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LAMINAR BOUNDARY LAYER ON A CIRCULAR CYLINDER IN AXIAL FLOW OF AN INCOMPRESSIBLE FLUID

By Jun-ichi OKABE

The velocity distributions across sections of the laminar boundary layer on a circular cylinder in axial flow of an incompressible fluid are assumed in the form of a polynomial of degree four with respect to non-dimensional radial length and having the coefficients determined from associated boundary conditions. Making use of the momentum equation, relationship between thickness of the boundary layer and axial distance is obtained. Finally, corresponding to five values of axial distance, velocity profiles computed by various methods are shown for comparison: those of YOUNG, SEBAN-BOND-KELLY, GLAUERT-LIGHTHILL, and the present writer. It is concluded that results of the calculation mentioned last agree practically with those of S-B-K based upon series expansions, at least in the range of axial distance under examination.

1. Introduction

So far as the present writer is aware, the laminar boundary layer on a circular cylinder in axial flow of an incompressible fluid was discussed by YOUNG,[1] SEBAN and BOND,[2] KELLY,[3] and later by GLAUERT and LIGHTHILL.[4] Other contributions were made, for example, by COOPER and TULIN,[5] BATCHELOR,[6] STEWARTSON,[7]

1) YOUNG, A. D., The Calculation of the Total and Skin Friction Drags of Bodies of Revolution at Zero Incidence, *Reports and Memoranda*, no. 1874 (1939), Aeronautical Research Committee.

2) SEBAN, R. A. and BOND, R., Skin-Friction and Heat-Transfer Characteristics of a Laminar Boundary Layer on a Cylinder in Axial Incompressible Flow, *Journal of the Aeronautical Sciences*, vol. 18 (1951), no. 10, pp. 671-675.

3) KELLY, H. R., A Note on the Laminar Boundary Layer on a Circular Cylinder in Axial Incompressible Flow, *Journal of the Aeronautical Sciences*, vol. 21 (1954), no. 9, Readers' Forum, p. 634.

4) GLAUERT, M. B. and LIGHTHILL, M. J., The Axisymmetric Boundary Layer on a Long Thin Cylinder, *Proceedings of the Royal Society*, A, vol. 230 (1955), pp. 188-203.

5) COOPER, R. D. and TULIN, M. P., The Laminar Flow about Very Slender Cylinders in Axial Motion, Including the Effect of Pressure Gradients and Unsteady Motions, *Report* 838 (1953), The David Taylor Model Basin.

6) BATCHELOR, G. K., The Skin Friction on Infinite Cylinders Moving Parallel to Their Length, *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. VII (1954), pt. 2, pp. 179-192.

7) STEWARTSON, K., The Asymptotic Boundary Layer on a Circular Cylinder in Axial Incompressible Flow, *Quarterly of Applied Mathematics*, vol. XIII (1955), no. 2, pp. 113-122.

PROBSTEIN and ELLIOTT,[1] PAI,[2] and YASUHARA,[3] but they are not concerned primarily with the topics of the present note.

First of all, for the velocity profiles inside the boundary layer upon the surface of a circular cylinder, YOUNG (1939) assumed the same polynomial expression as upon a flat plate (see equation (33) of his paper and (3.10) in this note), and by virtue of the momentum equation, he obtained the formula for the growth of the laminar boundary layer thickness δ in the form

$$0.1175\frac{\delta^2}{d^2}+0.106\frac{\delta^3}{d^3}=\frac{4\nu x}{Ud^2}, \tag{1.1}$$

where x is distance from the front of the cylinder, whose diameter is written d or $2a$, submerged in a uniform stream of velocity U parallel to the axis of the cylinder, and ν is the kinematic viscosity of the fluid. However, neglecting $(\delta/d)^3$ compared with $(\delta/d)^2$ in equation (1.1), he was led to the result[4]

$$\delta=5.83\sqrt{\nu x/U}\,. \tag{1.2}$$

It should be noticed that (1.2) is just the same formula as for the laminar boundary layer along a two-dimensional flat plate,[5] and it was concluded, therefore, that axial flow upon a circular cylinder was similar substantially to two-dimensional flow along a flat plate.

SEBAN and BOND (1951) derived the simultaneous equations governing the stream function f and the enthalpy g, and in particular in the neighborhood of the leading edge of the cylinder by assuming series expansions in the forms

$$f(\xi',\eta')=f_0(\eta')+\xi'f_1(\eta')+\xi'^2f_2(\eta')+\cdots,^{6)} \tag{1.3}$$

and
$$g(\xi',\eta')=g_0(\eta')+\xi'g_1(\eta')+\xi'^2g_2(\eta')+\cdots, \tag{1.4}$$

in terms of
$$\xi'=\frac{4}{a}\sqrt{\frac{\nu x}{U}}, \tag{1.5}$$

and
$$\eta'=\sqrt{\frac{U}{\nu x}}\frac{r^2-a^2}{4a}, \tag{1.6}$$

r being radial distance from the axis of the cylinder, they arrived finally at the

1) PROBSTEIN, R. F. and ELLIOTT, D., The Transverse Curvature Effect in Compressible Axially Symmetric Laminar Boundary-Layer Flow, *Journal of the Aeronautical Sciences*, vol. 23 (1956), no. 3, pp. 208-224.

2) PAI, S. I., On the Boundary-Layer Equations of a Very Slender Body of Revolution, *Journal of the Aeronautical Sciences*, vol. 23 (1956), no. 8, Readers' Forum, pp. 795-796.

3) YASUHARA, M., Axisymmetric Viscous Flow Past Very Slender Bodies of Revolution, *Journal of the Aerospace Sciences*, vol. 29 (1962), no. 6, pp. 667-679.

4) YOUNG, A. D., *ibid.*, equations (35) and (36), p. 68.

5) GOLDSTEIN, S., *Modern Developments in Fluid Dynamics*, vol. I (1938), IV, **60**, p. 156 *et seq.*

6) In this note for convenience of notations appearing in later sections, ξ and η in the original paper are replaced by ξ' and η' respectively, while on the other hand, primes attached to f denote differentiations with respect to η'.

ordinary differential equations with regard to the sequence of unknowns f_n and g_n (n = 0, 1, 2,...). Then f_0 is found to agree with the non-dimensional stream function for the flow along a flat plate, and its behavior is well-known.[1] Approximate values of f_1 and f_2 were obtained by numerical integration using first-order approximations with finite intervals of 0.1 in the independent variable. Curves of the functions f_0', f_1', and f_2' so determined are shown on FIGURE 1 of their paper (p. 675, *op. cit.*), and the velocity distribution on the cylinder is given by the expression

$$\frac{u}{U} = \frac{1}{2}\left\{f_0'(\eta') + \xi' f_1'(\eta') + \xi'^2 f'_2(\eta') + \cdots\right\}, \tag{1.7}$$

where u denotes the velocity component in the direction of x increasing.

In connection with another problem, KELLY (1954) solved the differential equations containing the functions f_0, f_1, and f_2, using an analog computer that allowed continuous integration with a high degree of accuracy. Then he found that f_0 was duplicated quite accurately, while the solutions for f_1 and f_2 varied somewhat from those given by SEBAN and BOND. He also remarked that the difference was probably due to the finite interval size used in the former integration. Because unfortunately, however, results of KELLY's computations were not published except the corrected values of $f_1''(0)$, $f_2''(0)$, $f_1(\infty)$, and $f_2(\infty)$, we shall still refer to the solution by SEBAN and BOND as the exact one, bearing KELLY's remarks in mind at the same time. From FIGURE 1 of SEBAN-BOND's paper, values of $f_1'(\eta')$ and $f_2'(\eta')$ were read for η' ranging from 0 to 4 at intervals of 0.1, while $f_0'(\eta')$ was copied from TABLE 1 of HOWARTH's paper.[2] Then u/U was computed with the aid of the formula (1.7) for the following values of ξ':

$$\xi' = 0,\ 0.1,\ 0.2,\ 0.4,\ \text{and}\ 1.0\ .^{3)} \tag{1.8}$$

The result of this calculation is shown in TABLE I at the end of this note.

Finally, GLAUERT and LIGHTHILL (1955) assumed the velocity profiles in the form

$$u = \frac{U}{\alpha} \ln\left(1 + \frac{y}{a}\right), \tag{1.9}$$

where y signifies distance from the solid boundary; $(a+y)$ is therefore distance from the axis and so is equal to r mentioned before, α being a non-dimensional parameter defined in terms of the boundary layer thickness δ by the equation

1) GOLDSTEIN, S., *op. cit.*, IV, **53**, p. 135 *et seq.*, esp. cf. equations (43) and (45). f_0 corresponds to GOLDSTEIN'S f.

2) HOWARTH, L., On the Solution of the Laminar Boundary Layer Equations, *Proceedings of the Royal Society*, A, vol. 164 (1938), pp. 547–578.

3) From relative magnitude of the functions f_0', f_1', and f_2', we may suppose that these three functions are not sufficient most likely to construct for themselves velocity profiles for values of ξ' not so small. In fact, in FIGURE 8 which will be shown later, the velocity profile for $\xi' = 1$ (SEBAN-BOND-KELLY method) was drawn by smoothing out the computed points, especially in outer region of the boundary layer.

$$\delta = a(e^{\alpha}-1)\,, \tag{1.10}$$

or

$$\alpha = \ln(1+\delta/a)\,. \tag{1.11}$$

As obvious from (1.9) and (1.11), we have

$$u = U \qquad \text{at} \qquad y = \delta\,, \tag{1.12}$$

so at the outer edge of the boundary layer a join is made between the profile mentioned and the constant value U which u must take for large y. Further, with the aid of the momentum equation, the authors were enabled to derive the relation

$$\frac{4\nu x}{Ua^2} = e^{2\alpha}+3-\frac{2(e^{2\alpha}-1)}{\alpha}+\mathrm{Ei}(2\alpha)-\ln(2\alpha)-\gamma\,, \tag{1.13}$$

where γ is EULER's constant. For small values of α the right-hand side may be expanded as

$$\frac{4\nu x}{Ua^2} = \frac{1}{3}\alpha^2+\frac{4}{9}\alpha^3+\frac{3}{10}\alpha^4+O(\alpha^5)\,. \tag{1.14}$$

Defining the local skin friction τ_0 by the equation

$$\tau_0 = \mu(\partial u/\partial y)_{y=0}\,, \tag{1.15}$$

where μ denotes the coefficient of viscosity, they showed the relations between τ_0 and x derived by various authors:

$$\frac{a\tau_0}{\mu U} = 0.289\left(\frac{\nu x}{Ua^2}\right)^{-1/2}+0.667-0.750\left(\frac{\nu x}{Ua^2}\right)^{1/2}+\cdots, \tag{1.16}$$

(GLAUERT-LIGHTHILL)

$$= 0.332\left(\frac{\nu x}{Ua^2}\right)^{-1/2}+0.697-0.797\left(\frac{\nu x}{Ua^2}\right)^{1/2}+\cdots, \tag{1.17}$$

(SEBAN-BOND-KELLY)

$$= 0.564\left(\frac{\nu x}{Ua^2}\right)^{-1/2}+0.500-0.141\left(\frac{\nu x}{Ua^2}\right)^{1/2}+\cdots, \tag{1.18}$$

(the RAYLEIGH method)

$$= 0.332\left(\frac{\nu x}{Ua^2}\right)^{-1/2}+1.328+\cdots\,.^{1)} \tag{1.19}$$

(YOUNG)

2. Velocity profile

The boundary layer equations for the axisymmetric laminar flow of a uniform stream U over a long cylinder of radius a may be written

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{a+y} = 0\,, \tag{2.1}$$

---

1) GLAUERT, M. B. and LIGHTHILL, M. J., *op. cit.*, pp. 192–193. Concerning (1.19), however, see footnote 1) on p. 22.

and
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{1}{a+y}\frac{\partial u}{\partial y}\right), \tag{2.2}$$

where x and y are the directions parallel, and perpendicular, to the axis of the cylinder, u and v being the velocity components, axial and radial, respectively. Associated boundary conditions read

$$u=v=0 \quad \text{at} \quad y=0\,, \tag{2.3}$$

and
$$u\to U \quad \text{as} \quad x\to 0 \quad \text{or as} \quad y\to\infty\,. \tag{2.4}$$

Putting $y=0$ in (2.2), we see that

$$\frac{\partial^2 u}{\partial y^2}+\frac{1}{a}\frac{\partial u}{\partial y}=0 \quad \text{at} \quad y=0\,. \tag{2.5}$$

Differentiating (2.2) with respect to y, and then putting $y=0$ again, we obtain further

$$\frac{\partial^3 u}{\partial y^3}+\frac{1}{a}\frac{\partial^2 u}{\partial y^2}-\frac{1}{a^2}\frac{\partial u}{\partial y}=0 \quad \text{at} \quad y=0\,. \tag{2.6}$$

Equations (2.5) and (2.6) show that as $y\to 0$, u may be expressed in the form

$$u=A(x)\left\{\frac{y}{a}-\frac{y^2}{2a^2}+\frac{y^3}{3a^3}+O\left(\frac{y^4}{a^4}\right)\right\},$$

or
$$=A(x)\left\{\ln\left(1+\frac{y}{a}\right)+O\left(\frac{y^4}{a^4}\right)\right\}. \tag{2.7}$$

If the terms of $O(y^4/a^4)$ may be omitted, (1.9) can be derived by determining $A(x)$ from condition (1.11). Combining both of these, we may write also

$$\frac{u}{U}=\frac{\ln(1+\sigma\eta)}{\ln(1+\sigma)}, \tag{2.8}$$

where for short we put

$$\sigma=\delta/a \quad \text{and} \quad \eta=y/\delta\,. \tag{2.9}$$

FIGURE 1 shows u/U given by (2.8) for various values of σ. Now it should be remarked that:

i) In contrast with simplification used by YOUNG, the velocity profile proposed by GLAUERT and LIGHTHILL changes its form gradually as δ or, in other words, as x increases, which is in accordance with the results obtained from the exact computations by SEBAN-BOND and KELLY; see TABLE I.

ii) In the limit $\sigma\to 0$, namely at the front of the cylinder, the velocity distribution across the section of the boundary layer is found to vary linearly as η (or y), according to assumption (2.8). However, as readily supposed physically, and in fact verified mathematically by SEBAN and BOND in (1.7), at the front of the cylinder the boundary layer should start without being affected by transverse curvature. So at the initial stage of development of the boundary layer the velocity profile specified by (2.8) might have the same order of accuracy as the linear profile ($u/U=\eta$) exemplified by POHLHAUSEN with a view to comparing results

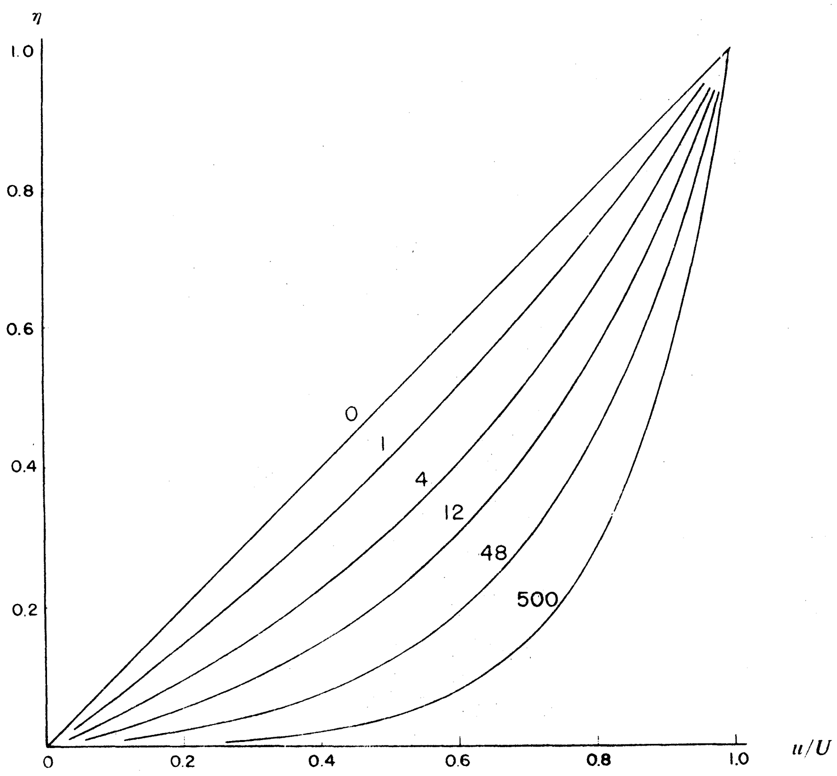

FIGURE 1. Family of velocity profiles due to GLAUERT and LIGHTHILL.
The number attached to each curve denotes the corresponding value of σ.

yielded from various forms of assumed velocity profiles.[1]

iii) In determining the form of the assumed velocity profile, GLAUERT and LIGHTHILL took account of only condition (1.12) at the outer edge of the boundary layer. Other conditions, namely smooth join of the first, and the second, derivatives of the profile at the outer edge, are both disregarded by these authors.[2]

Considering three points mentioned above, there arises a question which follows: "Can we find a velocity profile which starts at the front of the cylinder from two-dimensional flat-plate-flow-pattern more accurate than the linear profile, satisfies the condition of smooth join at $y = \delta$, and still remains convenient for subsequent computations?"

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1) GOLDSTEIN, S., *op. cit.*, IV, **60**, p. 157, TABLE. In connection with this table, the numerical coefficient of the first term of (1.16) comes from $u/U = \eta$. Further, according to the personal opinion of the present writer, (1.19) should read $0.343\,(\nu x/Ua^2)^{-1/2}+\cdots$.

2) They pointed out, however, that this in itself does not necessarily lead to serious errors in POHLHAUSEN treatments, cf. GLAUERT, M. B. and LIGHTHILL, M. J., *op. cit.*, p. 191.

With a view to generalizing YOUNG's formula, let us assume the velocity distributions across sections of the boundary layer in the form of a polynomial

$$u/U \equiv F(\eta) = k\eta + l\eta^2 + m\eta^3 + n\eta^4 \,, \tag{2.10}$$

where η denotes non-dimensional radial length, while k, l, m, and n are functions of x, axial distance, or, in other words, of σ, the non-dimensional parameter. η and σ were both defined by (2.9). At the outer edge of the boundary ($\eta = 1$), it should be that

$$\text{i)} \quad u = U \quad \text{or} \quad F\,(1) = 1\,, \tag{2.11}$$

$$\text{ii)} \quad \partial u/\partial y = 0 \quad \text{or} \quad F'\,(1) = 0\,, \tag{2.12}$$

and

$$\text{iii)} \quad \partial^2 u/\partial y^2 = 0 \quad \text{or} \quad F''(1) = 0\,, \tag{2.13}$$

where primes attached to F denote differentiations with respect to η; (2.12) and (2.13) are requirements of so-called smooth join discarded by GLAUERT and LIGHTHILL. The remaining one condition is concerned with the surface of the cylinder. Namely from (2.5) we obtain

$$F''(0) + \sigma F'(0) = 0\,. \tag{2.14}$$

These four equations (2.11)–(2.14) altogether suffice to determine four coefficients k, l, m, and n; namely

$$\begin{aligned} k &= \frac{12}{6-\sigma}\,, & l &= -\frac{6\sigma}{6-\sigma}\,, \\ m &= \frac{8\sigma-12}{6-\sigma}\,, \quad \text{and} & n &= \frac{6-3\sigma}{6-\sigma}\,. \end{aligned} \tag{2.15}$$

When $\sigma \to 0$, we have accordingly

$$k = 2,\quad l = 0,\quad m = -2,\quad \text{and} \quad n = 1\,, \tag{2.16}$$

so in this case $F(\eta)$ is found to agree with the polynomial of degree four used by POHLHAUSEN in approximating the velocity profile inside the boundary layer along a flat plate,[1] and our expression might be expected more accurate than the logarithmic distribution (2.8), near the front of the cylinder at least. In FIGURE 2, u/U thus determined is shown for various values of σ.

As readily noticed in this figure:

i) The series of profiles produced in this way varies monotonically with σ, just like those of GLAUERT and LIGHTHILL; see FIGURE 1.

ii) For values of σ exceeding about 4, a region of η takes place in each of the profile in which $u > U$.[2] This is by no means acceptable, and so our expression embodied in (2.10) and (2.15) can be appropriate only for the region of σ sufficiently less than 4. In practical cases, however, σ is usually quite small and this feature will not be so objectionable.

If, on the other hand, k, l, m, and n were determined tentatively in virtue

1) GOLDSTEIN, S., *op. cit.*, IV, **60**, p. 156 *et seq.*

2) This is caused by assuming a polynomial of fixed degree for the velocity profile.

of the condition (2.6) used by GLAUERT and LIGHTHILL instead of (2.13), then we should obtain the followings:

$$
\begin{aligned}
k &= \left(\frac{3}{4} - \frac{\sigma}{4} + \frac{\sigma^2}{12}\right)^{-1}, \\
l &= -\frac{\sigma}{2}\left(\frac{3}{4} - \frac{\sigma}{4} + \frac{\sigma^2}{12}\right)^{-1}, \\
m &= \frac{\sigma^2}{3}\left(\frac{3}{4} - \frac{\sigma}{4} + \frac{\sigma^2}{12}\right)^{-1},
\end{aligned} \tag{2.17}
$$

and
$$n = -\frac{1-\sigma+\sigma^2}{4}\left(\frac{3}{4} - \frac{\sigma}{4} + \frac{\sigma^2}{12}\right)^{-1}.$$

Corresponding to the limiting cases $\sigma \to 0$ and $\to \infty$, we have

$$F = \frac{4}{3}\eta - \frac{1}{3}\eta^4, \tag{2.18}$$

and
$$F = 4\eta^3 - 3\eta^4, \tag{2.19}$$

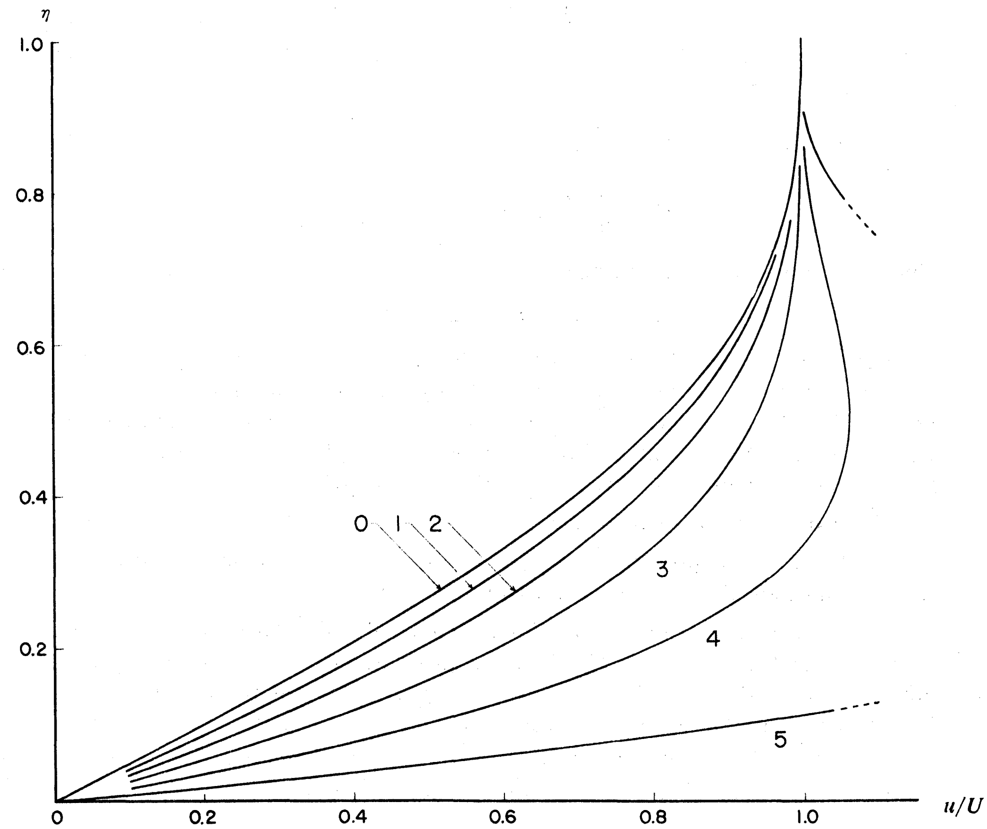

FIGURE 2. Family of velocity profiles constructed by the present method. The number attached to each curve denotes the corresponding value of σ.

respectively. So the polynomial thus selected cannot be said suitable to approximate the velocity profiles in the boundary layer.

3. Momentum equation

The equation of momentum of the axial flow of a fluid surrounding the cylinder is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_0^{\delta}\rho u(U-u)2\pi(a+y)\mathrm{d}y\right\}=2\pi a\tau_0\,, \tag{3.1}$$

where ρ and τ_0 are the density of the fluid and the skin friction defined by (1.15), respectively. However, by virtue of (2.9) and (2.10), we have readily

$$\tau_0=\rho\nu Uk/\delta\,, \tag{3.2}$$

and (3.1) may be rewritten

$$\frac{U\delta}{\nu k}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\delta\int_0^1\frac{u}{U}\left(1-\frac{u}{U}\right)\left(1+\frac{\delta}{a}\eta\right)\mathrm{d}\eta\right\}=1\,, \tag{3.3}$$

or, in terms of σ introduced in (2.9),

$$\frac{Ua^2}{\nu}\frac{\sigma}{k}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sigma\int_0^1\frac{u}{U}\left(1-\frac{u}{U}\right)\left(1+\sigma\eta\right)\mathrm{d}\eta\right\}=1\,. \tag{3.4}$$

After some elementary computations we can derive, through (2.10) and (2.15), the result

$$\int_0^1\frac{u}{U}\left(1-\frac{u}{U}\right)(1+\sigma\eta)\mathrm{d}\eta=\frac{888-12\sigma-92\sigma^2+10\sigma^3}{210(6-\sigma)^2}\,, \tag{3.5}$$

and finally equation (3.4) is integrated into

$$\begin{aligned}\frac{\nu x}{Ua^2}&=\frac{1}{2520}\int_0^{\sigma}\left\{\frac{19872-7344\sigma}{(6-\sigma)^2}\right.\\&\qquad\left.-552+168\sigma+92\sigma^2-20\sigma^3\right\}\mathrm{d}\sigma\\&=\frac{1}{2520}\left[7344\,\{\ln 6-\ln(6-\sigma)\}-24192\left(\frac{1}{6-\sigma}-\frac{1}{6}\right)\right.\\&\qquad\left.-552\sigma+84\sigma^2+\frac{92}{3}\sigma^3-5\sigma^4\right]\,.\end{aligned} \tag{3.6}$$

When σ is small, however, by expanding the right-hand side,

$$\frac{\nu x}{Ua^2}=\frac{1}{2520}\left(74\sigma^2+\frac{70}{3}\sigma^3-\frac{241}{36}\sigma^4+\cdots\right), \tag{3.7}$$

approximately. Further if we take only the first term of (3.7), we have

$$\frac{\nu x}{Ua^2}=\frac{74}{2520}\frac{\delta^2}{a^2}\,, \tag{3.8}$$

or

$$\delta=5.83\sqrt{\nu x/U}\,, \tag{3.9}$$

which is the well-known formula of POHLHAUSEN for the flow along a flat plate.

This shows once again that the boundary layer starts at the front of the cylinder just like in two-dimensional flow unaffected by its transverse curvature.

As mentioned before (cf. section 1), YOUNG assumed

$$u/U = 2\eta - 2\eta^3 + \eta^4 , \tag{3.10}$$

invariably inside the boundary layer irrespective of its growth with increasing x, and by substituting (3.10) in the momentum integral (3.1), we may show quite easily that

$$\frac{\nu x}{Ua^2} = \frac{74}{2520}\sigma^2 + \frac{10}{756}\sigma^3 , \tag{3.11}$$

or approximately[1])

$$\frac{\nu x}{Ua^2} = \frac{74}{2520}\frac{\delta^2}{a^2} , \tag{3.8$'$}$$

which are essentially the same equations as (1.1) and (1.2) respectively.

On the other hand, the formula derived by GLAUERT and LIGHTHILL is (1.13), or (1.14) provided that α be very small. The connection between α and σ can be established with the aid of (1.11), namely

$$\alpha = \ln(1+\sigma) , \tag{3.12}$$

or approximately $$\alpha = \sigma - \frac{\sigma^2}{2} + \frac{\sigma^3}{3} - \frac{\sigma^4}{4} + \cdots . \tag{3.13}$$

If use is made of this relation, (1.14) can be rewritten

$$\frac{\nu x}{Ua^2} = \frac{\sigma^2}{12} + \frac{\sigma^3}{36} - \frac{11}{720}\sigma^4 + \cdots , \tag{3.14}$$

and in contrast with (3.9), the boundary layer should begin in the form

$$\delta = 3.46\sqrt{\nu x/U} , \tag{3.15}$$

at the front of the cylinder.[2])

In FIGURE 3, the relation between σ and $\nu x/Ua^2$ are shown for various formulas: present calculation (3.6), its series expansion up to the order of σ^4 (3.7), two-dimensional POHLHAUSEN (3.8) or (3.8)$'$, YOUNG (3.11) together with GLAUERT-LIGHTHILL (1.13), as I, II, III, IV, and V respectively.

In addition to these, the simplified formula due to GRANVILLE for estimating boundary layer thickness δ on a surface of revolution as a function of arc length x measured from the forward stagnation,[3]) namely

1) As readily supposed, YOUNG's theory should produce the same solution as the present method in the immediate neighborhood of the leading edge, cf. (3.8) and (3.8)$'$.

2) GOLDSTEIN, S., *op. cit.*, IV, **60**, p. 157. See the first line of the TABLE.

3) GRANVILLE, P. S., The Calculation of the Viscous Drag of Bodies of Revolution, *Report* 849 (1953), The David Taylor Model Basin. Equation (3.16) is derived by integrating equation (27) of the original paper (p. 8).

$$(r_w\delta)^2 = 16\,\frac{\nu}{U^6}\int_0^x r_w{}^2U^5\mathrm{d}x\,, \tag{3.16}$$

r_w being the transverse radius to the surface of the body of revolution, may be applied to our problem by specifying

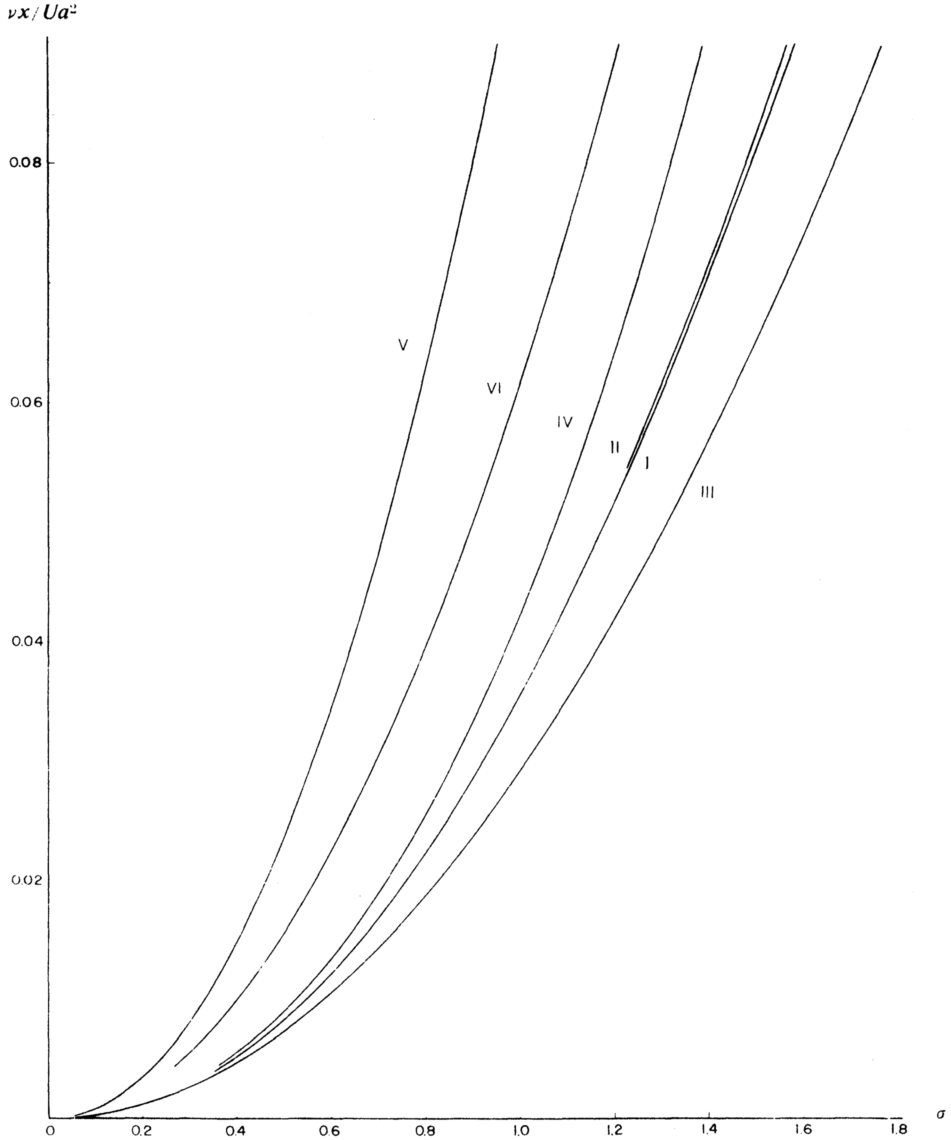

FIGURE 3. Relation between σ and $\nu x/Ua^2$.

I (present method), II (its approximate expression), III (POHLHAUSEN's method, two-dimensional), IV (YOUNG), V (GLAUERT-LIGHTHILL), and VI (GRANVILLE).

$$r_w = a \quad \text{and} \quad U = \text{constant}\,. \tag{3.17}$$

After integration we obtain

$$\frac{\nu x}{Ua^2} = \frac{\sigma^2}{16}, \tag{3.18}$$

in terms of our notations. In FIGURE 3, the curve corresponding to (3.18) is denoted by VI.

In concluding this section, it would be of some interest to derive from our POHLHAUSEN's polynomial method the expression corresponding to (1.16) through (1.19). From the definition of τ_0, the skin friction, $a\tau_0/\mu U$ may be readily found to be equivalent with k/σ, or $12/\sigma(6-\sigma)$ from (2.15), and expanding in terms of σ, we obtain

$$\frac{a\tau_0}{\mu U} = \frac{2}{\sigma} + \frac{1}{3} + \frac{\sigma}{18} + \cdots\,. \tag{3.19}$$

On the other hand, equation (3.7) may be inverted into the form

$$\sigma = 5.836\left(\frac{\nu x}{Ua^2}\right)^{1/2} - 5.369\left(\frac{\nu x}{Ua^2}\right) + 21.338\left(\frac{\nu x}{Ua^2}\right)^{3/2} + \cdots, \tag{3.20}$$

and by combining (3.19) with (3.20), after some computations we can arrive at

$$\frac{a\tau_0}{\mu U} = 0.343\left(\frac{\nu x}{Ua^2}\right)^{-1/2} + 0.649 - 0.639\left(\frac{\nu x}{Ua^2}\right)^{1/2} + \cdots, \tag{3.21}$$

an expression starting with the same coefficient as in YOUNG's formula, but on the whole quite similar to that due to SEBAN, BOND, and KELLY.

4. Comparison of velocity profiles

For the non-dimensional axial distances

$$\xi' = 0,\ 0.1,\ 0.2,\ 0.4,\ \text{and}\ 1.0\,, \tag{4.1}$$

the velocity profiles in the boundary layer worked out from various theories are shown in FIGURES 4, 5, 6, 7, and 8 respectively for the purpose of comparison. From definition of ξ', cf. (1.5), the corresponding values of x may be found respectively in the form

$$\nu x/Ua^2 = 0,\ 0.000625,\ 0.0025,\ 0.01,\ \text{and}\ 0.0625\,. \tag{4.2}$$

In the first place, the velocity profiles computed by the procedure due to SEBAN, BOND, and KELLY are reproduced from TABLE I; they are expressed in non-dimensional form in terms of u/U and η', cf (1.6), chosen as the abscissa and the ordinate respectively. In all other methods mentioned in the preceding sections, however, u/U is assumed as a function of another variable η defined by the equation (2.9), so we have to establish the connection between η and η' in order to carry out comparison.

Writing
$$r = a + y \tag{4.3}$$
in (1.6), we have

$$\eta' = \sqrt{\frac{U}{\nu x}}\left(\frac{y}{2} + \frac{y^2}{4a}\right), \tag{4.4}$$

and by introducing σ and η, we obtain further

$$\eta' = \sqrt{\frac{U}{\nu x}}\frac{\sigma a}{2}\eta\left(1+\frac{\sigma}{2}\eta\right).$$

Making use of (1.5), we can arrive finally at the relation

$$\eta' = \frac{2}{\xi'}\sigma\eta\left(1+\frac{\sigma}{2}\eta\right). \tag{4.5}$$

For given ξ' and η corresponding η' may be found from this equality, provided σ is known which is appropriate to the prescribed value of ξ' (or x). This can be calculated by virtue of the momentum equation, namely from (3.11) or approximately (3.8)′ (YOUNG), from (1.13) or approximately (3.14) (GLAUERT and LIGHTHILL), and from (3.6) or approximately (3.7) (the present writer). The values of σ computated in this way are shown altogether in the following table; however only the exact formulas were employed in order to retain conformity.

ξ'	YOUNG	GLAUERT and LIGHTHILL	OKABE
0	0	0	0
0.1	0.14145	0.08545	0.14284
0.2	0.27522	0.16890	0.28060
0.4	0.52481	0.33120	0.54553
1.0	1.17903	0.79423	1.30771
formulas	(3.11)	(1.13)	(3.6)

The followings are readily remarked in these figures:

i) Both of the methods of SEBAN-BOND-KELLY and of the present writer yield practically the same velocity profiles in the range of ξ' under examination.

ii) On the other hand, YOUNG's theory in which the flat-plate-type of velocity profiles is assumed invariably along the surface is found to produce the profiles coincident almost completely with those from the above two. Incidentally, it is quite interesting to note that in YOUNG's method the error of neglect of variation of velocity profile is just compensated eventually by that involved in evaluating σ for a given value of x.

iii) Finally the velocity profiles derived from GLAUERT and LIGHTHILL's formula show some discrepancy from the other three, especially in outer region of boundary layer, as was expected by the authors.

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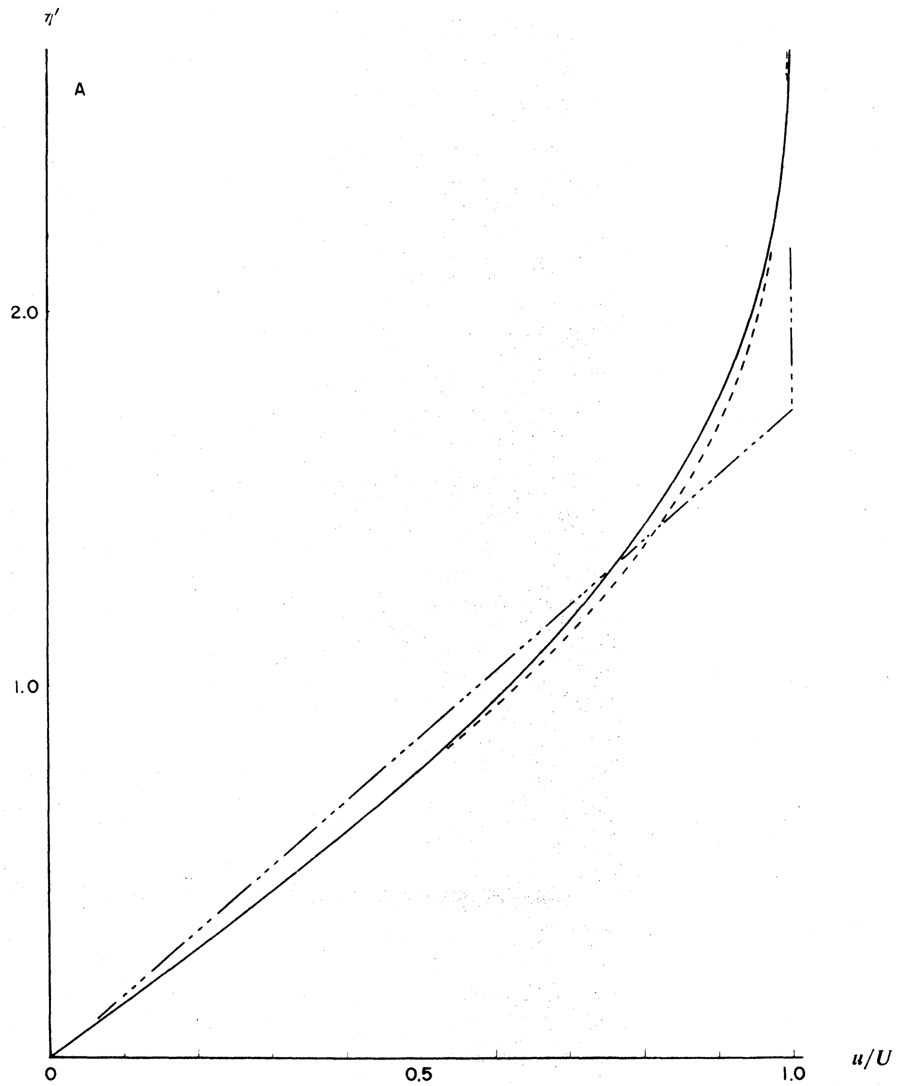

FIGURE 4. Comparison of velocity profiles ($\xi' = 0$).

– – – – – – – – – – – SEBAN, BOND, and KELLY,

· · —— · YOUNG,

—— · · · —— · · · —— GLAUERT and LIGHTHILL,

———————— Present writer.

In FIGURE 4, YOUNG's curve coincides completely with that of the present writer. In this and the following figures, curves or parts of curves practically coincident with full lines are omitted to avoid confusions.

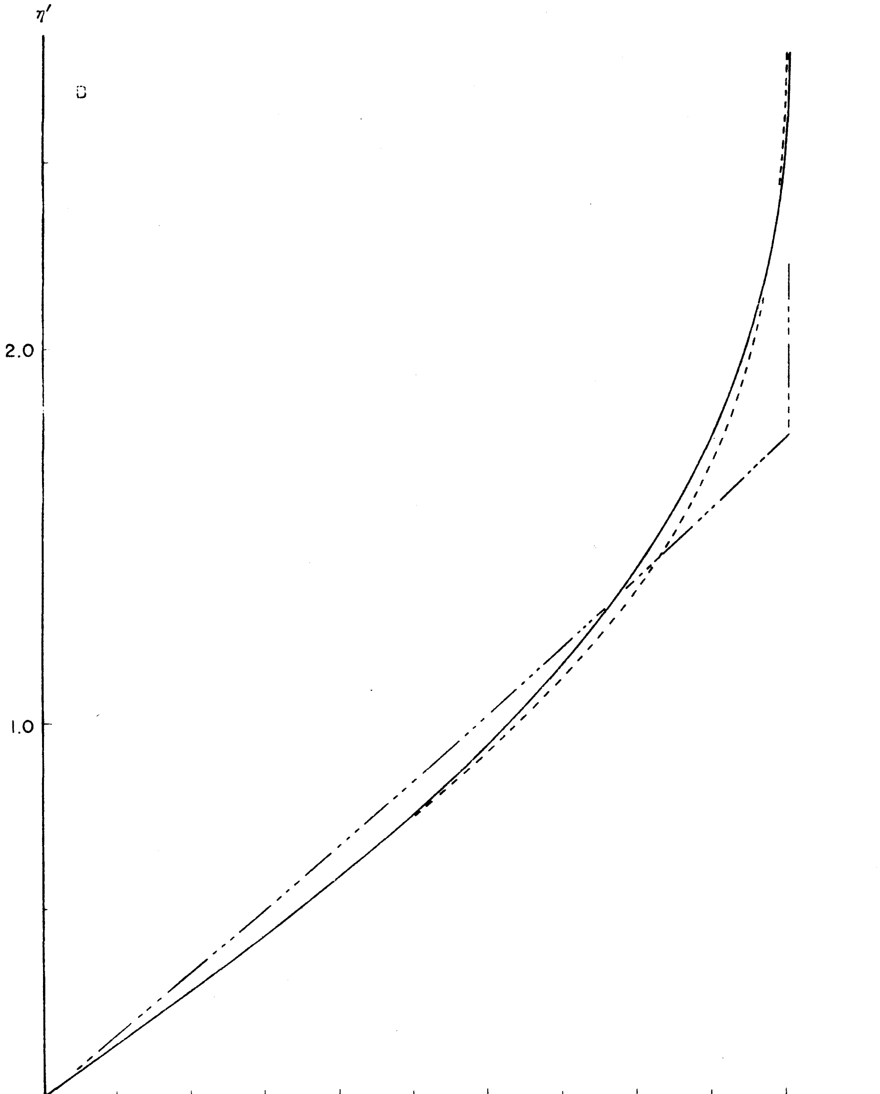

FIGURE 5. Comparison of velocity profiles ($\xi' = 0.1$).

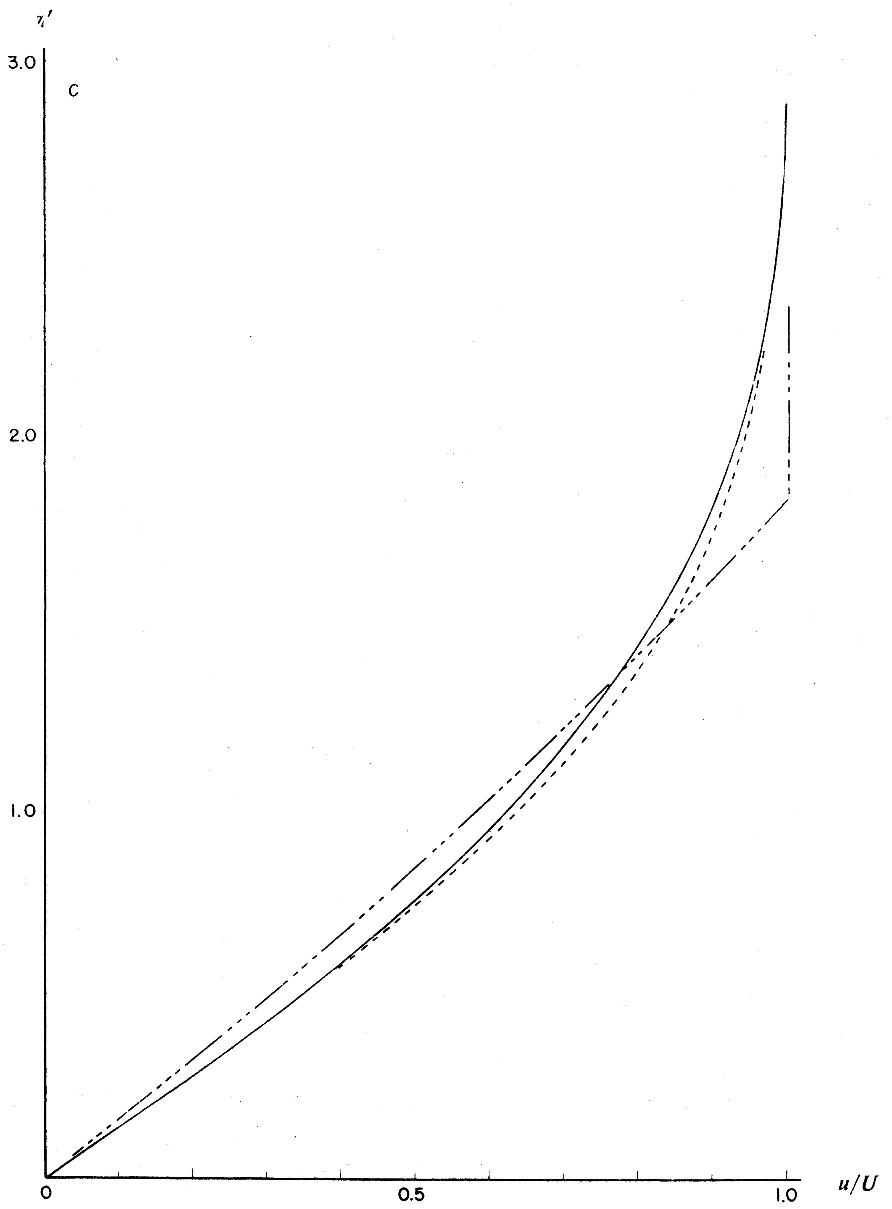

FIGURE 6. Comparison of velocity profiles ($\xi' = 0.2$).

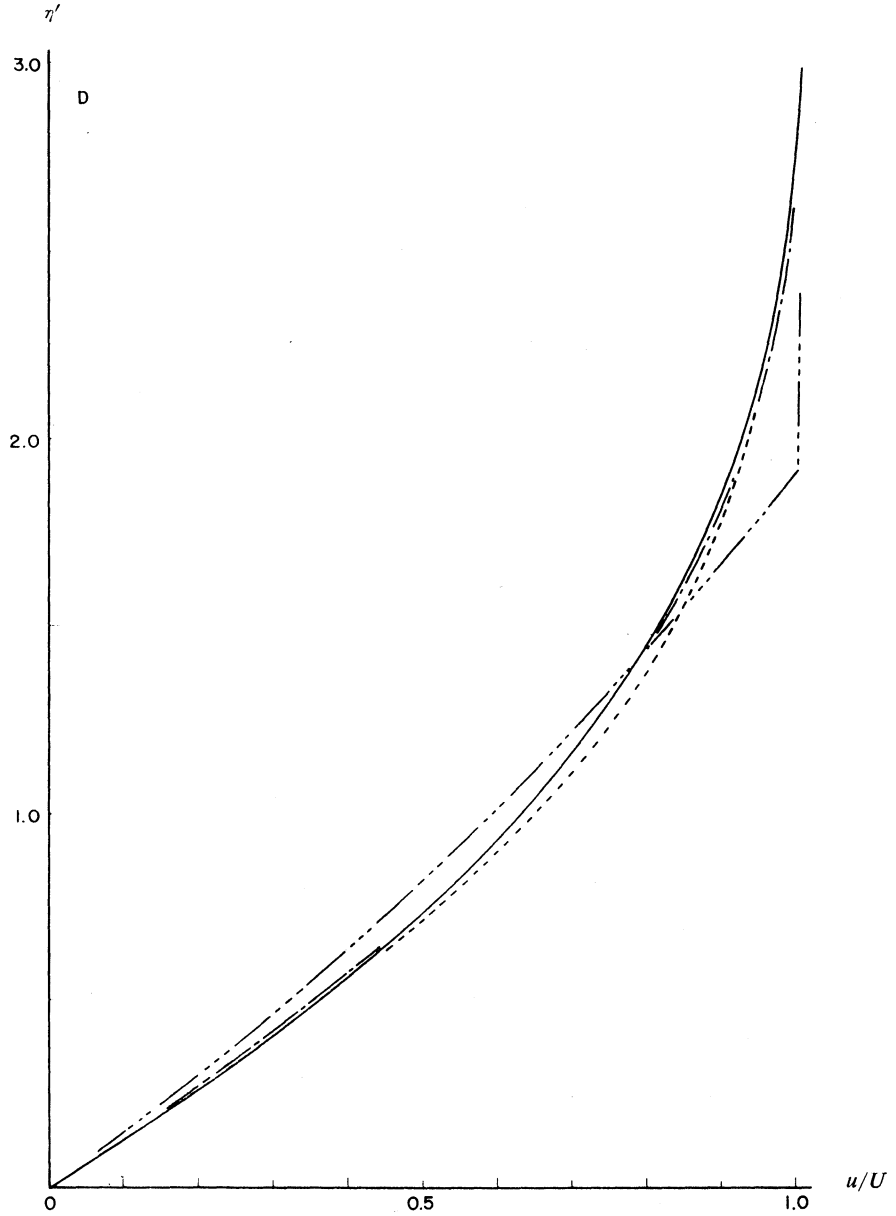

FIGURE 7. Comparison of velocity profiles ($\xi' = 0.4$).

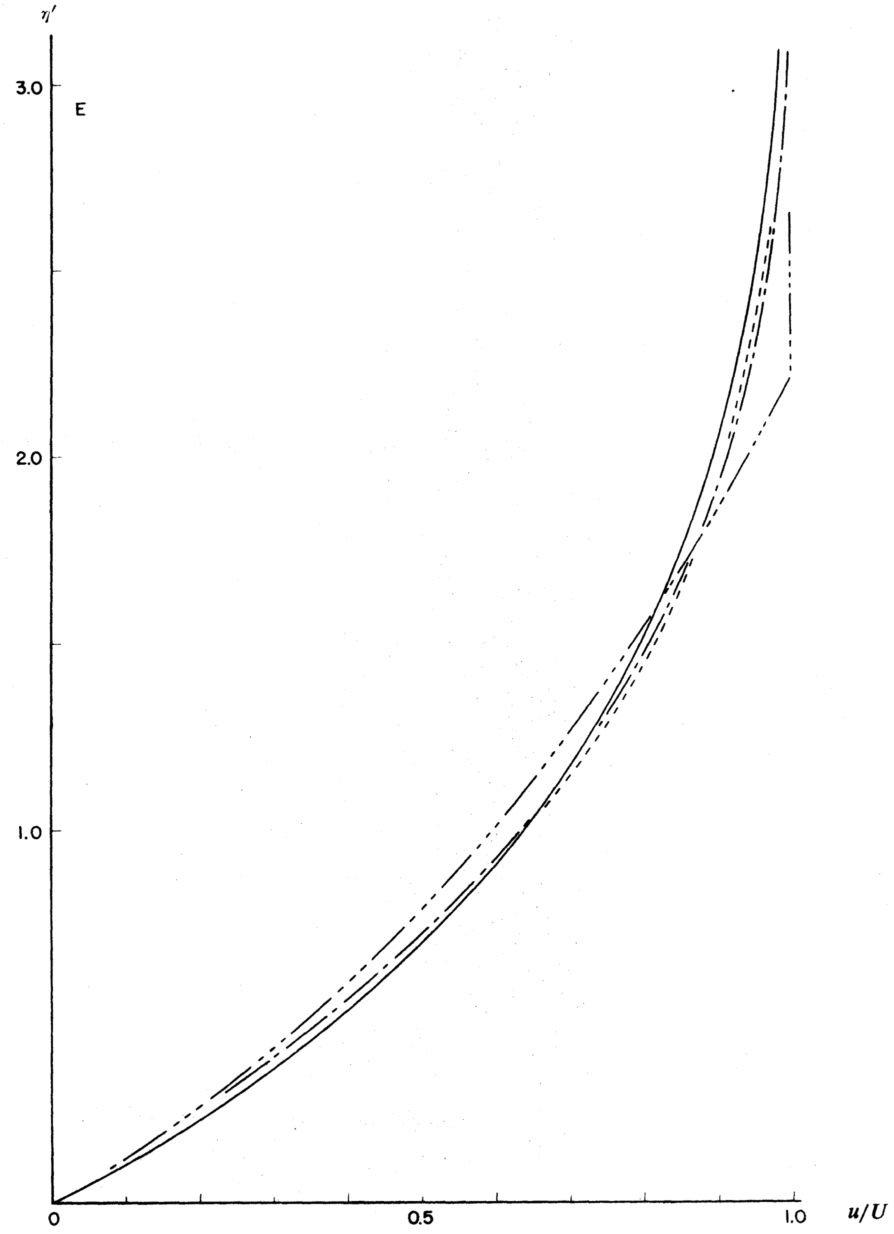

FIGURE 8. Comparison of velocity profiles ($\xi' = 1$).

η' \ ξ'	0	0.1	0.2	0.4	1.0
0.0	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0664	0699	0732	0796	0964
0.2	1328	1391	1452	1564	1828
0.3	1989	2072	2149	2289	2589
0.4	2647	2743	2831	2983	3247
0.5	0.3298	0.3403	0.3498	0.3658	0.3898
0.6	3938	4042	4134	4282	4438
0.7	4563	4661	4747	4879	4963
0.8	5168	5251	5323	5428	5443
0.9	5748	5822	5884	5972	5948
1.0	0.6298	0.6353	0.6398	0.6458	0.6398
1.1	6813	6844	6868	6893	6788
1.2	7290	7302	7310	7310	7190
1.3	7725	7724	7723	7717	7675
1.4	8115	8096	8077	8043	7965
1.5	0.8461	0.8422	0.8387	0.8325	0.8211
1.6	8761	8703	8651	8561	8411
1.7	9018	8951	8890	8786	8618
1.8	9233	9161	9095	8981	8783
1.9	9411	9339	.9273	9159	8961
2.0	0.9555	0.9483	0.9415	0.9295	0.9055
2.1	9670	9602	9538	9422	9170
2.2	9759	9698	9640	9533	9284
2.3	9827	9778	9731	9643	9427
2.4	9878	9833	9788	9698	9428
2.5	0.9915	0.9877	0.9838	0.9757	0.9490
2.6	9942	9911	9878	9806	9542
2.7	9962	9936	9908	9846	9612
2.8	9975	9959	9943	9907	9775
2.9	9984	9974	9964	9944	9884
3.0	0.9990	0.9985	0.9982	0.9978	0.9990
3.1	9994	9989	0.9986	0.9982	0.9994
3.2	9996	9997	0.9998	1.0004	1.0046
3.3	9998	9998	1.0000	1.0006	1.0048
3.4	9999	9999	0.9999	0.9999	0.9999
3.5	0.9999	0.9999	0.9999	0.9999	0.9999
3.6	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE I. Values of u/U computed by the method due to SEBAN, BOND, and KELLY.