ON THE STABILITY TO LONG WAVES OF TWO-DIMENSIONAL LAMINAR JET

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ON THE STABILITY TO LONG WAVES OF TWO-DIMENSIONAL LAMINAR JET

By Hikoji YAMADA

Synopsis Approximation of laminar jet by a discontinuous flow model in the problem of two-dimensional stability to long waves of disturbance is investigated, and with this model an infinite number of neutral stability curves, including the one which has been known up to now, is found. Accuracy of those curvcs as approximations to the ones of a jet of continuous velocity profile is discussed.

§1 Approximation by discontinuous flow In this paper we investigate the stability characteristics of a jet flow disturbed by the superposition of infinitesimal amplitude waves of small wave number. In this case the base flow profile of a jet in the differential equation governing stability can be simplified as in the followings:

In its non-dimensional form the stream function

$$\psi = \varphi(y)e^{i\alpha(x-ct)}$$

of the two-dimensional perturbation superposed on a parallel base flow satisfies the Orr-Sommerfeld equation:

$$\ddot{\ddot{\varphi}} - 2\alpha^2\ddot{\varphi} + \alpha^4\varphi - i\alpha R\{(w-c)(\ddot{\varphi}-\alpha^2\varphi) - \ddot{w}\varphi\} = 0\,; \tag{1}$$

here $w(y)$ is the velocity distribution of the base flow, its value $w(o)=1$ at $y=0$ decreases monotonically and vanishes sufficiently rapidly as y tends to $\pm\infty$. α is the wave number of the periodic disturbance; R is the Reynolds number composed of the half width of the jet, the velocity on the center line of the jet, and the kinematic viscosity of the fluid. Dot over letter denotes the differentiation with respect to y. If we change the coordinate y into η by $\eta=\alpha y$, (1) becomes

$$L(\varphi) \equiv (\varphi'''' - 2\varphi'' + \varphi) - iK\{(u-c)(\varphi''-\varphi) - u''\varphi\} = 0\,, \tag{1'}$$

where

$$u(\eta) = w(y)_{\,y=\eta/\alpha}\,, \quad u(0) = 1,\ K = R\alpha^{-1}\,,$$

and dash over letter denotes the differentiation with respect to η. The domain of existence of $u(\eta)$ is, by the particular choice of the characteristic length, restricted almost in the domain $\eta(-\alpha,\ +\alpha)$.

As is mentioned above our study is in the case of small wave number α. The width 2α of the jet is then small and, if the extent of the amplitude function $\varphi(\eta)$ is sufficiently large and even, the functional form of $u(\eta)$ will have little effect on the character of $\varphi(\eta)$. Therefore we assume the simplest form:

$$u(\eta)=\begin{cases}1 & (|\eta|\leq\alpha-\varepsilon),\\ 0 & (|\eta|\geq\alpha+\varepsilon),\end{cases} \tag{2}$$

and between $\alpha-\varepsilon$ and $\alpha+\varepsilon$ $u(\eta)$ drops from 1 to zero monotonically in some way or another, ε being comparable with or smaller than the thickness α.

In the domains $|\eta|\leq\alpha-\varepsilon$ and $|\eta|\geq\alpha+\varepsilon$ (1′) reduces separately into

$$\left.\begin{aligned}&\varphi''''-2\varphi''+\varphi-iK(1-c)(\varphi''-\varphi)=0 \quad (|\eta|\leq\alpha-\varepsilon),\\ &\varphi''''-2\varphi''+\varphi+iKc(\varphi''-\varphi)=0 \quad (|\eta|\geq\alpha+\varepsilon),\end{aligned}\right\} \tag{2′}$$

and the separate solutions φ's of these two equations have to be joined by the φ-functions of the intermediate domains $(\alpha-\varepsilon,\ \alpha+\varepsilon)$ and $(-\alpha-\varepsilon,\ -\alpha+\varepsilon)$. Rather we intend to join up our φ-functions directly, jumping the intermediate domains. In this case the joining condition is obtained as follows:

We consider a single transition layer $\eta(-\varepsilon,\ +\varepsilon)$; under and above this layer uniform flows $u=U_1$ and $u=U_2$ exist. Then

$$\eta=-\varepsilon:\quad u=U_1,\quad u'=0,$$
$$\eta=+\varepsilon:\quad u=U_2,\quad u'=0,$$

and φ-functions to be joined up are

$$\eta\leq-\varepsilon:\quad \varphi=A_1e^{\eta}+B_1^{-\eta}+C_1e^{k_1\eta}+D_1e^{-k_1\eta},$$
$$\eta\geq\varepsilon:\quad \varphi=A_2e^{\eta}+B_2e^{-\eta}+C_2e^{k_2\eta}+D_2e^{-k_2\eta},$$

where

$$k_1=\sqrt{1+iK(U_1-c)},\quad k_2=\sqrt{1+iK(U_2-c)}.$$

Here we assume that $K\gtrsim1$, but ε is so small that $\sqrt{K}\,\varepsilon\ll1$ holds. We assume also the order estimation:

$$|\varphi|\sim1,\ |\varphi'|\sim\sqrt{K},\ |\varphi''|\sim K,\ |\varphi'''|\sim\sqrt{K^3}, \tag{3}$$

over the transition layer, which is expected only when even extent of φ-function is realised.

Velocity of the trasition layer $u(\eta)$ satisfies

$$|u(\eta)|\sim1,\qquad u'(\eta)\sim\varepsilon^{-1},$$

and the equation for φ in this layer is

$$L(\varphi)\equiv\varphi''''-2\varphi''+\varphi-iK\{(\overline{u-c}\varphi'-u'\varphi)'-\overline{u-c}\varphi\}=0.$$

We integrate this equation multiplied by η^n between $\eta(-\varepsilon,\ \varepsilon)$ and estimate the orders of obtained terms one by one. Taking the lowest terms only we have the estimation:

$$\left.\begin{aligned}&n=3:\quad [\varphi]=0(\sqrt{K}\varepsilon),\\ &n=2:\quad [\varphi']=0(K\varepsilon),\end{aligned}\right. \tag{4}$$

$$\left.\begin{aligned} n=1: &\quad [\varphi'']+iK(U_2-U_1)\varphi(\varepsilon)=0(\sqrt{K^3}\varepsilon)\,, \\ n=0: &\quad [\varphi''']-iK(U_2-U_1)\varphi'(\varepsilon)=0(K^2\varepsilon)\,, \end{aligned}\right\}$$

the symbol $[f]$ being used for $f(\varepsilon)-f(-\varepsilon)$.

Transforming then the terms on the left-hand sides of these equations into quantities of the zero order, taking (3) into consideration and dividing by the powers of $\sqrt{K}$, we have

$$\left.\begin{aligned} &[\varphi]=0(\sqrt{K}\varepsilon)\,, \\ &K^{-1/2}[\varphi']=0\sqrt{K}\varepsilon)\,, \\ &K^{-1}[\varphi'']+i(U_2-U_1)\varphi(\varepsilon)=0(\sqrt{K}\varepsilon)\,, \\ &K^{-3/2}[\varphi''']-i(U_2-U_1)K^{-1/2}\varphi'(\varepsilon)=0(\sqrt{K}\varepsilon)\,. \end{aligned}\right\} \tag{4'}$$

If the breadth of the transition layer ε is δ in the original y-length, $\varepsilon=\alpha\delta$ and then $\sqrt{K}\,\varepsilon$ is equal to $\sqrt{\alpha R}\,\delta$ in the original notation. We thus have the results: if $\delta\ll(\alpha R)^{-1/2}$ we can take $\delta\doteqdot 0$, i. e. $\varepsilon\doteqdot 0$, and neglect the existence of the transition layer, the joining condition becoming now

$$\left.\begin{aligned} &[\varphi]=0\,, \\ &[\varphi']=0\,, \\ &[\varphi'']+iK[U]\varphi=0\,, \\ &[\varphi'']-iK[U]\varphi'=0\,. \end{aligned}\right\} \tag{4''}$$

To solve (2′) under the condition (4″) is the approximate procedure for the cases of small wave number, which has been introduced by G. P. Drazin.[(1)] The derivation here given differs a little from his and, we think, deserves mentioning anew. The condition for good approximation is $\delta\ll(\alpha R)^{-1/2}$ under the condition $R\gtrsim\alpha$, and then good in the cases:

(i) $\quad \delta\ll\dfrac{1}{\sqrt{R}}\cdot\alpha^{-1/2}$ when R is const.,

(ii) $\quad \delta\ll\dfrac{1}{\sqrt{\alpha R}}\cdot R^{1/2}$ when $\sqrt{\alpha}\,R$ is const.,

(iii) $\quad \delta\ll\dfrac{1}{\sqrt{\alpha R}}$ when αR is const.

As the breath δ of the transition layer cannot be so small from physical grounds, and R is not so small in ordinary practices, that cases (i) and (iii) have ***in general*** only mathematical interests. On the contrary the case (ii) has sufficient physical meanings and the results of our approximate calculations are applicable to ordinary jet problems.

§2 **Antisymmetrical disturbances (1).** As the antisymmtrical disturbance has

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(1) G. P. Drazin: Discontinuous velocity profiles for the Orr-Sommerfeld equation, J. Fluid Mech. **10** (1961), pp. 571–583.

the amplitude $\varphi(y)$ which is symmetrical in y, we obtain the solution of (2′) in the form:

$$\varphi=\left\{\begin{array}{ll}\varphi e^{-\eta}+Be^{-k_1\eta} & (\eta>\alpha),\\ A'\cosh\eta+B'\cosh(k_2\eta) & (\alpha>\eta>-\alpha),\\ Ae^{\eta}+Be^{k_1\eta} & (-\alpha>\eta),\end{array}\right\} \tag{5}$$

where

$$k_1=\sqrt{1-iKc}\,,\qquad k_2=\sqrt{1+iK(1-c)}\,. \tag{6}$$

Applying (4″) constants are determined as follows:

$$\left.\begin{aligned}A&=\frac{1}{c}(1-c)\sinh\alpha e^{\alpha}\cdot A',\\ B&=\frac{1}{2c}\{c+(c-2)\Delta\}(1-c)\cosh\alpha e^{k_1\alpha}\cdot A',\\ B'&=\frac{1}{2}\operatorname{cosech}(k_2\alpha)\{-1-c+(1-c)\Delta\}\cosh\alpha\cdot A',\\ &\qquad\qquad\qquad\qquad\Delta=\tanh\alpha,\end{aligned}\right\} \tag{7}$$

and we have the characteristic equation:

$$-2\Delta+\{-c+(2-c)\Delta\}(1-c)k_1+\{1+c-(1-c)\Delta\}ck_2\tanh(k_2\alpha)=0\,, \tag{8}$$

which, irrespective of its different expression, is identical with that due to Drazin.[1]

When c is real (8) reduces into two equations $F_1(\alpha, c, K)=0$ and $F_2(\alpha, c, K)=0$, and gives c and K as functions of α, which are so-called neutral curves. In the following we show that these neutral curves are more complicated than those contemplated by Drazin in his paper.[1]

When α is nearly equal or greater than 1, in which case our approximation fails, points on the neutral curve are determined by trial-and-error method. When α tends to ∞ $c=1/2$ and $K=8\sqrt{3}$ $(R=\infty)$, these results not agreeing with the well-known inviscid limit of continuous velocity profiles, e. g. $c=2/3$ and $R=\infty$ when α approaches to 2 in the case of velocity distribution $w(y)=\operatorname{sech}^2 y$.[2] When $\alpha\ll 1$ it is known that there are two branches of our neutral curves,[1] and their expressions, now found in detail, are

$$\left.\begin{aligned}\rho&=K(1-c)=\rho_0\alpha^{-3/2}(1+a_1\alpha+a_2\alpha^2+\cdots)\,,\\ c&=c_0\alpha^2(1+b_1\alpha+b_2\alpha^2+\cdots),\end{aligned}\right\} \tag{9}$$

where, for the first branch

$$\rho_0=1.179\,;\quad a_1=2.150,\ a_2=0.7556,\cdots,$$
$$c_0=1.543\,;\quad b_1=-1.421,\ b_2=1.094,\cdots,$$
$$\alpha\sqrt{R}\to 1.179\,;$$

(2) T. Tatsumi and T. Kakutani: The stability of a two-dimensional laminar jet, J. Fluid Mech. **4** (1958), pp. 261–275.

and for the second branch

$$\rho_0 = 5.689\,; \quad a_1 = 14.15,\ a_2 = -86.39,\cdots,$$
$$c_0 = -1.210\,; \quad b_1 = -2.001,\ b_2 = 410.9,\cdots,$$
$$\sqrt{\alpha}\, R \to 5.689\,.$$

For these two branches the admissible thickness δ of the transition layers of the velocity distribution can be broad enough when R is large, i. e. α is small, by the condition (ii) at the end of the last section. Our results given above must then be good approximations for a continuous profile of laminar jet. The first branch is joined up to the branch $\alpha \gtrsim 1$ above mentioned, forming one complete curve of neutral stability, which corresponds to the ordinary one known up to now.[2] Numerical values are in Table 1 and inscribed as the curves *A–1* in Figs. 1 and 2. Critical Reynolds number is about 4.0 corresponding to $\alpha = 0.25$ and $c = 0.07$; this critical state is, strictly speaking, beyond the limit of applicability of our method, and yet these values are in good agreement with the numerical values, $R = 3.37 \sim 4.00$, $\alpha \doteqdot 0.25$, $c = 0.05 \sim 0.10$, by Clenshaw and Elliott,[3] calculated for a smooth velocity profile $w = \mathrm{sech}^2 y$.

Table 1

α	c	R
0.00886	0.000120	12.77
0.03545	0.00184	6.71
0.0886	0.0107	4.805
0.222	0.0560	4.01
0.252	0.0697	4.00
0.318	0.100	4.13
0.514	0.200	4.95
0.724	0.300	6.53
1.017	0.400	10.16
1.414	0.460	17.03
1.767	0.480	22.94
∞	0.500	∞

As we could not conjecture beforehand the curve into which the second branch stated above joins, we traced step by step, starting from this branch, by a trial-and-error method. The outline of this method is as follows. Taking c real we use λ instead of K by the relation

$$K(1-c) = \sinh(2\lambda) \quad \text{i. e.} \quad k_2 = \cosh\lambda + i\sinh\lambda\,,$$

and write:

$$\left.\begin{aligned} &k_2\tanh(k_2\alpha) = C + iD\,, \\ &C = \frac{\cosh\lambda\sinh(2\alpha\cosh\lambda) - \sinh\lambda\sin(2\alpha\sinh\lambda)}{\cosh(2\alpha\cosh\lambda) + \cos(2\alpha\sinh\lambda)}\,, \\ &D = \frac{\cosh\lambda\sin(2\alpha\sinh\lambda) + \sinh\lambda\sinh(2\alpha\cosh\lambda)}{\cosh(2\alpha\cosh\lambda) + \cos(2\alpha\sinh\lambda)}\,. \end{aligned}\right\} \tag{10}$$

When, in addition, we use the notations

$$A_0 = -c + (2-c)\Delta, \quad B_0 = 1 + c - (1-c)\Delta\,, \tag{10'}$$

and decompose the characteristic equation into real and imaginary parts, we obtain

$$F_1 \equiv \frac{1-c}{c}\left(\frac{A_0}{B_0}\right)^2 - \left(\frac{2\Delta}{cB_0} - C\right)\frac{D}{\sinh\lambda\cosh\lambda} = 0\,, \tag{11}$$

(3) C. W. Clenshaw and D. Elliott: A numerical treatment of the Orr-Sommerfeld equation in the case of a laminar jet, Quat. J. Appl. Math. **13** (1961), pp. 302–313.

$$F_2 \equiv \left(\frac{1-c}{c}\right)^2\left(\frac{A_0}{B_0}\right)^2 - \left(\frac{2\Delta}{cB_0} - \overline{C+D}\right)\left(\frac{2\Delta}{cB} - \overline{C-D}\right) = 0 .$$

Assuming a certain value for λ we find α and c from these equations. Results of this not simple calculation are listed in order in the Table 2 and as the curves *A-2* in Figs. 1 and 2.

Table 2

α	c	$R(\alpha R)$	α	R	$R(\alpha R)$
0.00000	0.00000	∞(0.000)	0.08485	−0.1250	41.3
0.00050	−0.00000	256	0.0571	−0.209	66.3
0.01250	−0.00021	50.6	0.0448	−0.242	85.1
0.0266	−0.00107	37.3	0.0345	−0.270	110.0
0.0540	−0.00590	29.4	0.01280	−0.326	289
0.0700	−0.01260	27.5	0.00287	−0.352	1275
0.0935	−0.0850	34.3	0.00000	−0.3592	∞(3.631)

The other end of this second neutral curve seems, from this result, to correspond to $\alpha \to 0$ and in this limit c and αR not to vanish, i. e. $c \to c_0(\neq 0)$ and $\alpha R \to A_0(\neq 0)$. Therefore we assume anew the expansions:

$$\left.\begin{aligned} \rho = K(1-c) &= \rho_0\alpha^{-2}(1+a_1\alpha+a_2\alpha^2+\cdots) , \\ c &= c_0(1+b_1\alpha+b_2\alpha^2+\cdots) , \end{aligned}\right\} \tag{12}$$

and using these in the characteristic equation we have at first

$$(1+c_0)\tanh\sqrt{i\rho_0} - \sqrt{c_0(c_0-1)} = 0 ,$$

whose real and imaginary parts giving the values:

$$\left.\begin{aligned} \rho_0 = \frac{1}{2}\pi^2 n^2 , \quad (1+c_0)T = \sqrt{c_0(c_0-1)} , \\ (n = 1, 2, \cdots), \end{aligned}\right\} \tag{13}$$

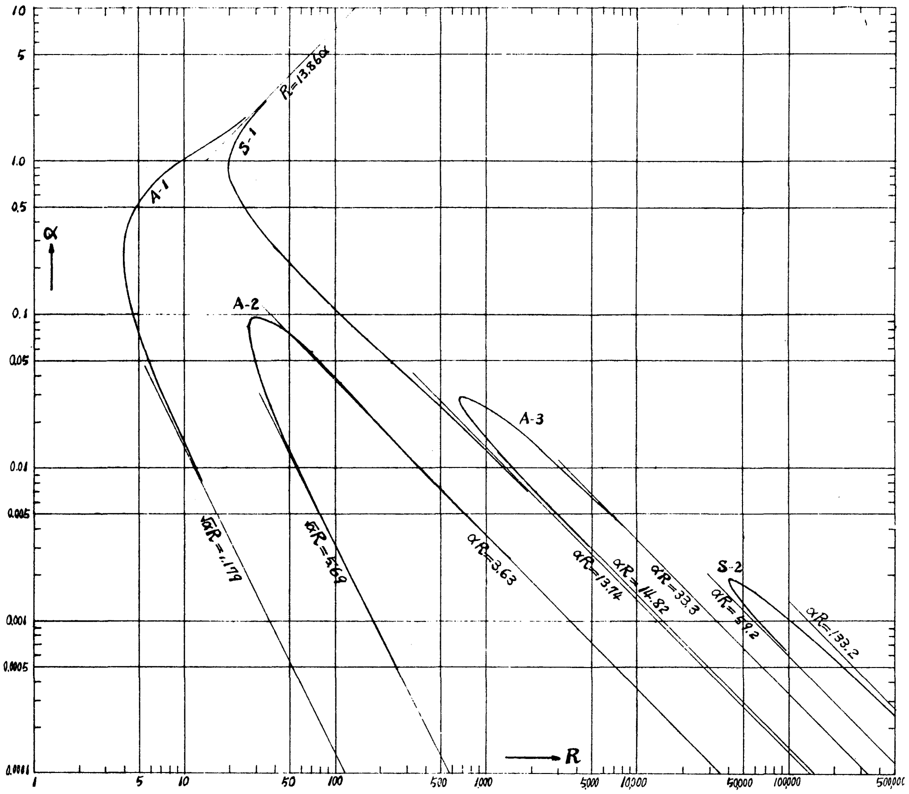
where $\rho_0 = A_0(1-c_0)$ and $T = \sinh(n\pi)/\{\cosh(n\pi)+(-1)^n\}$. Thus the number of branches with constant A_0 is infinite! To every n we find a pair of values ρ_0, c_0, and expansions:

$$\left.\begin{aligned} a_1 = 0, \ a_2 = -\frac{1}{\rho_0}\left\{1+\frac{\sqrt{\frac{2}{\rho_0}T - \frac{4}{1+c_0}}}{c_0(1-T^2)}\right\}, \quad \cdots, \\ b_1 = \frac{4(1-c_0)}{c_0(1-3c_0)}, \ b_2 = -\frac{1}{2}b_1\left\{\begin{aligned} &\frac{2}{c_0(1-3c_0)^2}(1-4c_0-12c_0^2+9c_0^3) \\ &+\frac{1+c_0}{2\rho_0}\left(1-a_2c_0\sqrt{2\rho_0^3}\cdot\frac{1-T^2}{T}\right) \end{aligned}\right\}, \cdots . \end{aligned}\right\} \tag{13}$$

First three of these are as follow:

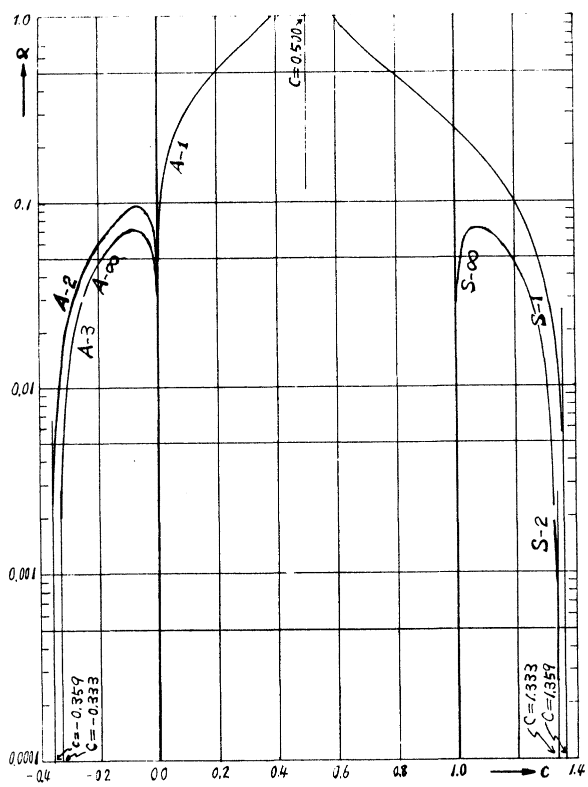
$$n=1: \quad T=1.0903; \ \rho_0=4.935, \ c_0=-0.3592;$$

$$A_0=3.631; \quad \begin{aligned} &a_1=0, \ a_2=-16.38,\cdots; \\ &b_1=-7.285, \ b_2=1.755,\cdots; \end{aligned}$$



R=13.86α
S-1
A-1
A-2
A-3
S-2
α
R
αR=3.63
αR=13.74
αR=14.82
αR=33.3
αR=59.2
αR=133.2

Fig. 1.



α
C
C=0.500
A-1
A-2
A-3
A-∞
S-∞
S-1
S-2
C=-0.359
C=-0.333
C=1.333
C=1.359

Fig. 2.

$$n=2: \quad T=0.9963; \ \rho_0=19.74, \ c_0=-0.3322;$$
$$A_0=14.82; \quad \begin{array}{l} a_1=0, \ a_2=116.3,\cdots; \\ b_1=-8.033, \ b_2=-1.609,\cdots; \end{array}$$
$$n=3: \quad T=1.0002; \ \rho_0=44.41, \ c_0=-0.3334;$$
$$A_0=33.31; \quad \begin{array}{l} a_1=0, \ a_2=-1211,\cdots; \\ b_1=-7.999, \ b_2=-2.330,\cdots. \end{array}$$

§3 **Antisymmetrical disturbances** (2) It is almost evident that the third of neutral curves is the one which joins up the two branches $n=2$ and $n=3$. To obtain its middle part we simplify (11) by use of the following approximations:

(i) λ is so large that cosh λ is equal to sinh λ,

(ii) $\xi=2\alpha\cosh\lambda$ is so large that $\cosh\xi$ is equal to $\sinh\xi$,

(iii) α is so small that $\tanh\alpha=\Delta$ is equal to α.

We obtain then from (11)

$$\left.\begin{array}{l} \alpha R=\xi^2/2(1-c), \quad c\doteqdot-\dfrac{1}{3}(1-8\alpha), \\ \alpha^2(1+6\alpha)\doteqdot\dfrac{\xi^2\sin\xi}{(9\xi-4)\cosh\xi}. \end{array}\right\} \qquad (11')$$

For given ξ values of α, c, and αR are calculated easily. The results are tabulated in Table 3, and inscribed as A-3 in Figs. 1 and 2.

Table 3

α	c	$\alpha(\alpha R)$	α	c	$\alpha(\alpha R)$
0.000×10^{-2}	-0.3322	$\infty(14.82)$	2.871×10^{-2}	-0.257	723
0.315 〃	-0.324	474×10	2.820 〃	-0.258	768
0.470 〃	-0.320	319×10	2.609 〃	-0.264	898
0.574 〃	-0.317	262×10	2.300 〃	-0.272	109.8×10
0.702 〃	-0.3135	215×10	1.927 〃	-0.282	140.3×10
0.858 〃	-0.309	178×10	1.515 〃	-0.293	190.5×10
1.051 〃	-0.304	146×10	1.303 〃	-0.2985	229×10
1.163 〃	-0.301	133×10	1.075 〃	-0.305	286×10
1.426 〃	-0.294	109.5×10	0.823 〃	-0.311	378×10
1.844 〃	-0.284	876	0.683 〃	-0.315	466×10
2.375 〃	-0.270	721.5	0.565 〃	-0.318	573×10
2.662 〃	-0.262	679	0.465 〃	-0.321	703×10
2.812 〃	-0.258	675	0.000 〃	-0.3334	$\infty(33.31)$
2.874 〃	-0.257	692			

There is no doubt that the branches $n=4$ and $n=5$ combine into a neutral curve, and so on for $n\geq6$.

A certain interest is for the case $n\to\infty$. In this case ρ_0, and accordingly $K(1-c)$, tends to ∞ proportionally to n^2 irrespective of α, but c, from (12), (13) and (13′), becomes

$$c = -\frac{1}{3}(1-8\alpha-4\alpha^2\cdots) , \tag{14}$$

which is a finite value dependent on α. R is then ∞. On the contrary if we make $R\to\infty$ in the characteristic equation (8) preserving α constant, then we have

$$3(1+\Delta)^2c^2+(1+\Delta)(1-7\Delta)c+4\Delta^2 = 0 , \tag{15}$$

and this equation, as noted by Drazin, has two real roots c_1, c_2 in the domain $0<\alpha<\tanh^{-1}(7-4\sqrt{3})\doteqdot 0.07192$, which is a neutral region of the flow. Expressed in power series in α these roots are

$$\left.\begin{aligned} c_1 &= -\frac{1}{3}(1-8\alpha-4\alpha^2-\cdots) , \\ c_2 &= -4\alpha^2(1+\cdots) , \end{aligned}\right\} \tag{16}$$

the former being nothing but the limiting form (14) and the latter having the same character as the first and the second branches. $c_1(\alpha)$ and $c_2(\alpha)$ join up into a neutral curve, which is indicated by A–∞ in Fig. 2.

Existence of such a neutral region of inviscid jet flow is contrary to the well-known result due to Rayleigh and are considered doubtful. Rayleigh's result seems, however, to be not more accurate than (15) here, for both theories assume as the base flow the same discontinuous one. In our case there is no more neglect, and we let the solution tend to the limit $R\to\infty$, whereas in Rayleigh's case $(\alpha R)^{-1}$ is at first put equal to zero in the fundamental equation (1), obtaining

$$L_1(\varphi) \equiv (w-c)(\ddot{\varphi}-\alpha^2\varphi)-\ddot{w}\varphi=0 . \tag{17}$$

Treating this equation by our method we are equivalent with Rayleigh as the followings indicate.

The joining condition at $y=1$ is obtained as before by

$$\lim \int L_1(\varphi)y^n dy=0 \quad (n=0,\ 1) ,$$

and gives:

$$\left.\begin{aligned} n=0:&\quad [(w-c)\dot{\varphi}]=0 , \\ n=1:&\quad \{w(1+0)-c\}\varphi(1-0)=\{w(1-0)-c\}\varphi(1+0) . \end{aligned}\right\} \tag{18}$$

To get the second relation ($n=1$) we take the integral in the form

$$[(w-c)\varphi]-2\int \varphi\dot{w}dy=0 ,$$

and by means of the relations:

$$\begin{aligned} \dot{w}(y) &= [w]\delta(y-1), \\ \varphi(y) &= \varphi(1-0)+[\varphi]\int_{1-0}^{y} \delta(y_1-1)dy_1 , \end{aligned}$$

where δ is the Dirac function, rewrite the second term on the left-hand side into

$$\int \varphi \dot{w} dy = \varphi(1-0)[w] + [\varphi][w] \int_{1-0}^{1+0} \delta(y-1)dy \int_{1-0}^{y} \delta(y_1-1)dy_1$$
$$= \varphi(1-0)[w] + \frac{1}{2}[\varphi][w] .$$

With this the relation for $n=1$ is evident.

Now (17) reduces to the single equation:

$$\ddot{\varphi} - \alpha^2 \varphi = 0 \tag{17'}$$

in all the domains $y>1$, $|y|<1$, and $y<-1$. Symmetrical solution is then

$$\varphi = Ae^{-\alpha y}(y>1), \quad \varphi = A' \cosh(\alpha y) \quad (|y|<1) , \tag{19}$$

and by use of the relations (18) we have

$$A = \frac{1-c}{c} \sinh \alpha e^{\alpha} \cdot A' , \tag{20}$$

and

$$c = \frac{\Delta \pm \sqrt{-\Delta}}{1+\Delta} , \tag{21}$$

which are the very results due to Rayleigh.

Amplitude function $\varphi(y)$ corresponding to (15) is obtained from (5), (6) and (7) as the limiting case $K \to \infty$, and is easily seen to be identical with (19), (20). Here then we are in an interesting situation. Identical eigenfunctions and non-accordant eigenvalues! The reason for this is the existence of an infinite number of neutral curves in our case, the last one of which, i. e. that which corresponds to the limit $n \to \infty$, having made its appearance in the inviscid limit.

Which of the two alternatives is then more real? This question has probably not any meaning, for for any real fluid discontinuous velocity profiles are unrealizable and, by means of the condition (iii) at the end of §1, admissible breadth δ of transition layers must be infinitely small, in order our approximation to have some suggestion of the case of continuous velocity profiles. Contrary to this limiting case the branches of $n=1$, 2, 3 or so have surely some meaning to continuous flow profiles. The condition (iii) becomes now

$$n=1: \delta \leqq 0.524, \quad n=2: \delta \leqq 0.260, \quad n=3: \delta \leqq 0.173 ;$$

and then sufficiently short range of transition, e. g. $\delta = 0.04$, will match to this requirement. For given δ larger n means, of course, cruder approximation.

§4 Symmetrical disturbances In the same way as in the preceding sections neutral oscillations of symmetrical type are obtained. In this case

$$\left.\begin{aligned} \eta > \alpha : &\quad \varphi = Ae^{-\eta} + Be^{-k_1 \eta} , \\ |\eta| < \alpha : &\quad \varphi = A' \sinh \eta + B' \sinh(k_2 \eta), \\ \eta < -\alpha : &\quad \varphi = -Ae^{\eta} - Be^{k_1 \eta} , \end{aligned}\right\} \tag{22}$$

and the joining condition at $\eta = \pm \alpha$, which is identical with (4'), determines the coefficients and the characteristic value:

$$\left.\begin{aligned} &A=\frac{1}{c}(1-c)\cosh\alpha e^{\alpha}\cdot A', \quad B=\frac{1}{2c}(c-2+c\Delta)(1-c)\cosh\alpha\, e^{k_1\alpha}\cdot A', \\ &B'=\frac{1}{2}\operatorname{cosech}(k_2\alpha)\{1-c-(1+c)\Delta\}\cosh\alpha\cdot A', \end{aligned}\right\} \tag{23}$$

and

$$2+(2-c-c\Delta)(1-c)k_1+\{-1+c+(1+c)\Delta\}ck_2\coth(k_2\alpha)=0, \tag{24}$$

which are already known.[1]

After some examinations we knew that all the neutral curves from (24) have the branches which, as $\alpha\to 0$, tend to

$$c\to c_0 : \text{const}(\neq 0,\ 1), \quad \alpha^2K(1-c)\to\rho_0 : \text{const}(\neq 0);$$

the values c_0, ρ_0 are derived from (24), multiplied by α and made $\alpha\to 0$:

$$(2-c_0)\tanh\sqrt{i\rho_0}-\sqrt{c_0(c_0-1)}=0;$$

from which we have

$$\rho_0=\Lambda_0(c_0-1)=\frac{1}{2}\pi^2n^2, \quad (2-c_0)T=\sqrt{c_0(c_0-1)}, \tag{25}$$

where $\Lambda_0=\lim(\alpha R)$, and T is that in (13)*. There is then infinite neutral branches in this case too. First few examples are

$$n=1: \quad c_0=1.359, \quad \Lambda_0=13.74,$$
$$n=2: \quad c_0=1.332, \quad \Lambda_0=59.24,$$
$$n=3: \quad c_0=1.333, \quad \Lambda_0=133.2.$$

The first neutral curve starts from the branch $n=1$. When α is small and $c>1$ the estimation $\Delta\doteqdot\alpha$, $cK\gg 1$, $(c-1)K\gg 1$ holds, and then k_1, k_2 can be replaced by $\sqrt{-iKc}$, $\sqrt{-iK(1-c)}$. The characteristic equation (24) separates then into real and imaginary parts:

$$\left.\begin{aligned} &A_1\sqrt{c(c-1)}(2-c-c\alpha)=2\alpha B_1, \\ &A_1c\{c-1+(c+1)\alpha\}=2\alpha C_1, \end{aligned}\right\} \tag{26}$$

where

$$\left.\begin{aligned} &A_1=\xi\sin\xi, \quad B_1=\sinh\xi-\sin\xi, \\ &C_1=\frac{\sinh^2\xi+\sin\xi}{\cosh\xi+\cos\xi}, \quad \alpha R=\frac{1}{2(c-1)}\xi^2, \end{aligned}\right\} \tag{26'}$$

and we obtain α and ξ (i. e. αR) assuming values of c. When α is large and $c<1$, tanh $(k_2\alpha)$ is equal to unity, and (24) can be solved for Δ and K for given c.

For intermediate values of α extrapolated values from the case of smaller or larger α are corrected to within some few percents errors. At the point $c=1$ (24)

* Solution given by Drazin, his (55) on p, 581, seems to contain some trifle mistakes; c. f. footnote (1).

vanishes identically in α, and then we substitute $c=1+\xi$ in this equation and make the coefficient of the first degree term in the ξ-expansion zero, with the result:

$$-(1-\Delta)\sqrt{1-iK}+3+\frac{1}{\Delta}-iK+\frac{1-\Delta^2}{\Delta}iK\alpha=0\,;$$

from which we obtain

$$c=1: \qquad \alpha=0.2493, \qquad K=175.2.$$

The second neutral curve has as its branches $n=2$ and 3 above, and lies wholly in the region of large R, where the approximations (26) and (26′), and moreover

$$B_1=C_1=\sinh\xi \ (2\pi\leq\xi\leq 3\pi),$$

hold, other curves of higner order being obtainable likewise. Results thus obtained are indicated in Tables 4 (the first curve) and 5 (the second curve), and inscribed as S–1 and S–2 in Figs. 1 and 2.

Table 4

α	c	$R(\alpha R)$
0.0000	1.3592	∞(13.74)
0.0341	1.300	355
0.0963	1.200	114.6
0.1675	1.100	63.6
0.249	1.000	43.68
0.485	0.800	25.1
0.665	0.700	21.1
0.964	0.600	19.6
1.57	0.530	24.1
2.44	0.505	34.4
∞	0.500	∞(∞)

Table 5

α	c	$R(\alpha R)$
0.000×10^{-3}	1.332	∞(59.24)
0.713 〃	1.331	878×10^2
1.231 〃	1.330	536 〃
1.581 〃	1.329	438 〃
1.789 〃	1.329	406 〃
1.881 〃	1.328	405 〃
1.880 〃	1.328	423 〃
1.808 〃	1.329	459 〃
1.528 〃	1.329	589 〃
1.355 〃	1.330	689 〃
1.065 〃	1.331	936 〃
0.751 〃	1.331	1410 〃
0.484 〃	1.332	2322 〃
0.271 〃	1.333	4392 〃
0.000 〃	1.333	∞(133.2)

Let $K\to\infty$ in (24), holding α constant, then we have

$$3(1+\Delta)^2c^2+(1+\Delta)(\Delta-7)c+4=0\,, \tag{27}$$

and solving for c:

$$c(\alpha)=(7-\Delta\pm\sqrt{1-14\Delta+\Delta^2})/6(1+\Delta)\,, \tag{27′}$$

which gives identical stability regions as in the case of antisymmetrical disturbances. $c=c(\alpha)$ is the curve of neutral stability of inviscid jet, corresponding to A–∞ in preceding section, and is inscribed as S–∞ in Fig. 2. Its two branches are, expanded into power series,

$$\left.\begin{aligned} c_1&=\frac{4}{3}(1-2\Delta-\Delta^2-\cdots)\,,\\ c_2&=1+4\Delta^2+24\Delta^3+\cdots\,; \end{aligned}\right\} \tag{28}$$

we see the latter branch having no counterpart in the finite R region.

§5 **Extent of disturbance waves** We have been assuming that the disturbance waves cover evenly over the jet and its neighborhood. If this broad and even extension is not satisfied our discontinuous approximation fails, and therefore we study in this section the magnitude of amplitude function $\varphi(y)$. Of course in the linear theory we have to interprete this magnitude by the relative one compared to that of a standard point. When α is large the extent is, as is well known, so narrow and comparable with the jet width that the discontinuous model is inappropriate.

We study then the cases of small α. First the antisymmetrical disturbance. In this case amplitude is given by (5)$\sim$(7) ; at the centre line of jet

$$\varphi(0)=A'+B'=\left[1+\frac{\{-1-c+(1-c)\Delta\}}{2\cosh(k_2\alpha)}\cosh\alpha\right]\cdot A', \tag{29}$$

and at the outside domain of the discontinuous layer

$$\varphi(y)=\frac{1-c}{2c}\left[2\sinh\alpha e^{-\alpha(y-1)}+\cosh\alpha\{c+(c-2)\Delta\}e^{-k_1\alpha(y-1)}\right]\cdot A', \quad (y\geq 1)\,. \tag{30}$$

As α is small now we use α-expansions of constants. On the branch of the first neutral curve we have

$$\alpha^2K=\alpha R=\rho_0\alpha^{1/2}+0(\alpha^{3/2})\,,$$

$$\alpha k_1=\alpha-i\frac{\rho_0c_0}{2}\alpha^{3/2}+\frac{1}{8}\rho_0{}^2c_0{}^2\alpha^2+0(\alpha^{5/2})\,,$$

$$\alpha k_2=\sqrt{i\rho_0}\,\alpha^{1/4}+0(\alpha^{5/4})\,,$$

$$\frac{1-c}{2c_0}=\frac{1}{2c_0}\alpha^{-2}(1-b_1\alpha+0(\alpha^2))\,,$$

and using these into (29) and (30)

$$\varphi(0)/A'=\frac{1}{2}+i\frac{\rho_0}{4}\alpha^{1/2}+0(\alpha)\,, \tag{29'}$$

$$\varphi(y)/A'=e^{-\alpha(y-1)}\left\{\frac{1}{2}-\frac{i}{2}\rho_0\alpha^{1/2}(y-1)+0(\alpha)\right\}, \quad (y\geq 1)\,. \tag{30'}$$

From these expansions we see that the neutral waves on the branch extend their domain of existence into y-direction according as α decreases, becoming far larger than the jet breadth. It results therefore that the use of a discontinuous approximation of base flow is legitimate for this branch and numerical results derived therefrom are reliable.

Circumstances are the same for the first branch of the second neutral curve, but are different for its second branch ($n=1$) and for all the branches of remaining neutral curves ($n=2, 3,\cdots$). In the latter cases (12), (13) and (13') hold and consequently

$$\alpha^2 K = \alpha R = \frac{\pi^2 n^2}{2(1-c_0)}\left(1+\frac{c_0 b_1}{1-c_0}\alpha+0(\alpha^2)\right),$$

$$\alpha k_1 = (1+i)\frac{\pi n}{2}\sqrt{\frac{-c_0}{1-c_0}}\left(1+\frac{b_1}{2(1-c_0)}\alpha+0(\alpha^2)\right),$$

$$\alpha k_2 = (1+i)\frac{\pi n}{2}\left(1+0(\alpha^2)\right),$$

$$\frac{1-c}{2c} = \frac{1-c_0}{2c_0}\left(1-\frac{b_1}{1-c_0}\alpha+0(\alpha^2)\right).$$

Using these in (29), (30) we have

$$\varphi(0)/A' = 1 - \frac{1+c_0}{2\cosh\left\{(1+i)\frac{n\pi}{2}\right\}} + 0(\alpha) \,, \tag{29''}$$

$$\varphi(y)/A' = \left(\frac{1-c_0}{c_0}\alpha + 0(\alpha^2)\right)e^{-\alpha(y-1)}$$

$$+\left(-\frac{1-c_0}{2}+0(\alpha)\right)e^{-(1+i)\frac{\pi n}{2}\sqrt{\frac{-c_0}{1-c_0}}(y-1)}, \tag{30''}$$

$$(y \geq 1) \,,$$

and by means of the numerical values given at the end of §2,

$$n=1: \quad \varphi(0)/A' = 1+0.139i+0(\alpha) \,,$$

$$\varphi(y)/A' = (-3.784\alpha+0(\alpha^2)e^{-\alpha(y-1)}$$

$$+(0.680+0(\alpha))e^{-0.820(1+i)(y-1)}(y \geq 1) \,,$$

$$n=2: \quad \varphi(0)/A' = 1.029+0(\alpha) \,,$$

$$\varphi(y)/A' = (-4.010\alpha+0(\alpha^2))e^{-\alpha(y-1)}$$

$$+(0.666+0(\alpha))e^{-1.569(1+i)(y-1)}(y \geq 1) \,,$$

and so forth. These amplitudes do not differ appreciably from

$$\left.\begin{aligned} &\varphi(0)/A' = 1 \,, \\ &\varphi(y)/A' = \frac{2}{3}e^{-(1+i)\frac{\pi n}{2}(y-1)} \\ &\qquad\qquad\qquad (y \geq 1) \,, \\ &\qquad (n=1,\, 2,\, \cdots) \,, \end{aligned}\right\} \tag{31}$$

the larger n the more accurate being this simplified amplitude.

All these branches have therefore the character totally different from the previous ones. Now each branch has a long range term and a short one. The amplitude of the former vanishes as α and the principal part of φ is the latter, which damps more quickly and oscillates more rapidly as n increases, in the outside domain of transient layer. Thus our basic assumption for the adoption of discontinuous velocity profile is invalidated, except first few branches, and is worse n being larger. The positions and forms of these branches seem therefore to be appreciably erroneous, and it is preferable to study a smooth velocity profile in the case of small α and large R, which is now in practice in our circle. And yet, we think, the very existence of such neutral curves in the small α-region is haply not denied, a thin transition layer not being expected to differ qualitatively from a moderate transition layer.

Turning to symmetrical disturbances (22) and (23) give the amplitude for $y \geq 1$:

$$\phi(y)/A' = \frac{1-c}{2c} \cosh\alpha \left\{ 2e^{-\alpha(y-1)} + (c-2+c\Delta)e^{-k_1\alpha(y-1)} \right\}, \tag{32}$$

which, by use of (25) and numerical values there given, becomes

$$\varphi(y)/A' \doteqdot -\frac{1}{2} e^{-\alpha(y-1)} + \frac{1}{6} e^{-(1-i)\frac{n\pi}{2}(y-1)} \qquad (y \geq 1). \tag{33}$$

Here then the amplitude of long range term does not vanish and extends much further when α becomes smaller. The other term of relatively smaller amplitude oscillates more rapidly but vanishes exponentially as n becomes larger. The joining condition (4″) is then not so inappropriate as for an antisymmetrial disturbance and the resulting stability curves are more reliable; existence of infinite neutral curves in cases of smooth velocity profile is most probable.

Some words ought to be said here about the wall effect on the stability. The neutral waves to which our approximation adequately applies extend their existence domains far beyond the jet, and to such waves the wall effect seems to be severe. Thus the first several branches of antisymmetrical disturbance and all branches of symmetrical disturbance are sensitive to the existence of walls. Moreover their different reactions are expected because of their different states of extention into the y-direction. The problem of wall effect is thus in the range of our approximation.

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