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SIMILARITY IN BOUNDARY LAYER SEPARATIONS OF TWO-DIMENSIONAL FLOWS

By Jun-ichi OKABE

In a group of two-dimensional, steady flow patterns defined by the equation of the form $u_1 = \alpha\varphi(x)$ or $x = \beta\psi(u_1)$, where φ and ψ are arbitrary functions, x, the arc length measured along the surface of the body, u_1, the velocity in the main stream just outside the boundary layer, while α and β denote the parameters variable within that group, it is shown that the separation of flow, laminar or turbulent, from the wall occurs at the point $x =$ constant or $u_1 =$ constant, respectively, according to the form of relationship between x and u_1.

1. Equations of Laminar Boundary Layers

According to L. HOWARTH,[1] the differential equation governing the laminar boundary layer in a two-dimensional steady flow may be put in the form

$$\frac{d\xi}{dx} = -(1-\xi)\frac{u_1'}{u_1} + \frac{1}{2}\frac{\chi}{d\chi/d\xi}\frac{u_1''}{u_1'}, \tag{1.1}$$

while, on the other hand, if we transform the momentum equation derived by VON KÁRMÁN and POHLHAUSEN, we obtain

$$\frac{dA}{dx} = g(A)\frac{u_1'}{u_1} + \{A^2h(A)+A\}\frac{u_1''}{u_1'}, \tag{1.2}$$

alternatively.[2]

In both of these equations, ξ and A are the variables characterizing the development of the boundary layer, u_1, the velocity in the main stream just outside it, x being the arc length measured along the surface of the body; dashes denote differentiations with respect to x. Further, $\chi \div (d\chi/d\xi)$, $g(A)$ and $h(A)$ are known functions of ξ and A respectively, and are tabulated against various values of the variables. Thus representing the unknown quantities generally by σ, these equations may be expressed in unified manner as

$$\frac{d\sigma}{dx} = F(\sigma)\frac{u_1'}{u_1} + G(\sigma)\frac{u_1''}{u_1'}, \tag{1.3}$$

where all of $u_1(x)$, $u_1'(x)$, $u_1''(x)$, $F(\sigma)$, and $G(\sigma)$ are known functions with regard to the argument x or σ.

In equation (1.1), the initial value of ξ has to be given at $x = 0$ in advance of the computation; for example, if we start from the pressure minimum we should have $\xi = 0$ there, and, in the end, the boundary layer is expected to separate from

1) GOLDSTEIN, S., *Modern Developments in Fluid Dynamics*, vol. I (1938), **63**, eqn. (193), p. 176.
2) GOLDSTEIN, S., *ibid.*, vol. I, **60**, footnote, p. 160.

the wall at the point where ξ takes the value 0.120. Likewise, in equation (1.2), Λ begins with some value at the start (for example, at the forward stagnation Λ is found to be 7.052) and the separation takes place at $\Lambda = -12$. Speaking generally, σ starts from the initial value σ_i at $x = 0$ and finally reaches the value σ_s at the point of separation $x = x_s$. It should be noted incidentally that, in the following sections in which will be investigated two kinds of similarity of the boundary layers among the groups of flows produced by varying the values of the parameter, if we make the problem clear-cut by fixing the range of integration with respect to x, for example, from the pressure minimum to the separation in the former equation, or from the forward stagnation to the separation in the latter, then both of σ_i and σ_s may be regarded as the universal constants independent of the detailed situation of the particular problem.

2. Similarity in Laminar Boundary Layers

1) In a given series of flow patterns, let us suppose the velocity u_1 be expressed in the form

$$u_1 = \alpha\varphi(x), \tag{2.1}$$

where φ and α denote an arbitrary function and a parameter, respectively. Then α will be cancelled from the terms u_1'/u_1 and u_1''/u_1', and so equation (1.3) is found free from the parameter. Under the above-mentioned assumption, therefore, we may conclude that the separation of the boundary layer should occur at a definite value of x independent of α.

2) In the second place, by making use of an arbitrary function and another parameter, ψ and β respectively, if the field be defined as

$$x = \beta\psi(u_1) \tag{2.2}$$

for the velocity distribution, then putting

$$x/\beta \equiv w, \tag{2.3}$$

we may write

$$w = \psi(u_1), \tag{2.4}$$

or inversely,

$$u_1 = \omega(w), \text{ say.} \tag{2.5}$$

The following relations are obtained from (2.3):

$$\frac{d\sigma}{dx} = \frac{1}{\beta}\frac{d\sigma}{dw}, \quad u_1' \equiv \frac{du_1}{dx} = \frac{1}{\beta}\frac{du_1}{dw},$$

and

$$u_1'' \equiv \frac{d^2u_1}{dx^2} = \frac{1}{\beta^2}\frac{d^2u_1}{dw^2}, \tag{2.6}$$

so we are led to the equation

$$\frac{d\sigma}{dw} = F(\sigma)\frac{du_1}{dw}\Big/u_1 + G(\sigma)\frac{d^2u_1}{dw^2}\Big/\frac{du_1}{dw}. \tag{2.7}$$

Similarly to the preceding discussions, therefore, we should have the separation at a constant value of w, namely at a constant value of u_1, independent of the parameter.

A few years ago, the present writer published an approximate calculation of the laminar boundary layer in which for a series of velocity distributions

$$u_1 = (1-x)^n \quad\text{with}\quad n = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ \text{and } 1.2, \tag{2.8}$$

where u_1 and x have been reduced non-dimensional for convenience, the separation point, and $u_1(x_s)$, the velocity in the main stream at that point, were computed; the result is shown by the table.[1] It is noticed that the values of $u_1(x_s)$ are very uniform irrespective of the variation of n. In fact, as long as x remains small compared with unity, we have from (2.8)

n	x_s	$u_1(x_s)$
0.2	0.412	0.899
0.4	0.255	0.889
0.6	0.184	0.885
0.8	0.144	0.883
1.0	0.119	0.881
1.2	0.101	0.881

$$u_1 = 1-nx,$$

or

$$x = (1-u_1)/n, \tag{2.9}$$

approximately, which is just the form treated in the second part of this section.

3. Similarity in Turbulent Boundary Layers

Owing to the theory developed by A. BURI and later by L. HOWARTH, turbulent boundary layers in two-dimensions may be described, at least approximately, in terms of a single parameter Γ defined by

$$\Gamma \equiv \frac{\vartheta}{u_1}\frac{du_1}{dx}R_\vartheta^{1/4}, \tag{3.1}$$

ϑ being the momentum thickness and

$$R_\vartheta \equiv u_1\vartheta/\nu. \qquad (3.2)^{2)}$$

Furthermore their studies revealed that we should expect the separation at the point determined by the equation

$$\Gamma = -0.06. \qquad (3.3)^{3)}$$

However, from (3.1) and (3.2), Γ may also be written

$$\Gamma = \nu^{-1/4}\vartheta^{5/4}u_1^{-3/4}du_1/dx. \tag{3.4}$$

In order to evaluate the magnitude of ϑ, let us assume that we may put

$$\vartheta = k\delta, \tag{3.5}$$

where δ and k are the overall thickness of the boundary layer and a constant less than unity, respectively. For the flow along a flat plate, for example, we have approximately

$$k = 7/72, \tag{3.6}$$

a well-known result deduced from the 1/7th power law.[4] Substituting (3.5) for

1) OKABE, J., A Note on the Laminar Separation (*in Japanese*), *Bulletin of Research Institute for Applied Mechanics*, no. 19 (1962), pp. 125–133.

2) GOLDSTEIN, S., *ibid.*, vol. II (1938), **166**, eqn. (83), p. 374.

3) GOLDSTEIN, S., *ibid.*, vol. II, **194**, p. 438.

4) GOLDSTEIN, S., *ibid.*, vol. II, **163**, eqn. (63), p. 361. Strictly speaking, of course, k introduced in (3.5) is not an invariant but changes more or less gradually as the boundary layer develops and the velocity profile is deformed. In order to calculate turbulent boundary layers approximately, however, it was found (*continued on the following page*)

(3.4), we obtain

$$\Gamma = k^{5/4}\nu^{-1/4}\delta^{5/4}u_1^{-3/4}\mathrm{d}u_1/\mathrm{d}x. \tag{3.7}$$

Afterwards it was shown by C. B. MILLIKAN, however, that the thickness of the boundary layer may be computed by the aid of the following formula:[1)]

$$\delta = 0.38\,\nu^{1/5}u_1^{-23/7}\left(\int_0^x u_1^{27/7}\mathrm{d}x\right)^{4/5}. \tag{3.8}$$

If we suppose that this expression would be valid in the whole range of the flow up to the point of separation, then by combining (3.7) and (3.8) we might have

$$\Gamma = l\,u_1^{-34/7}(\mathrm{d}u_1/\mathrm{d}x)\int_0^x u_1^{27/7}\mathrm{d}x, \tag{3.9}$$

where l denotes another constant which is assumed to contain no parameter.

Now for the case when the velocity be given in the form (2.1), we may readily verify that Γ becomes independent of α. From equation (3.3), we must conclude, therefore, that the separation would be found at a constant value of x irrespective of the parameter.

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very convenient to assume the proportionality relationship between ϑ and δ. So we have to understand k in the sense of its average value, say, from the beginning to the separation of the turbulent boundary layer.

According to the past experiences of the writer, in most cases Γ decreases (with minus sign) very rapidly as the separation point x_s is approached, and so the error involved in estimating the value of k has most likely no sensible effects in determining x_s. As to the legitimacy of substituting 7/72 for k in the flows containing the separation, see the following: OKABE, J., On Frictional Resistance of Ships (*in Japanese*), *Reports of Research Institute for Fluid Engineering*, vol. 4, no. 2 (1947), pp. 1-20, and *Bulletin of Institute*, no. 7 (1955), pp. 39-42 (cf. the next footnote).

1) MILLIKAN, C. B., The Boundary Layer and Skin Friction for a Figure of Revolution, *Trans. A. S. M. E.*, Applied Mechanics Section, vol. 54 (1932), pp. 29-39. In deriving the formula, it is assumed for the sake of identity of the initial conditions that the boundary layer begins with vanishing thickness at the start $x=0$. Also see the following: RESEARCH COMMITTEE FOR HYDROLOGY, Tables of $u^{27/7}$ and $u^{-23/7}$, ($u = 0.500\sim 1.500$) (*in Japanese*), *Bulletin of Research Institute for Applied Mechanics*, no. 5 (1954), pp. 1-16; *ibid.*, no. 7 (1955) pp. 39-42; *ibid.*, no. 18 (1961), pp. 19-38.

The formula is based upon the so-called 1/7th power law for the velocity profile and is valid for moderate REYNOLDS numbers less than 3×10^6. On the other hand, if nth power law is assumed more generally for the profile, then (3.8) has to be rewritten

$$\delta^{1-m} = \kappa\frac{(1+2n)(1+3n)}{n}\nu^{-m}u_1^{-(3+2n)(1-m)}\int_0^x u_1^{3+6n}\mathrm{d}x,$$
$$\kappa \equiv 0.0232,\quad m \equiv -2n/(1+n), \tag{3.8$'$}$$

and in place of (3.6) we have

$$k = n/(1+n)(1+2n) \tag{3.6$'$}$$

for wider range of REYNOLDS numbers; see GOLDSTEIN, S., *ibid.*, vol. II, **163**, p. 362 and also **226**, TABLE 17, p. 516. Through slight modification in the definition of Γ, however, it may be shown quite easily that we are led to the same conclusion as in the text: necessary generalization on the part of Γ is given by

$$\Gamma \equiv \frac{\vartheta}{u_1}\frac{\mathrm{d}u_1}{\mathrm{d}x}R_\vartheta^{-m}. \tag{3.2$'$}$$

Formulas (3.8) and (3.8)$'$ correspond to equations (2.10) and (2.9), respectively, in *Bulletin*, no. 5, p. 3. Minor differences between both of these expressions are due to the fact that the notations u_1 and x are used to indicate quantities *with* dimensions in this note, whereas they were employed for those *without* dimensions in *Bulletin*.

On the other hand, when the second relation, cf. equation (2.2), does hold for the velocity, we have from (2.3), (2.5), and (2.6),

$$\Gamma = l\omega(w)^{-34/7}(\mathrm{d}\omega/\mathrm{d}w)\int_0^w \omega(w)^{27/7}\mathrm{d}w, \tag{3.10}$$

an expression not depending upon β explicitly. From the form of (3.10), we know that Γ takes a constant value when w or, what amounts to the same thing, u_1 attains some definite value. And so we must infer that the separation points would be situated at the points of u_1 constant, irrespective of the parameter. In fact, in *Bulletin of Research Institute for Applied Mechanics*, no. 18 (1961), cf. footnote on p. 36, sensible uniformity of $u_1(x_s)$ was exemplified for the turbulent separations of the flow patterns

n	x_s	$u_1(x_s)$
2	0.278	0.521
3/2	0.346	0.529
1	0.455	0.545
1/2	0.656	0.587
1/3	0.761	0.621

$$u_1 = (1-x)^n$$

with $$n = 2,\ 3/2,\ 1,\ 1/2,\ \text{and}\ 1/3; \tag{3.11}$$

cf. the table. These two alternative similarities correspond exactly to the circumstances taking place within the laminar boundary layers.

4. Conclusion

The similarity condition (2.1) or (2.2) ensures that the separation of the boundary layer, laminar or turbulent, from the wall arises at the points x = constant or u_1 = constant, respectively. If we know in advance, therefore, among a series of flow patterns, either of the similarity conditions is satisfied for the velocity, at least approximately, then by finding, theoretically or experimentally, only one of the points of separation of the flows, we should be enabled to estimate those points for the remaining members of the group without labor of computing their boundary layers.

While taking a walk with Professor YAMADA of Kyoto University a few years ago, he and the writer discussed problems of boundary layer separations. This note is a outgrowth of this little walk through the park. The writer takes this chance to express his thanks to the professor.

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Note added in proof: According to SCHLICHTING, H., *Boundary Layer Theory* (1960), XXI, p. 537, eqn. (21.11), the range of REYNOLDS numbers in which the 1/7th power law holds is

$$5\times10^5 < R_L < 10^7;$$

cf. footnote 1) on p. 36.