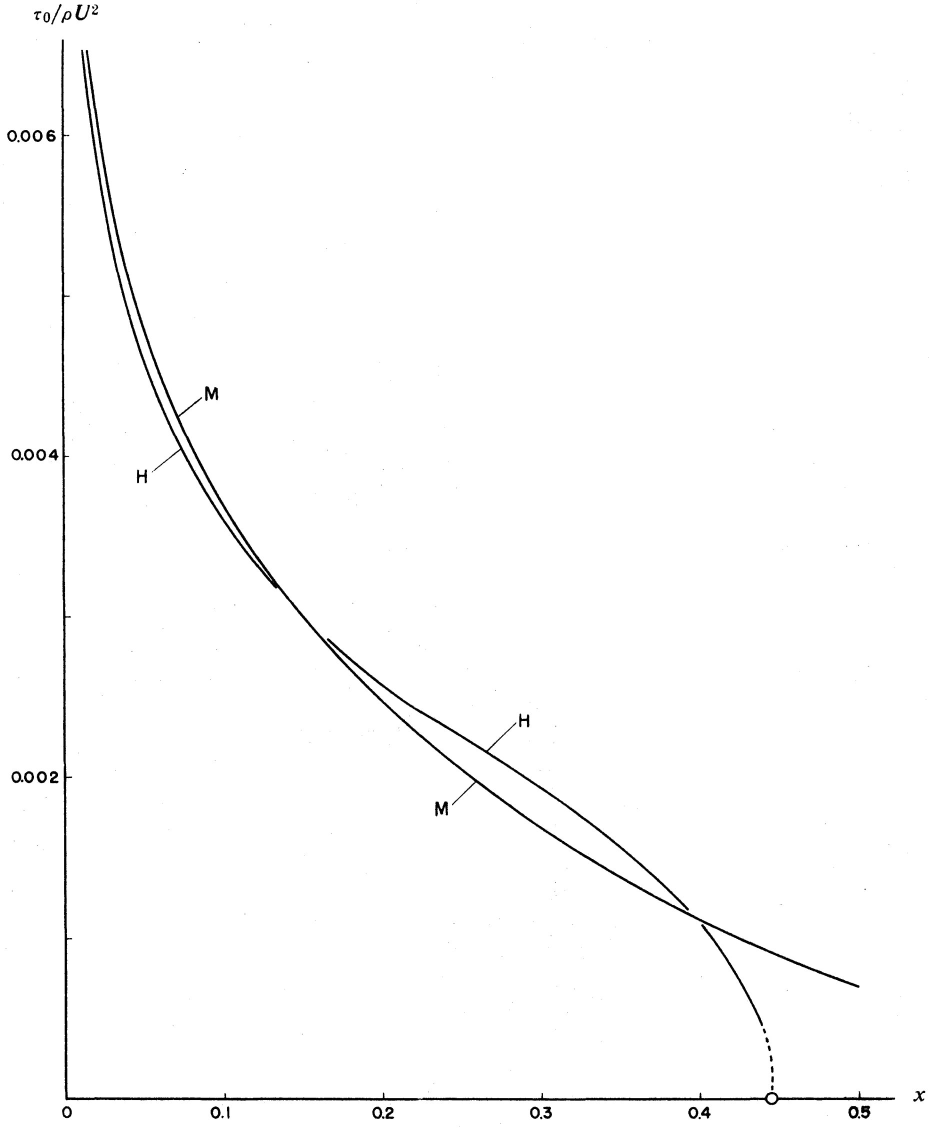

FIGURE 2. Intensity of skin friction $\tau_0/\rho U^2$.
M : MILLIKAN's method, H : HOWARTH's method.
The white circle shows the position of separation.

3) If we should assume, after HOWARTH, that the separation from the wall occurs at the point x where the critical equation (2.5) is satisfied, then there could be found no appreciable difference between the values of x_s calculated by

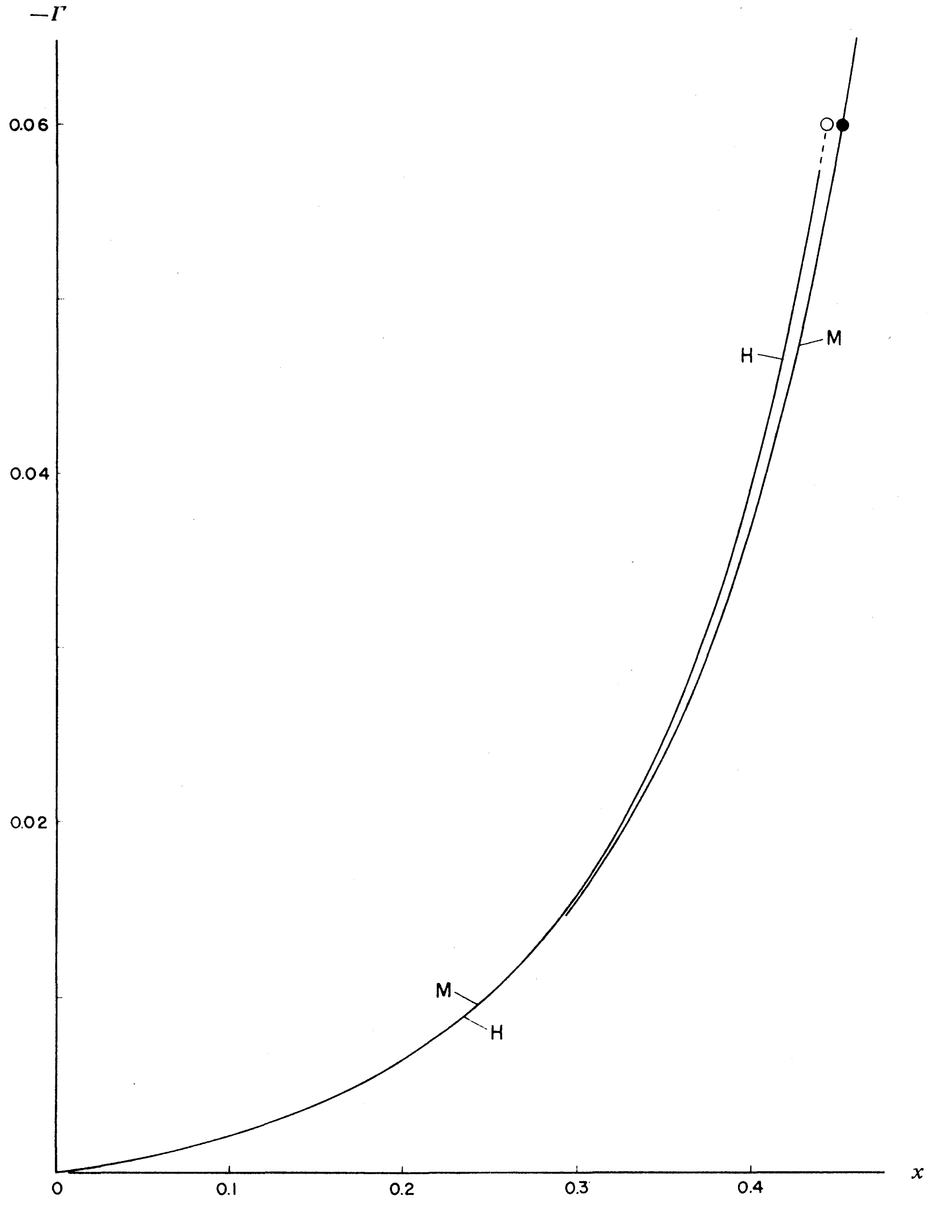

FIGURE 3. Parameter Γ.

M: MILLIKAN's method, H: HOWARTH's method.

The circles indicate the positions of separation predicted by the equaton $\Gamma(x_s) = -0.06$.

means of these two theories.

To sum up, from 1), 2), and 3), we find that the following notices become necessary in calculating boundary layers of retarded flows:

Through MILLIKAN's formula together with the Tables, ϑ, τ_0, and at the same time Γ may be calculated for various values of x. Defining the separation by the point where Γ is equal to -0.06, it may be supposed that these computations would possess the accuracy almost comparable with that of HOWARTH's method at least in the upstream boundary layer sufficiently distant from the separation. Considering the necessary computational labor is quite small, we might find, as a matter of fact, a number of problems of practical importance which may be treated in this way to draw some useful conclusions. In the immediate neighborhood of the separation point, however, results from MILLIKAN's formula become unreliable.[1] In practice, therefore, we should leave the problems, especially those concerning the skin friction, *unsolved* in this region, or alternatively correct the calculated values in an appropriate manner by means of more elaborate theory or by experiences.

4. Comparison with other methods (accelerated flow)

Let us next examine the accuracy of MILLIKAN's method when applied to an accelerated flow.[2] Suppose the velocity of the basic flow is given in the form

$$u_1 = (x+1)/2 \tag{4.1}$$

in terms of the notations introduced in (3.1). In other words, the velocity outside the boundary layer takes the values $U/2$ and U on the leading, and the trailing, edges respectively, increasing linearly in the intermediate region between them. Then the boundary layer thickness δ can be calculated by combining (4.1) with (2.1):

$$\delta(x) = 0.3799 R_L^{-1/5}\left(\frac{x+1}{2}\right)^{-23/7}\left(\frac{7}{17}\right)^{4/5}\left\{\left(\frac{x+1}{2}\right)^{34/7} - \left(\frac{1}{2}\right)^{34/7}\right\}^{4/5}. \tag{4.2}$$

With the aid of the relations (2.2), (2.3), and (2.4), the momentum thickness ϑ, the intensity of skin friction $\tau_0/\rho U^2$, together with the parameter Γ are readily found for successive values of x. In FIGURES 4, 5, and 6 the curves denoted as 'MILLIKAN' correspond to their values for the same REYNOLDS number (10^5) as in the preceding section.

1) Almost all the solutions already published seem to become unreliable, more or less, as separation point is approached, cf., for example, SCHUBAUER, G. B. and KLEBANOFF, P. S., Investigation of Separation of the Turbulent Boundary Layer, *NACA Technical Report* 1030 (1951), esp. p. 11. With regard to the position of separation in particular, however, we may expect it would be determined considerably accurately owing to the before-mentioned property of Γ-curve. Only, as 'accurately' we mean that the value computed with MILLIKAN's formula does not differ sensibly from that obtained with HOWARTH's method, and do not necessarily mean that the computed value corresponds precisely to the value observed in experiment.

2) RESEARCH COMMITTEE FOR HYDROLOGY, Tables of $u^{27/7}$ and $u^{-23/7}$ (the 3rd supplementary note), *Bulletin*, no. 22, (1963), pp. 7–33.

On the other hand, turning now to HOWARTH's solution, in the empirical representations of $H(\Gamma)$ and $\zeta(\Gamma)$ appearing in his differential equation (3.6)(see the figures cited in footnote 2) on p. 44), the points plotted after the results of BURI's measurements in the region of acceleration (namely $\Gamma > 0$) show considerable scatterings among themselves, and the present writer cannot help suspecting that reliability of the curves drawn through those points would not be so high here as in the domain of negative Γ. In this sense, according to his personal opinion, this method might be inappropriate for accelerated flows. As a tentative procedure, however, we shall resort to the following expedient: provided that the result computed by means of MILLIKAN's method is not so far from reality, the maximum value of Γ attainable in this example would be of order $+0.003$, and the value of Γ in which we are now interested lies in a range adjacent to the ordinate ($\Gamma = 0$) on the H-, and the ζ-, planes. Besides considering, therefore, the fact that the variations of H and ζ are not so large near the ordinate, we shall put approximately

$$H = H(0) = 1.37 \,,$$

and

$$\zeta = \zeta(0) = 0.0122 \,, \tag{4.3}$$

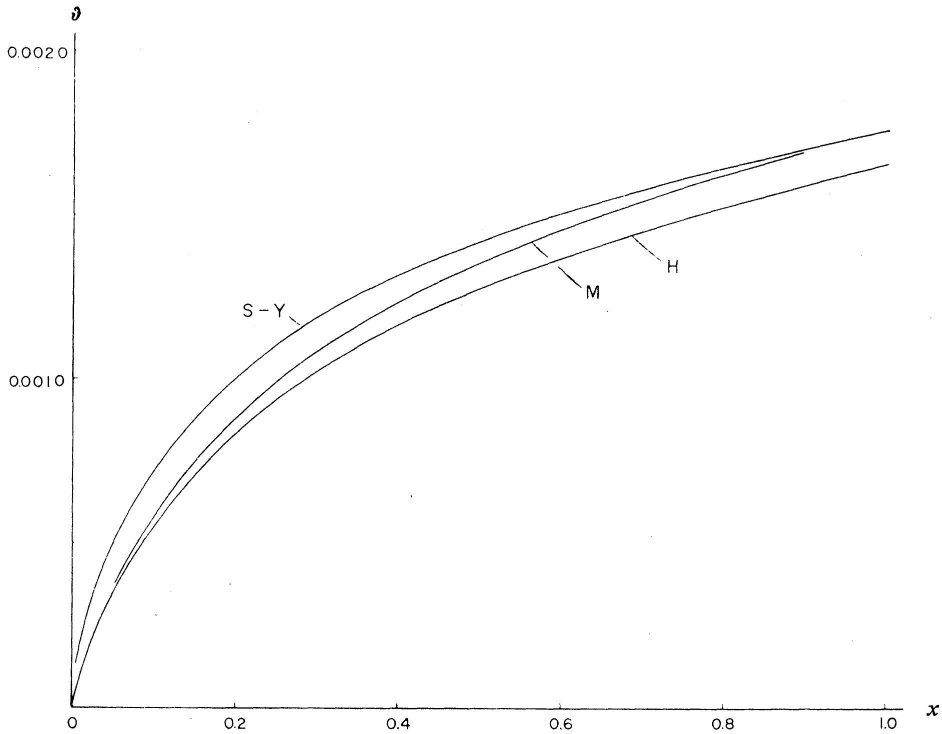

FIGURE 4. Momentum thickness ϑ.
M : MILLIKAN's method, H : HOWARTH's method, S-Y : SQUIRE-YOUNG's method.

throughout the computation, where the values of H and ζ corresponding to $\Gamma=0$ are written $H(0)$ and $\zeta(0)$ respectively. Thus, assuming H as well as ζ might be put approximately invariable, equation (3.6) can be readily integrated into the form

$$\chi=\frac{\beta}{\alpha+3/4}\left\{(x+1)^{3/4}-(x+1)^{-\alpha}\right\}, \tag{4.4}$$

where α and β are given respectively by

$$\alpha\equiv 5(2+H)/4=4.21\ ,$$

and

$$\beta\equiv 5\sqrt[4]{2}\,\zeta/4=0.0181\ . \tag{4.5}$$

We are now enabled to calculate ϑ, $\tau_0/\rho U^2$, and Γ through the formulas (3.5), (3.10), and (3.7), respectively, for values of x ranging from 0 to 1 ; they are rep-

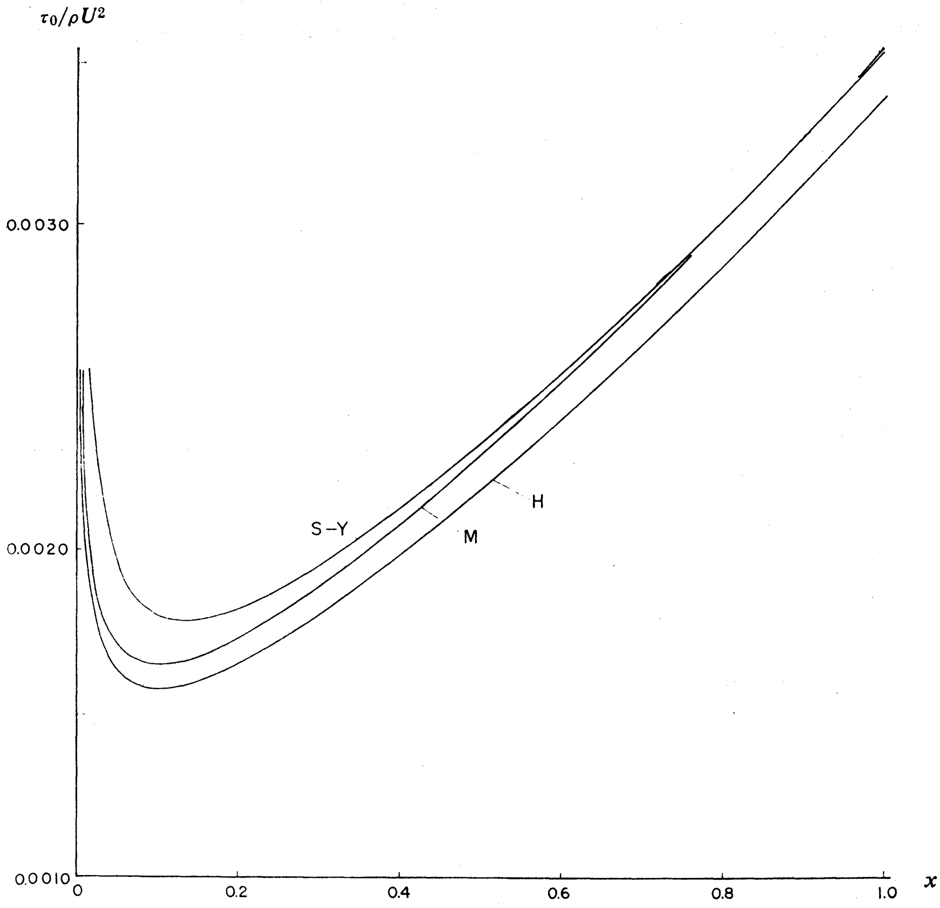

FIGURE 5. Intensity of skin friction $\tau_0/\rho U^2$.
M: MILLIKAN's method, H: HOWARTH's method, S-Y: SQUIRE-YOUNG's method.

roduced by the curves 'HOWARTH' in those figures mentioned above.

Finally, since the criterion of the accuracy of MILLIKAN's method by comparing with that of HOWARTH cannot be mentioned in our case as sufficiently reliable for the reason explained above, let us proceed further and solve the same example again using the procedure invented by SQUIRE and YOUNG.[1] The reason of this particular choice is that just like the method of HOWARTH this is also reduced to solving a differential equation of the first order and they are supposed to possess the accuracy almost comparable with each other.

According to the authors, if a non-dimensional parameter z is defined by

$$1/z^2 \equiv \tau_0/\rho U_1^2 , \tag{4.6}$$ [2]

then we may induce the following relationship out of extensive experimental data already published:

$$R_L u_1 \vartheta = 0.2454\, e^{0.3914z} , \tag{4.7}$$

R_L being REYNOLDS number. When both of (4.6) and (4.7) are substituted in the momentum equation of the boundary layer, we are led finally to the equation

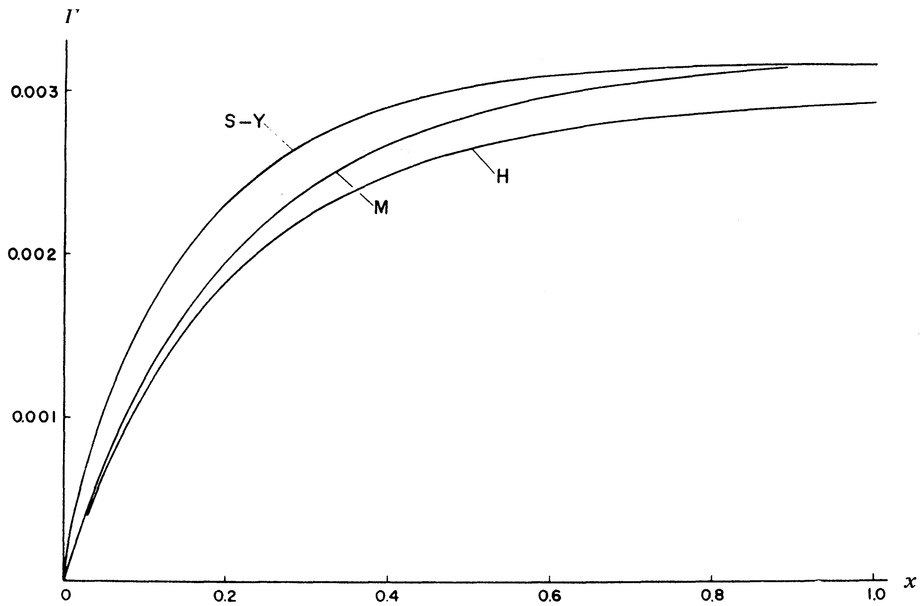

FIGURE 6. Parameter Γ.
M: MILLIKAN's method, H: HOWARTH's method, S-Y: SQUIRE-YOUNG's method.

1) SQUIRE, H. B. and YOUNG, A. D., The Calculation of the Profile Drag of Aerofoils, *Reports and Memoranda*, no. 1838 (1937).

2) In the original paper the notation ζ is used instead of z. In this note, however, the new symbol is introduced in order to avoid confusions with ζ employed in the preceding section.

$$\frac{dz}{d\xi} = \frac{10.411 R_L \xi^2 z^{-2} e^{-0.3914z} - 12.264}{2\xi}, \tag{4.8}$$

where we write

$$\xi \equiv x+1 \tag{4.9}$$

for short. For the same REYNOLDS number as before (10^5) this equation should be solved numerically starting with the initial condition

$$z = 0 \quad \text{at} \quad \xi = 1 . \tag{4.10}$$

However, since $dz/d\xi$ increases indefinitely at the point $\xi = 1$, in practice we should begin the integration with respect to the equation in which the independent, and the dependent, variables are interchanged, and transfer to the original form when appropriate. Once z is calculated, the quantities necessary to us may be evaluated in the forms

$$\vartheta = 0.4908 \times 10^{-5} \xi^{-1} e^{0.3914z}, \tag{4.11}$$

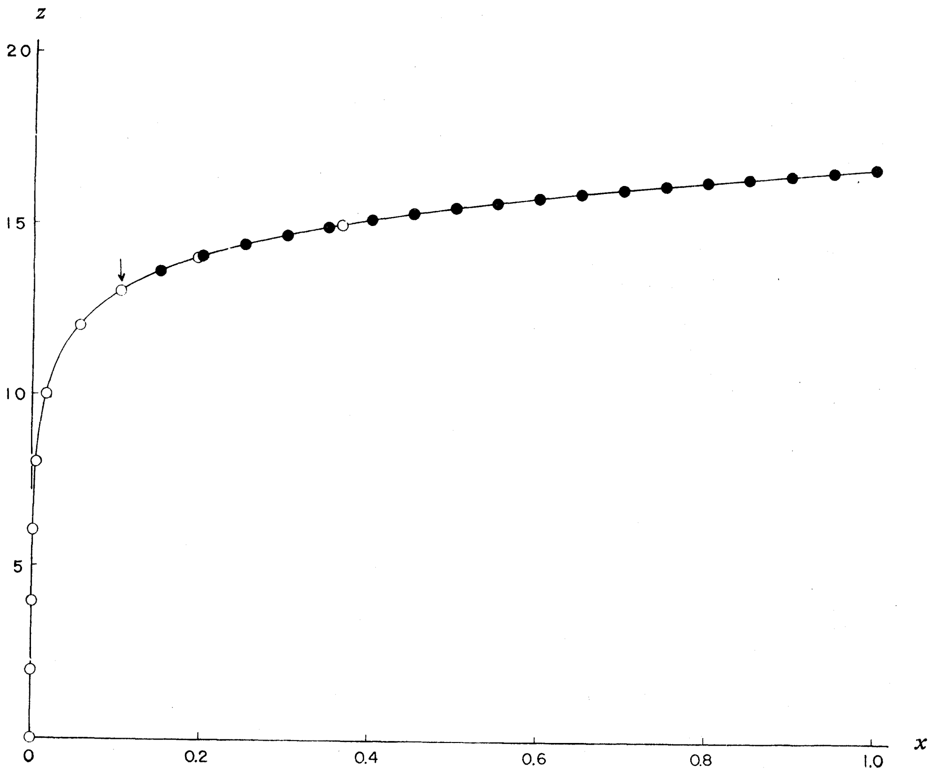

FIGURE 7. Solution by SQUIRE-YOUNG's method.

The black circle, the white circle, and the arrow indicate respectively the solution of equation (4.8), that of its inverted form, and the point of connection of these solutions.

and $$\tau_0/\rho U^2 = \xi^2/4z^2 \,, \tag{4.12}$$

furthermore Γ through (2.3) taking account of the relation

$$\xi = 2u_1 \,. \tag{4.13}$$

In the three figures above-mentiond these values are shown by the curves marked as 'SQUIRE-YOUNG'.

Equation (4.8) has been integrated using the procedure due to RUNGE and KUTTA. First, out of the inverted form of the equation the values of ξ correponding to $z = 0$, 2, 4, 6, 8, 10, 12, 13, 14,* and 15* have been computed. Starting from the point ($z = 13$, $\xi = 1.103$) lying on the integral curve, integration is continued making use of the original form (4.8) to work out the values of z for $\xi = 1.15$, 1.20, 1.25, ⋯, 1.95, and 2.00. The two figures to which asterisks are attached show that part of the first curve which overlaps with the second; see FIGURE 7. The complete overlapping as observed in the figure is an indication that our numerical integration would have the accuracy almost satisfactory for our purpose.

Now that we have constructed three kinds of approximate solutions for the boundary layer of a linearly accelerated flow, we are now in a position to compare them with each other with a view to arriving at some conclusion, which may be summarized as follows—

1) Among the three kinds of approximate values for each of ϑ, $\tau_0/\rho U^2$, and Γ, there exists an inequality which reads

$$⟪\text{SQUIRE-YOUNG}⟫ > ⟪\text{MILLIKAN}⟫ > ⟪\text{HOWARTH}⟫ \,, \tag{4.14}$$

for the same value of x, where as ⟪MILLIKAN⟫, for instance, we mean the numerical value, or the curve, of the solution calculated through the method of MILLIKAN.

2) In all of these FIGURES (4, 5, and 6), ⟪MILLIKAN⟫ branches off from ⟪HOWARTH⟫ near the origin and approaches to ⟪SQUIRE-YOUNG⟫ as x increases.

3) At any rate differences among results of these three methods are found not so serious, and any one of them might be chosen safely as an approximate treatment of boundary layers.

Of course, these are the conclusions obtained from only a single example, and in fact there is no evidence that all of them could be valid for all cases without fail. But among the above three statements, at least the last one is supposed to hold for REYNOLDS numbers of order 10^5 and for acceleration of similar magnitude. So it can be expected naturally that MILLIKAN's method simplified with the aid of the Tables would yield the result quite reasonable for a limited range of accelerated flows at the expense of computational labor negligible compared with other procedures.

5. Remarks on numerical integrations

The velocity distribution u_1 may be given as a function of the arc length x either analytically or empirically, according to the nature of a problem. Unless the functional form be sufficiently simple, however, there is very little hope that we should succeed in performing analytically the integration of $u_1^{27/7}$ with respec to x. But even if analytical treatment be difficult, $u_1^{27/7}$ can be found in the Tables for values of u_1 ranging from 0.500 to 1.500, and by means of numerical integration we may easily evaluate the integral. Then the question which arises next is the following: 'How small should be chosen Δx, the interval of x with which the integrand is to be read, in order that the error of numerical integration may be kept within a certain limit?' We shall make a series of tests employing SIMPSON's rule for numerical integrations consistently throughout this note to remove ambiguities resulting from other reasons.[1)]

For the velocity distributions

$$u_1 = (1-x)^n \,, \tag{5.1}$$

where

$$n = 2,\ 3/2,\ 1,\ 1/2, \text{ and } 1/3 \,, \tag{5.2}$$

of the main flow subject to retardations, the boundary layer thickness is estimated making use of (2.1). In this case, since the integral can be carried out analytically, we can find the 'exact' value of δ, say δ_{exact}, without difficulty, where incidentally as 'exact' we do mean that the formula is evaluated exactly and not necessarily that the boundary layer thickness is computed accurately. On the other hand, for those five values of n, u_1 is calculated for successive values of x at intervals of 0.005, and $u_1^{27/7}$ is read in the Tables, finally by means of the well-known procedure of SIMPSON's rule $\int_0^x u_1^{27/7} dx$ can be estimated.[2)] At the same location of x, $u_1^{-23/7}$ is found again in the Tables. Thus formula (2.1) enables us to evaluate δ; in contrast with δ_{exact} some of them are reproduced as δ_{approx} in TABLE I.[3)]

It will become evident from this table that the approximate calculations (with the Tables and SIMPSON's rule) yield the results practically the same as those from analytical integrations, provided that we should be able to make the interval of x with which the integrand is to be estimated as small as in this example. Besides this comparison presents a kind of check, although indirect and inextensive, of the Tables.

To scrutinize the effect of magnitude of the interval of x, say Δx, we repeat the above computation specifically for the case $n = 1$ in (5.1), namely

$$u_1 = 1 - x \,, \tag{5.3}$$

by taking

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1) Concerning a retarded flow, cf. RESEARCH COMMITTEE FOR HYDROLOGY, Tables of $u^{27/7}$ and $u^{-23/7}$ (the 2nd supplementary note), *Bulletin*, no. 18 (1961), pp. 19–38. As to an accelerated flow to be discussed later, see *ibid.* (the 3rd supplementary note), *Bulletin*, no. 22 (1963), pp. 7–33.

2) In using the Tables, taking account of the fraction of u_1 below the 3rd decimal place, values of $u_1^{27/7}$ were obtained by linear interpolation with the aid of their first differences printed together. This procedure might be unnecessary in calculations for practical purposes. Only, in this section since the focus of discussions lies in the error estimation of numerical works, we endeavored to remove the error as completely as possible of rounding off the figures. For the same reason, in TABLES I, II, and III more figures are shown than are necessary from practical or physical point of view.

3) The numbered tables are found at the end of the paper.

$$\Delta x = |\Delta u_1| = 0.01,\ 0.02,\ 0.04,\ 0.08,\ 0.16,\ 0.24, \text{ and } 0.48\ , \tag{5.4}$$

and calculate approximate values of $\int_0^x u_1^{27/7}\mathrm{d}x$ for successive values of x, Δu_1 being the step of u_1 corresponding to Δx.

In the next place, for the linearly accelerated flows expressed in the form

$$u_1 = (x+1)/2\ , \tag{5.5}$$

and

$$u_1 = (x+2)/2\ , \tag{5.6}$$

the former of which was dealt with in the preceding section, the integral is evaluated through the same process corresponding to the same choice of Δx ($=2|\Delta u_1|$ in this case) as (5.4). Results of these calculations are reproduced in TABLES II and III, where one half of the integral $\int_0^x u_1^{27/7}\mathrm{d}x$ computed by taking $|\Delta u_1|$ equal to 0.005, 0.01,$\cdots$ are denoted by $I_{0.005}$, $I_{0.01}$,$\cdots$, respectively. On the other hand, the integral $\int_0^x u_1^{27/7}\mathrm{d}x$, or to be more precise one half of it, performed analytically by substituting (5.5) or (5.6) for u_1, namely

$$\frac{7}{34}\left\{\left(\frac{x+1}{2}\right)^{34/7} - \left(\frac{1}{2}\right)^{34/7}\right\} \quad \text{for} \quad u_1 = \frac{x+1}{2}\ , \tag{5.7}$$

or

$$\frac{7}{34}\left\{\left(\frac{x+2}{2}\right)^{34/7} - 1\right\} \quad \text{for} \quad u_1 = \frac{x+2}{2}\ , \tag{5.8}$$

are both written I_{exact} in these tables. Furthermore, for the velocity distribution (5.5) in particular, we write

$$I(x) \equiv \frac{1}{2}\int_0^x \left(\frac{x+1}{2}\right)^{27/7}\mathrm{d}x\ , \tag{5.9}$$

then by putting

$$2y \equiv x+1\ ,$$

we have also

$$I(y) = \int_{0.5}^{y} y^{27/7}\mathrm{d}y \qquad (0.5 \leqq y \leqq 1)\ . \tag{5.10}$$

Again in the linearly retarded stream (5.3), if we write likewise

$$J(x) \equiv \frac{1}{2}\int_0^x (1-x)^{27/7}\mathrm{d}x\ , \tag{5.11}$$

then we may transform J through the substitution

$$z \equiv 1-x\ ,$$

into

$$J(z) = \frac{1}{2}\int_z^1 z^{27/7}\mathrm{d}z\ . \tag{5.12}$$

Now confining ourselves only in the region

$$0.5 \leqq z \leqq 1\ ,$$

and putting

$$z = y\ ,$$

we shall arrive at the result

$$J(y) = \frac{1}{2}\left\{I(1)-I(y)\right\}. \tag{5.13}$$

By virtue of this relation, for the retarded flow (5.3) a table similar to TABLE II or III may be constructed from TABLE II without further computations but is omitted to avoid unnecessary duplications.

In so far as these three examples are concerned, the same conclusion would be obtained from a case of a linearly retarded flow just as from a linearly accelerated, which might be summarized as follows—

1) For $|\Delta u_1|$ sufficiently small (say, less than 0.04), the approximate values of the integral are found practically coincident with the exact values. In other words, as is expected with certainty, so long as the decimal places of the integrand are kept invariable (in fact, $u_1^{27/7}$ are tabulated to the 5th decimal place for $u_1 \leqq 1.196$ ($u_1^{27/7} < 2$), and to the 4th for $u_1 \geqq 1.197$ ($u_1^{27/7} > 2$) in our Tables), we can not improve the accuracy of numerical integrations by decreasing $|\Delta u_1|$ (or Δx) indefinitely. This may be observed more clearly in Table 2 of the 2nd supplementary note, pp. 32–33, (omitted here) where numerical integrations were repeated for $\Delta x = |\Delta u_1| = 0.005$, 0.01, 0.02, 0.03, 0.04, 0.06, 0.08, 0.12, and 0.24.

2) Next, provided that u_1 changes linearly with x, even if $|\Delta u_1|$ (or Δx) be not so small, the accuracy of the integral is maintained in a level satisfactorily high for practical purposes. For instance, when $|\Delta u_1| = 0.24$, the relative error of the integral $\int_0^x u_1^{27/7}dx$ is found to be 0.1% for the case of (5.5), cf. TABLE II, and 0.01% for the flow pattern (5.6), TABLE III; the accuracy of both of these integrations would be much higher than that of the assumptions underlying MILLIKAN's formula.

From these tests we know that in velocity distributions, linearly retarded or accelerated, we can obtain approximate values of the integral $\int_0^x u_1^{27/7}dx$ sufficiently accurate for practical purposes by taking $|\Delta u_1| \cong 1/4$. Then, for velocity distributions other than a linear function of x, how should be modified this conclusion? So far there is no answer available to this question. But at any rate we may approximate a given distribution by means of a number of successive straight lines and by applying the above estimation to each of these segments we should arrive at some choice of Δu_1 (or Δx) not so far from being adequate.

6. Example 1 (calculation of separation points)[1]

For those five kinds of flow patterns which is defined by (5.1) together with (5.2), positions of separation of the boundary layer are computed according to the scheme explained in the preceding sections. Namely, δ is calculated from (2.1), ϑ from (2.4), and Γ through (2.3); finally the separation point x_s is determined by the critical equation (2.5) or

$$\Gamma(x_s) = -0.06. \tag{6.1}$$

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1) Tables of $u^{27/7}$ and $u^{-23/7}$ (the 2nd supplementary note), *Bulletin*, no. 18 (1961), pp. 19–38.

Furthermore, if we denote by $u_1(x_s)$ the velocity of the main flow just outside the boundary layer at the position of separation, then results are obtained as shown in the following table. Incidentally, all of these computations were performed with the aid of the Tables taking $\Delta x = 0.005$.

n	x_s	$u_1(x_s)$
2	0.278	0.521
3/2	0.346	0.529
1	0.455	0.545
1/2	0.656	0.587
1/3	0.761	0.621

It should be noticed that $u_1(x_s)$ is sensibly uniform irrespective of the variation among n, although there can be observed remarkable differences among the values of x_s. This property may be readily explained if we note that, according to our computation procedures, the position of separation would be independent of REYNOLDS number.[1] Namely, from (2.1) δ is found to vary as $R_L^{-1/5}$, and from (2.4) so does ϑ. Substituting it for ϑ in (2.3), Γ as well as the critical equation (6.1) are both reduced independent of REYNOLDS number. On the basis of this fact, we may deduce the followings:

In a group of two-dimensional flow patterns defined by the equation of the form

$$u_1 = \alpha\varphi(x) \,,$$

or

$$x = \beta\psi(u_1) \,, \tag{6.2}$$

where φ and ψ are arbitrary functions while α and β denote parameters variable within that group, separation of the flow from the wall occurs at the point x $=$ constant or $u_1 =$ constant respectively, according to the form of relationship between u_1 and x. In fact, so long as x remains small compared with unity, we have from (5.1)

$$u_1 = 1 - nx \,,$$

or

$$x = (1-u_1)/n \,, \tag{6.3}$$

approximately, which is just the second form of (6.2).

If, in cases encountered practically, either of these conditions of similarity is found satisfied, at least approximately, between u_1 and x, then by knowing, theoretically or experimentally, one of the points of separation among a series of systematic modification of flow, there exists possibility of predicting roughly the remaining positions of separation without labor of calculating the boundary layers for these flows.

7. Example 2 (velocity disturbances in sinusoidal form)[2]

Since we have illustrated in sections 3 and 4 that MILLIKAN's formula of cal-

1) At this point the writer expresses his gratitude to Professor H. YAMADA of Kyoto University for turning his attention to this fact. Likewise out of HOWARTH's equation, cf. equation (3.6) in this note or GOLDSTEIN, S., *op. cit.*, vol. II, **194**, pp. 436-439, we may deduce that there is no scale effect on the position of separation except in so far as u_1 alters.

2) Tables of $u^{27/7}$ and $u^{-23/7}$ (the 3rd supplementary note), ***Bulletin***, no. 22 (1963), pp. 7-33, esp. appendix.

culating boundary layers yields results quite comparable with those due to HOWARTH's or SQUIRE-YOUNG's method both for retarded, and accelerated, main flows except in the immediate neighborhood of separation point, we are now in a position to consider the boundary layer corresponding to the velocity distribution expressed in the form

$$u_1 = 1 + a \sin\left(\frac{\pi}{2}\frac{4x}{\lambda}\right), \tag{7.1}$$

as an example of flow patterns having both of retarded, and accelerated, regions at the same time. This is the velocity disturbance in sinusoidal form with the amplitude a (aU) and the wave length λ (λL) superposed upon a uniform stream 1 (U).[1] We might visualize the flow field by a uniform stream U past a plate of length L corrugated with wave length λL.

Now for the values of λ and a such that

$$\left.\begin{aligned} &\lambda = 0.2, \\ \text{and} \quad &a = 0,\ 0.1,\ 0.2, \text{ and } 0.4, \end{aligned}\right\} \tag{7.2}$$

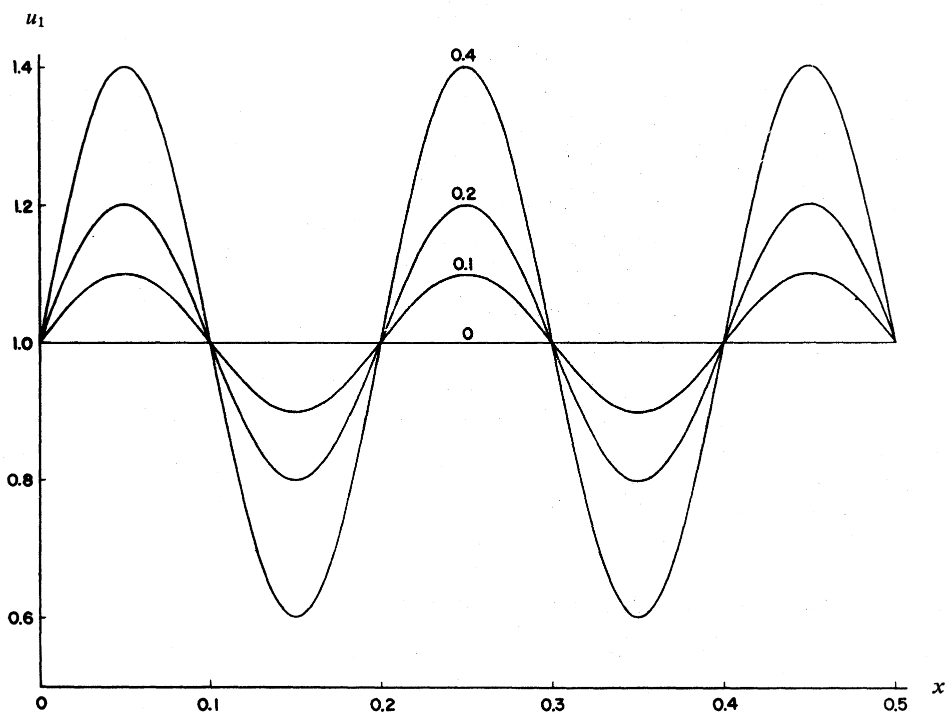

FIGURE 8. Velocity distribution u_1.
$\lambda = 0.2$; $a = 0, 0.1, 0.2, 0.4$.
The values of a are shown by the numbers above the curves.

1) For example, a (aU) stands for 'magnitude aU, or in its dimensionless form, a'.

u_1, δ and $\tau_0/\rho U^2$ may be calculated through (7.1), (2.1) together with the Tables,[1] and (2.2), respectively. In FIGURES 8, 9, and 10, u_1, $R_L^{1/5}\delta$, and $R_L^{1/5}\tau_0/\rho U^2$ are shown, respectively, for the numerical values of the constants given by (7.2).[2] Next with a view to examining the effect of phase of the undulation, assuming

$$\lambda = 0.2 \quad \text{and} \quad a = 0.2\,, \tag{7.3}$$

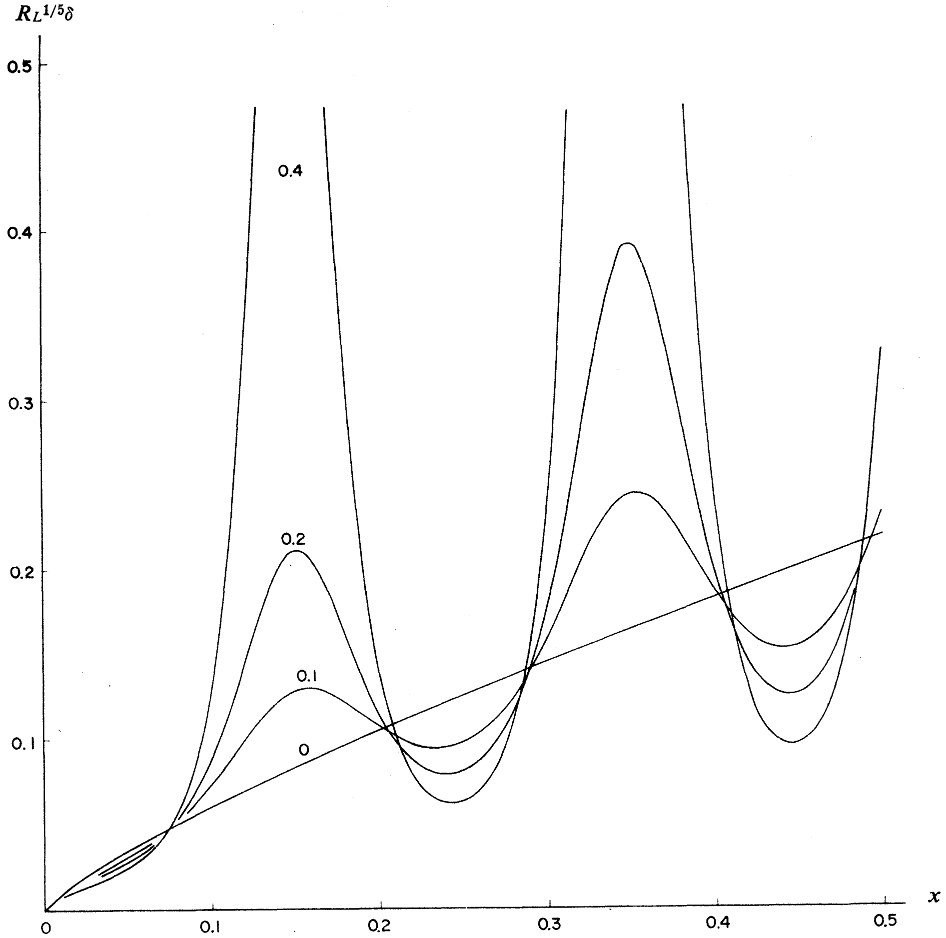

FIGURE 9. Boundary layer thickness $R_L^{1/5}\delta$.
$\lambda = 0.2$; $a = 0$, 0.1, 0.2, 0.4.
The values of a are shown by the numbers above the curves.

1) The following choices were made for the steps in numerical integrations: $\Delta x = 0.005$ for $\lambda = 0.2$, 0.0025 for 0.1, and 0.01 for 0.4.

2) In this section and the appendix, instead of assuming $R_L = 10^5$, REYNOLDS number is preserved as an indeterminate parameter. Since δ is proportional to $R_L^{-1/5}$, from (2.2) $\tau_0/\rho U^2$ also is found to vary as $R_L^{-1/5}$.

we put $$u_1 = 1-0.2 \sin\left(\frac{\pi}{2}\frac{4x}{0.2}\right) \tag{7.4}$$

in place of (7.1), and compute $R_L^{1/5}\delta$ by a similar procedure. Finally fixing the value of a at 0.2 in (7.1) while varying λ from

$$\lambda = 0.1 \quad \text{to} \quad 0.4\,, \tag{7.5}$$

u_1 and $R_L^{1/5}\delta$ are worked out. Results of these computations are all embodied in FIGURES 11 and 12; the former shows various kinds of u_1, while the latter gives corresponding values of $R_L^{1/5}\delta$. In both of these figures the symbols A, B, C, D, and E will become self-evident by the following table (see p. 62); E stands for the basic uniform flow.

Of course the number of examples computed here is too small for us to be able

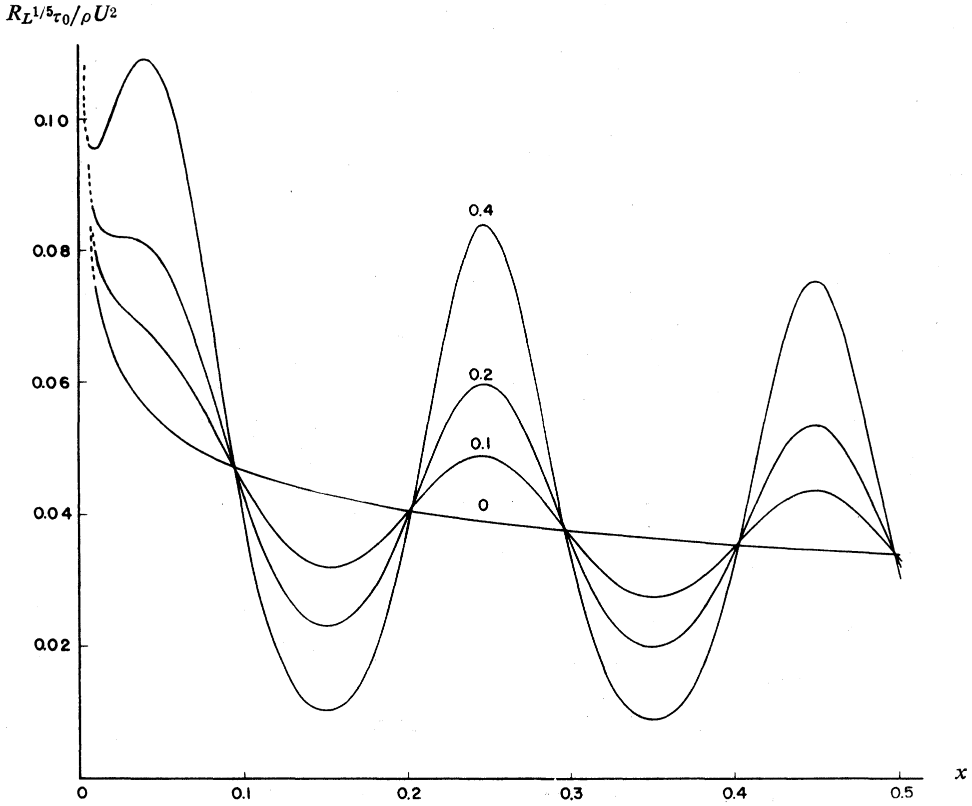

FIGURE 10. Intensity of skin friction $R_L^{1/5}\tau_0/\rho U^2$.

$\lambda = 0.2$; $a = 0$, 0.1, 0.2, 0.4.

The values of a are shown by the numbers above the curves.

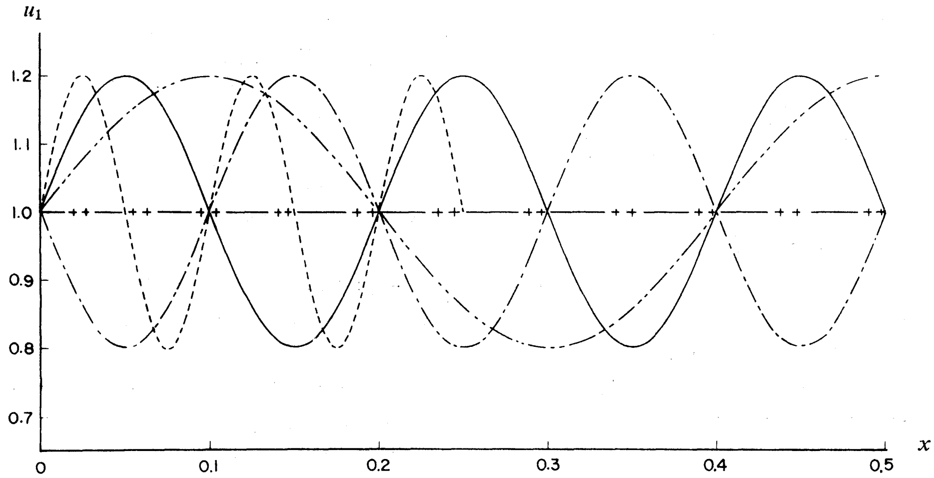

FIGURE 11. Velocity distribution u_1.

A B C
D + + E

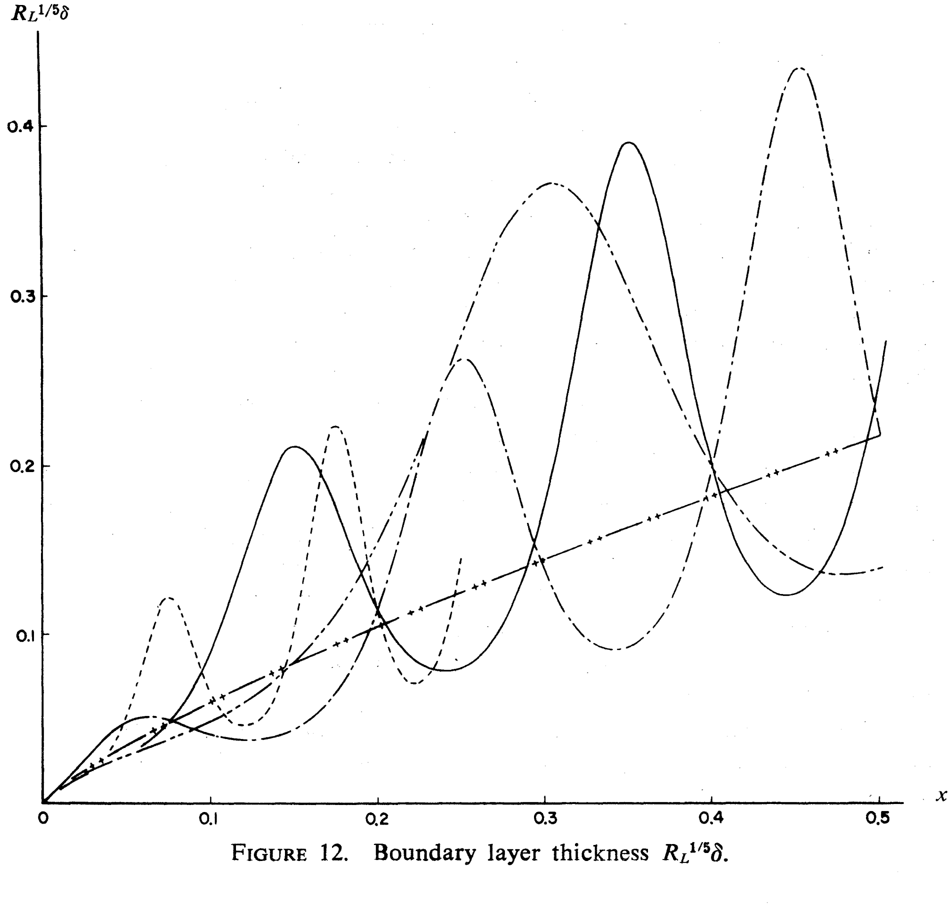

FIGURE 12. Boundary layer thickness $R_L^{1/5}\delta$.

	expression for u_1	λ	a
A	(7.1)	0.2	0.2
B	(7.4)	0.2	0.2
C	(7.1)	0.1	0.2
D	(7.1)	0.4	0.2
E	(7.1)		0

to induce the definite conclusions out of them. However, the followings would form, in spite of the restricted variety of tests, a part of more or less general statements—

1) Disturbance generated in δ, the thickness of the boundary layer, develops with the same wave length as the undulation contained in the velocity.

2) As the amplitude grows up of the superposed sinusoidal term of the velocity, degree of variation produced on the part of the boundary layer thickness increases much more markedly.

3) Denoting the boundary layer thickness corresponding to the basic uniform flow by δ_0, change of δ is more pronounced on the side $\delta > \delta_0$ than on $\delta < \delta_0$. In other words, the disturbed boundary layer appears to become *thicker*, on the average, than the original condition.

4) Variation of δ propagates along the plate amplifying in the manner roughly proportional to $\delta_0(x)$.

5) Of course, the phase and the wave length of the undulation in δ become different as the corresponding ones of the velocity distribution are varied. But still the amplitude of δ as a function of x is subjected to no essential modifications.

6) Circumstances are more complicated for friction. Namely, in contrast with the disturbance produced in the boundary layer thickness, the undulation generated in the intensity of local skin friction decays gradually as we go down along the plate. Situation near the leading edge of the plate, therefore, becomes the most important factor in the problem, and so the phase as well as the wave length of the fluctuating term in velocity distribution may play decisive parts in the total frictional resistance. Concerning the variation in the local skin friction, just as in the case of the boundary layer thickness, change is found larger on the positive side (increasing the non-dimensional local skin friction over the value for the uniform stream) than on the negative side, and if a large number of wave lengths are contained in the whole length of the plate, then the total skin friction will increase most likely compared with the uniform flow.

Corresponding to the flow patterns exemplified by (7.1) and (7.2), or illustrated in FIGURE 8, the non-dimensional parameter Γ calculated through (2.3) is reproduced in FIGURE 13. As remarked already, Γ is found to be independent of REYNOLDS number. In spite of HOWARTH's remark that separation should be determined by the critical equation (2.5), separation phenomenon in this case would be, according to the personal opinion of the writer, much more complicated than for ordinary flows in which separation can take place only once as we go downstream along the surface.

Finally to obtain a general idea for the flow in which the basic flow is not uniform along the plate, for the same values of the constants listed in (7.2), let us assume u_1 in the form

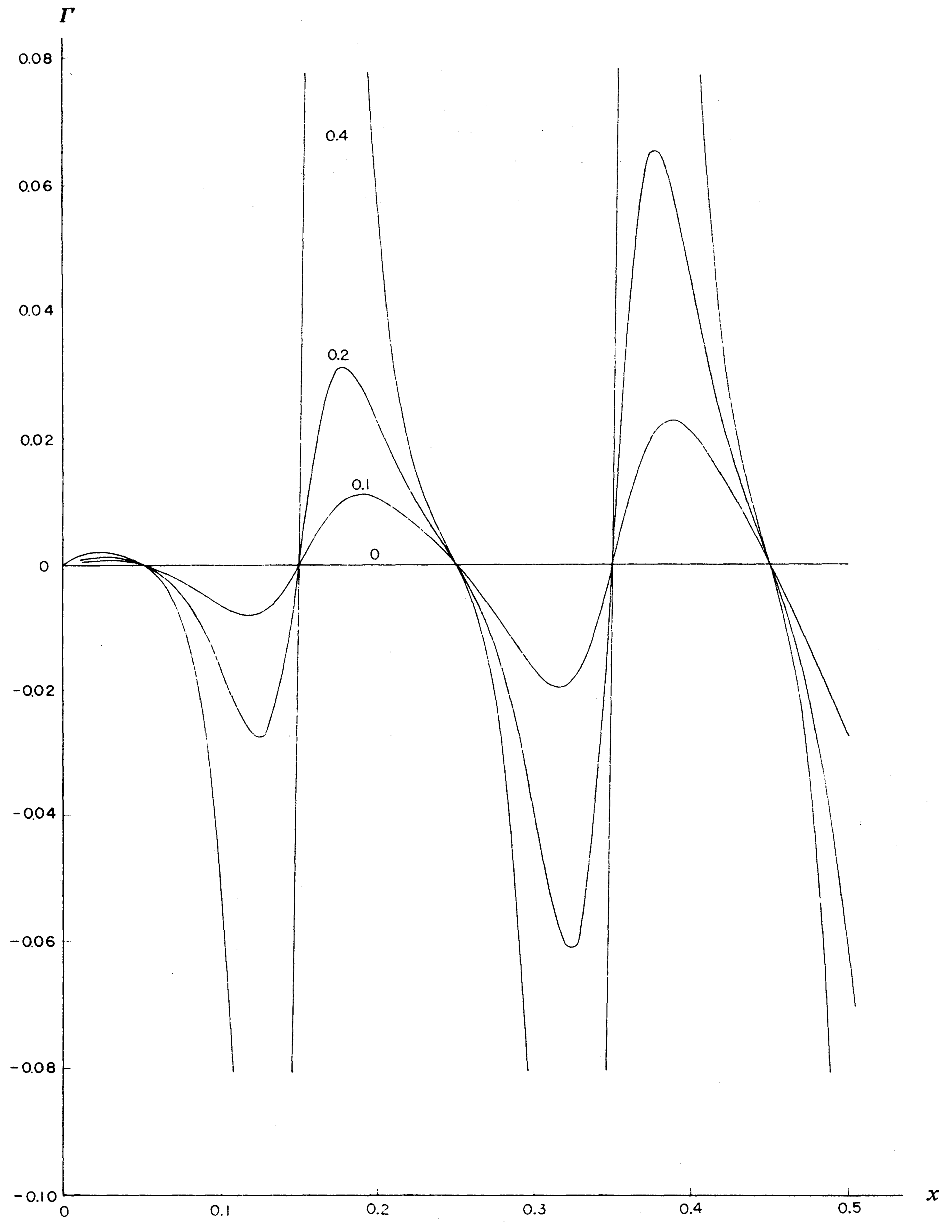

FIGURE 13. Parameter Γ.

$\lambda = 0.2$; $a = 0$, 0.1, 0.2, 0.4.

The values of a are shown by the numbers above the curves.

$$u_1 = (1-x)+a\sin\left(\frac{\pi}{2}\frac{4x}{\lambda}\right), \tag{7.6}$$

and calculate the value of $R_L^{1/5}\delta$ for successive values of x, both of which are shown by FIGURES 14 and 15 respectively. It will be observed that the items mentioned in 1), 2), 3), and 4) are reproduced far more conspicuously than before; this is natural consequence from the fact that as x increases, the general flow loses gradually its speed and so the relative importance of the fluctuating part becomes larger.

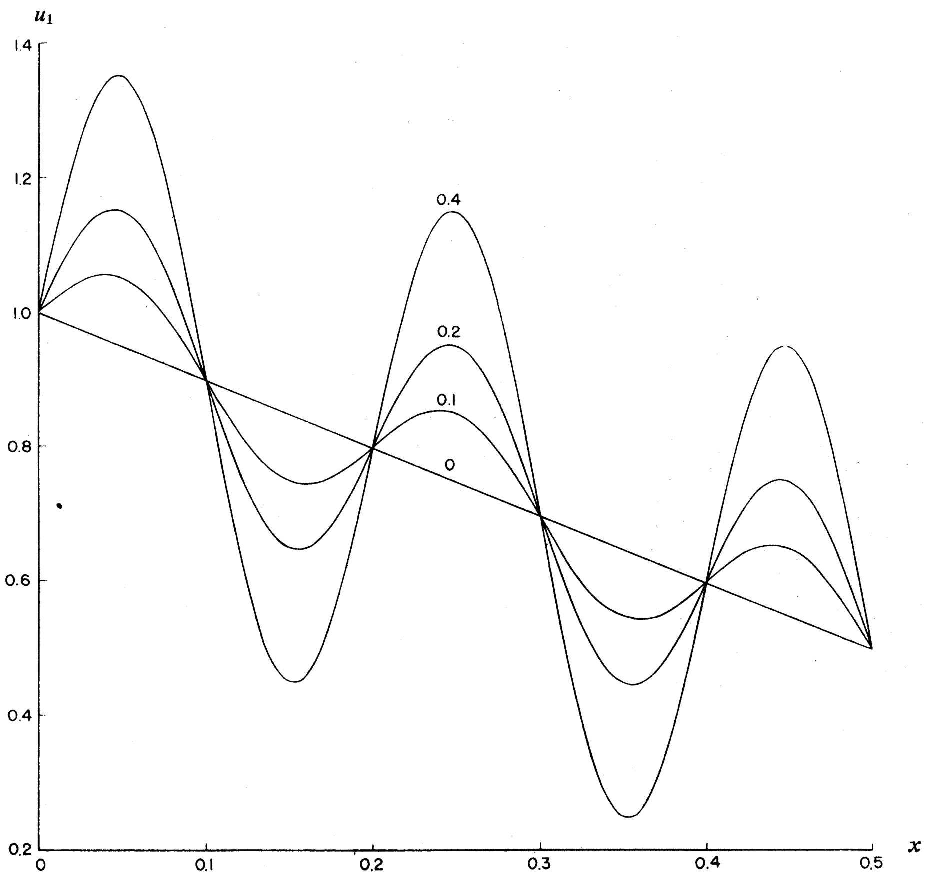

FIGURE 14. Velocity distribution u_1.
$\lambda = 0.2$; $a = 0$, 0.1, 0.2, 0.4.
The values of a are shown by the numbers above the curves.

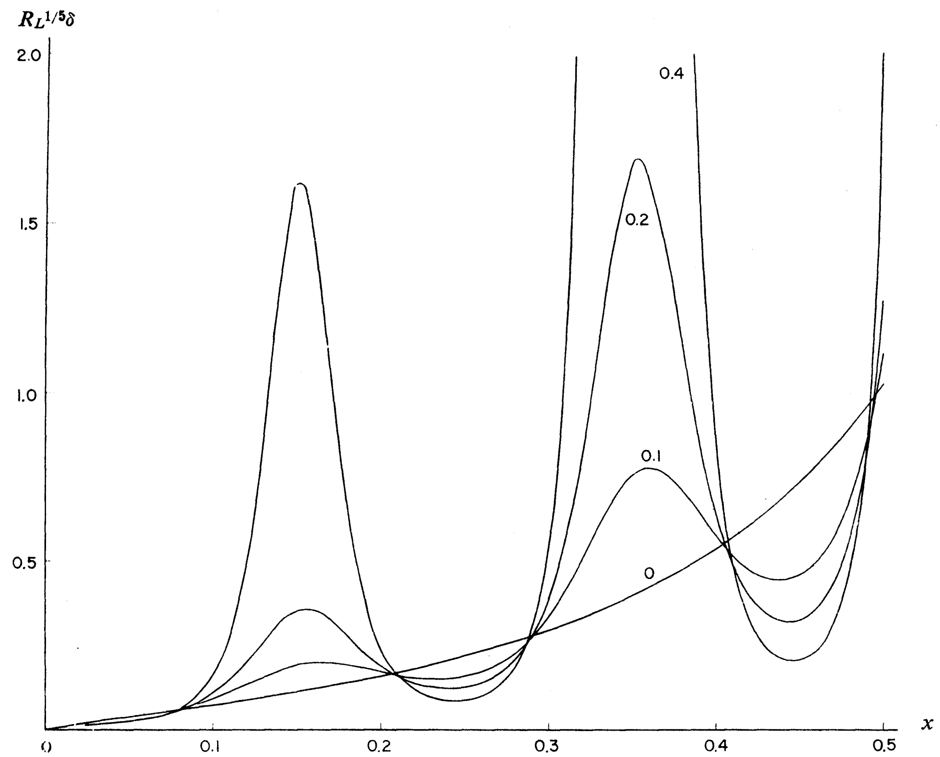

FIGURE 15. Boundary layer thickness $R_L^{1/5}\delta$.
$\lambda = 0.2$; $a = 0$, 0.1, 0.2, 0.4.
The values of a are shown by the numbers above the curves.

8. Appendix[1]

Although completely independent of the topics concerning MILLIKAN's formula nor the Tables, let us touch very briefly on the problem of those variations in laminar boundary layers which are produced by velocity fluctuations in sinusoidal form superposed upon a uniform stream. In terms of the same notations as before, the momentum thickness of laminar boundary layers may be estimated by making use of TANI's formula :[2]

1) Tables of $u^{27/7}$ and $u^{-23/7}$ (the 3rd supplementary note), *Bulletin*, no. 22 (1963), pp. 7-33, esp. appendix.

2) TANI, I., On a Simplified Calculation of the Laminar Separation Point, (in Japanese), *Journal of the Aeronautical Research Institute*, Tokyo Imperial University, no. 199 (1941), pp. 62-67, esp. 63, and TANI, I., On the Solution of the Laminar Boundary Layer Equations, *50 Jahre Grenzschichtforschung* (1955), S. 193-200. It is of interest to note that WALZ proposed independently a similar expression in the same year:

(*continued on the following page*)

$$R_L\vartheta^2 = \frac{0.44}{u_1^6}\int_0^x u_1^5 \mathrm{d}x. \tag{A.1}$$

Corresponding to the flow pattern generated from (7.1) together with (7.2), $R_L^{1/2}\vartheta$

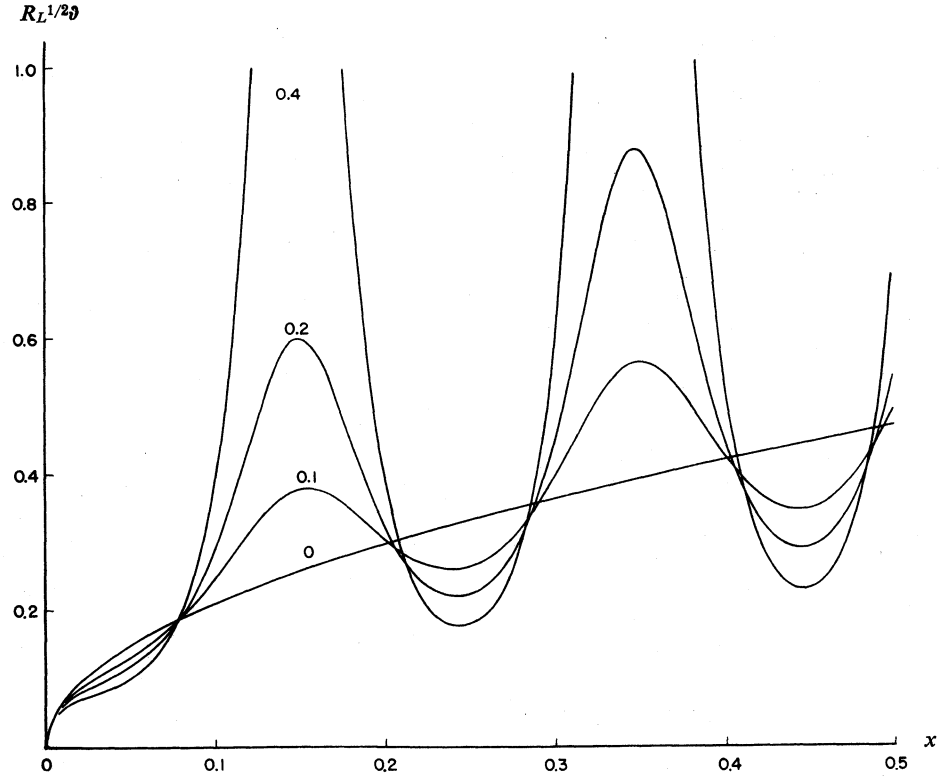

FIGURE 16. Momentum thickness $R_L^{1/2}\vartheta$ (laminar boundary layer).
$\lambda = 0.2$; $a = 0, 0.1, 0.2, 0.4$.
The values of a are shown by the numbers above the curves.

$$R_L\vartheta^2 = \frac{0.470}{u_1^6}\int_0^x u_1^5 \mathrm{d}x\,,$$

cf. SCHLICHTING, H., *op. cit.* (see footnote 4) on p. 41), XII, b, p. 251, equation (12.37). The formula obtained later by THWAITES reads

$$R_L\vartheta^2 = \frac{0.45}{u_1^6}\int_0^x u_1^5 \mathrm{d}x\,,$$

cf. THWAITES, B., Approximate Calculation of the Laminar Boundary Layer, *The Aeronautical Quarterly*, vol. I (1949), pp. 245-280. As to the comparison among these three formulas, see OKABE, J. and INOUE, S., Laminar Boundary Layers on Wedges, *Reports of Research Institute for Applied Mechanics*, vol. X, no. 38 (1962), pp. 45-51. Again concerning the criterion of laminar separation, see *Handbuch der Physik*, Bd. VIII/1, *Strömungsmechanik* I, S. 283-286.

calculated by virtue of this formula is shown in FIGURE 16, where separation caused possibly by velocity fluctuations is left out of consideration. The general behavior of these curves is essentially the same as in the turbulent boundary layer, and so the same conclusions as were stated there should be still valid also for the flow in the laminar boundary layer.

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TABLE I.

x	$n=2$		$n=3/2$		$n=1$		$n=1/2$		$n=1/3$	
	δ_{exact}	δ_{approx}	δ_{exact}	δ_{approx}	δ_{exact}	δ_{approx}	δ_{exact}	δ_{approx}	δ_{exact}	δ_{approx}
0.00	0	0	0	0	0	0	0	0	0	0
05	0.004 1627	0.004 1628	0.003 9704	0.003 9704	0.003 7893	0.003 7894	0.003 6190	0.003 6190	0.003 5644	0.003 5643
10	008 9364	008 9363	008 0637	008 0637	007 2960	007 2959	006 6190	006 6190	006 4117	006 4117
15	015 6598	015 6596	013 2477	013 2485	011 2767	011 2768	009 6598	009 6599	009 1875	009 1876
20	025 747	025 747	020 218	020 218	016 0530	016 0526	012 8948	012 8951	012 0191	012 0192
0.25	0.041 591	0.041 592	0.029 992	0.029 992	0.022 000	0.022 000	0.016 4420	0.016 4421	0.014 9861	0.014 9859
30			044 211	044 211	029 644	029 645	020 427	020 427	018 1601	018 1603
35			065 660	065 659	039 771	039 771	025 006	025 006	021 622	021 622
40					053 602	053 602	030 389	030 389	025 467	025 467
45					073 132	073 133	036 869	036 870	029 825	029 825
0.50					0.101 742	0.101 742	0.044 875	0.044 875	0.034 870	0.034 870
55							055 063	055 063	040 854	040 854
60							068 478	068 477	048 158	048 158
65							086 891	086 891	057 384	057 385
70							113 505	113 505	069 544	069 543
0.75									0.086 489	0.086 487
80									111 981	111 978
85									154 984	154 988

TABLE II.

u_1	x	I_{exact}	$I_{0.005}$	$I_{0.01}$	$I_{0.02}$	$I_{0.04}$	$I_{0.08}$	$I_{0.12}$	$I_{0.24}$
0.50	0.00	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000
51	02	000 717	000 717						
52	04	001 491	001 491	001 491					
53	06	002 324	002 324						
54	08	003 220	003 220	003 220	003 220				
0.55	0.10	0.004 182	0.004 182						
56	12	005 214	005 214	0.005 214					
57	14	006 320	006 320						
58	16	007 503	007 503	007 503	0.007 503	0.007 503			
59	18	008 768	008 768						
0.60	0.20	0 010 118	0.010 118	0.010 118					
61	22	011 557	011 558						
62	24	013 091	013 091	013 091	0.013 091				
63	26	014 723	014 723						
64	28	016 458	016 458	016 458					
0.65	0.30	0.018 301	0.018 301						
66	32	020 257	020 257	0.020 257	0.020 257	0.020 257	0.020 257		
67	34	022 330	022 330						
68	36	024 526	024 526	024 526					
69	38	026 850	026 850						
0.70	0.40	0.029 308	0.029 308	0.029 308	0.029 308				
71	42	031 905	031 905						
72	44	034 647	034 647	034 647					
73	46	037 540	037 540						
74	48	040 589	040 590	040 590	040 590	0.040 590		0.040 596	
0.75	0.50	0.043 803	0.043 803						
76	52	047 186	047 186	0.047 186					
77	54	050 745	050 745						
78	56	054 486	054 486	054 486	0.054 486				
79	58	058 416	058 417						
0.80	0.60	0.062 544	0.062 545	0.062 545					
81	62	066 876	066 877						
82	64	071 420	071 420	071 420	0.071 420	0.071 420	0.071 422		
83	66	076 182	076 182						
84	68	081 171	081 170	081 171					
0.85	0.70	0.086 394	0.086 393						
86	72	091 858	091 859	0.091 859	0.091 859				
87	74	097 575	097 575						
88	76	103 550	103 550	103 550					
89	78	109 792	109 792						
0.90	0.80	0.116 311	0.116 312	0.116 312	0.116 312	0.116 311			
91	82	123 117	123 116						
92	84	130 216	130 216	130 216					
93	86	137 619	137 619						
94	88	145 336	145 336	145 336	145 336				
0.95	0.90	0.153 376	0.153 376						
96	92	161 749	161 749	0.161 749					
97	94	170 466	170 466						
98	96	179 535	179 536	179 536	0.179 536	0.179 535	0.179 538	0.179 546	0.179 699
99	98	188 971	188 970						
1.00	1.00	0.198 779	0.198 779	0.198 779					

Table III.

u_1	x	I_{exact}	$I_{0.005}$	$I_{0.01}$	$I_{0.02}$	$I_{0.04}$	$I_{0.08}$	$I_{0.12}$	$I_{0.24}$
1.00	0.00	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000
01	02	010 195	010 195						
02	04	020 786	020 786	020 786					
03	06	031 786	031 786						
04	08	043 206	043 206	043 206	0.043 205				
1.05	0.10	0.055 057	0.055 056						
06	12	067 350	067 351	0.067 351					
07	14	080 101	080 101						
08	16	093 318	093 319	093 319	0.093 318	0.093 319			
09	18	107 018	107 017						
1.10	0.20	0.121 209	0.121 209	0.121 209					
11	22	135 907	135 908						
12	24	151 126	151 126	151 126	0.151 126				
13	26	166 878	166 877						
14	28	183 176	183 175	183 175					
1.15	0.30	0.200 04	0.200 03						
16	32	217 47	217 47	0.217 47	0.217 47	0.217 47	0.217 47		
17	34	235 49	235 49						
18	36	254 12	254 12	254 12					
19	38	273 37	273 37						
1.20	0.40	0.293 24	0.293 25	0.293 25	0.293 25				
21	42	313 79	313 78						
22	44	334 97	334 97	334 97					
23	46	356 86	356 85						
24	48	379 42	379 42	379 42	0.379 42	0.379 42		0.379 43	
1.25	0.50	0.402 71	0.402 71						
26	52	426 73	426 72	0.426 72					
27	54	451 48	451 49						
28	56	477 01	477 01	477 01	0.477 01				
29	58	503 32	503 32						
1.30	0.60	0.530 41	0.530 42	0.530 42					
31	62	558 35	558 34						
32	64	587 09	587 10	587 10	0.587 10	0.587 10	0.587 11		
33	66	616 70	616 71						
34	68	647 19	647 19	647 19					
1.35	0.70	0.678 57	0.678 56						
36	72	710 83	710 84	0.710 84	0.710 84				
37	74	744 04	744 05						
38	76	778 19	778 20	778 20					
39	78	813 32	813 32						
1.40	0.80	0.849 43	0.849 44	0.849 44	0.849 44	0.849 44			
41	82	886 55	886 56						
42	84	924 70	924 71	924 71					
43	86	963 90	963 91						
44	88	1.004 19	1.004 18	1.004 18	1.004 18				
1.45	0.90	1.045 55	1.045 55						
46	92	088 03	088 03	1.088 03					
47	94	131 65	131 65						
48	96	176 43	176 42	176 42	1.176 42	1.176 43	1.176 43	1.176 44	1.176 58
49	98	222 39	222 38						
1.50	1.00	1.269 55	1.269 55	1.269 55					