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NOTE

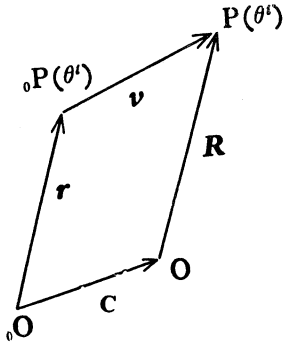
A Note on the Stress Function

By Kazuo KAWATATE

In considering the mechanics of continua, if the method initiated by Dorn and Schild is followed, the stress is to be expressed in terms of the stress function by means of the metric tensors in their original state.

1. Introduction

It is well known that the stress is represented by the stress function in the mechanics of continua[1)-5)]. The relations present a fine style when the metric tensors of deformed state are used, even in the case of finite deformation, to say nothing of infinitesimal deformation. In the present note, an attempt will be made to seek the said relations by means of the metric tensors in the state before deformation, though the results obtained will be rather complicated.



Hereafter, Green's notations[6),7)] will be mostly used. Let the position vector of undeformed state be $\boldsymbol{r}$, that of deformed $\boldsymbol{R}$, the displacement vector $\boldsymbol{v}$ and the shift of reference point $\boldsymbol{c}$, then $\boldsymbol{r}+\boldsymbol{v}=\boldsymbol{R}+\boldsymbol{c}$.

Taking tentatively the curvilinear coordinate system θ^i, which moves continuously with the body as one passes from the original state to the deformed state, one obtains base vectors, metric tensors, Christoffel symbols, Riemann-Christoffel curvature tensors etc. as follows :[6),8)]

$$\left.\begin{array}{ll} \boldsymbol{g}_i=\dfrac{\partial \boldsymbol{r}}{\partial \theta^i}=\boldsymbol{r}_{,i} & \boldsymbol{G}_i=\dfrac{\partial \boldsymbol{R}}{\partial \theta^i}=\boldsymbol{R}_{,i} \\ g_{ij}=\boldsymbol{g}_i.\boldsymbol{g}_j & G_{ij}=\boldsymbol{G}_i.\boldsymbol{G}_j \\ g^{ij}g_{mj}=\delta^i_j & G^{im}G_{mj}=\delta^i_j \\ \boldsymbol{g}^i=g^{im}\boldsymbol{g}_m & \boldsymbol{G}^i=G^{im}\boldsymbol{G}_m \\ g=\det|g_{ij}| & G=\det|G_{ij}| \\ I_3=\dfrac{G}{g} & \end{array}\right\}, \tag{1.1}$$

$$\left.\begin{array}{l} [ij,\ m]=[ji,\ m]=\dfrac{1}{2}(g_{jm,i}+g_{mi,j}-g_{ij,m}) \\ [\![ij,\ m]\!]=[\![ji,m]\!]=\dfrac{1}{2}(G_{im,i}+G_{mi,j}-G_{ij,m}) \end{array}\right|, \tag{1.2}$$

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$$\left.\begin{aligned}&\left\{ {r \atop ij} \right\} = \left\{ {r \atop ji} \right\} = g^{rm}[ij, m]\\&\left\{\left| {r \atop ij} \right|\right\} = \left\{\left| {r \atop ji} \right|\right\} = G^{rm}[|ij, m|]\end{aligned}\right)$$

$$\left.\begin{aligned}{}_0R^p_{.sij} &= \left\{ {p \atop sj} \right\}_{,i} - \left\{ {p \atop si} \right\}_{,j} + \left\{ {m \atop sj} \right\}\left\{ {p \atop mi} \right\} - \left\{ {m \atop si} \right\}\left\{ {p \atop mj} \right\}\\ R^p_{.sij} &= \left\{\left| {p \atop sj} \right|\right\}_{,i} - \left\{\left| {p \atop si} \right|\right\}_{,j} + \left\{\left| {m \atop sj} \right|\right\}\left\{\left| {p \atop mi} \right|\right\} - \left\{\left| {m \atop si} \right|\right\}\left\{\left| {p \atop mj} \right|\right\}\end{aligned}\right\}, \tag{1.3}$$

$$\left.\begin{aligned}{}_0R_{rsij} = g_{rp\,0}R^p_{.sij} &= \frac{1}{2}\left(g_{rj,si}+g_{si,rj}-g_{ri,sj}-g_{sj,ri}\right) + g_{mn}\left(\left\{ {m \atop rj} \right\}\left\{ {n \atop si} \right\} - \left\{ {m \atop ri} \right\}\left\{ {n \atop sj} \right\}\right)\\ R_{rsij} = G_{rp}R^p_{.sij} &= \frac{1}{2}\left(G_{rj,si}+G_{si,rj}-G_{ri,sj}-G_{sj,ri}\right) + G_{mn}\left(\left\{\left| {m \atop rj} \right|\right\}\left\{\left| {n \atop si} \right|\right\} - \left\{\left| {m \atop ri} \right|\right\}\left\{\left| {n \atop sj} \right|\right\}\right)\end{aligned}\right\}. \tag{1.4}$$

Putting,

$$\Lambda^r_{ij} = \Lambda^r_{ji} = \left\{\left| {r \atop ij} \right|\right\} - \left\{ {r \atop ij} \right\}, \tag{1.5}$$

which will be shown later to be a tensor, it follows from equation (1.3) that

$$\begin{aligned}R^p_{.sij} - {}_0R^p_{.sij} &= \Lambda^p_{sj}|_i - \Lambda^p_{si}|_j + \Lambda^m_{sj}\Lambda^p_{mi} - \Lambda^m_{si}\Lambda^p_{mj},\\ &= \Lambda^p_{sj}\|_i - \Lambda^p_{si}\|_j - \Lambda^m_{sj}\Lambda^p_{mi} + \Lambda^m_{si}\Lambda^p_{mj},\end{aligned} \tag{1.6}$$

where the stroke " $|$ " and " $\|$ " denote the covariant differentiation with respect to metric tensors in the original and strained state respectively.

If the displacement vector $\boldsymbol{v}$ is expressed as[6)]

$$\boldsymbol{v} = v_m\boldsymbol{g}^m = v^m\boldsymbol{g}_m = V_m\boldsymbol{G}^m = V^m\boldsymbol{G}_m, \tag{1.7}$$

it can be derived that

$$\delta^i_j = (\delta^i_m + v^i|_m)(\delta^m_j - V^m\|_j) = (\delta^m_j + v^m|_j)(\delta^i_m - V^i\|_m), \tag{1.8}$$

$$\left.\begin{aligned}v^i &= (\delta^i_m + v^i|_m)V^m \qquad & v_i &= (\delta^m_i - V^m\|_i)V_m\\ V^i &= (\delta^i_m - V^i\|_m)v^m & V_i &= (\delta^m_i + v^m|_i)v_m\end{aligned}\right\}, \tag{1.9}$$

$$\left.\begin{aligned}v^i|_j &= (\delta^i_m + v^i|_m)V^m\|_j \qquad & v_i|_j &= (\delta^m_i - V^m\|_i)V_m\|_j\\ V^i\|_j &= (\delta^i_m - V^i\|_m)v^m|_j & V_i\|_j &= (\delta^m_i + v^m|_i)v_m|_j\end{aligned}\right\}, \tag{1.10}$$

$$v^m|_i v_m|_j = V^m\|_i V_m\|_j. \tag{1.11}$$

The above relations plainly show that

$$v^i|_j = v^i|_{mj}V^m + (\delta^i_m + v^i|_m)V^m|_j = (\delta^i_m + v^i|_m)V^m\|_j.$$

From the definition of covariant differentiation, it results that

$$V^m\|_j - V^m|_j = \Lambda^m_{nj}V^n,$$

followed by a conclusion Λ^m_{nj} is a tensor,

so that

$$v^i|_{kj} = (\delta^i_m + v^i|_m)\Lambda^m_{kj}. \tag{1.12}$$

Evidently as usual[8)]

$$v^i|_{kj} - v^i|_{jk} = {}_0R^i_{.mjk}v^m,$$

so equations (1.5) and (1.12) demonstrate that

$$ {}_0R^i_{.mjk} = 0 \,. \tag{1.13}$$

In the similar way, one finds

$$ v^i|_{jkl} = (\delta^i_n + v^i|_n)\,(\Lambda^n_{jk}|_l + \Lambda^m_{jk}\,\Lambda^n_{ml}) \,. \tag{1.14}$$

Then equations (1.6), (1.13) and (1.14) lead to the relation

$$ R^i_{.mjk} = 0 \,. \tag{1.15}$$

Defining strain tensors[6] as

$$ \begin{aligned} r_{ij} &= \frac{1}{2}\Big(G_{ij}-g_{ij}\Big) = \frac{1}{2}\Big(v_i|_j + v_j|_i + v^m|_i\, v_m|_j\Big) \\ &= \frac{1}{2}\Big(V_i\|_j + V_j\|_i - V^m\|_i\, V_m\|_j\Big) \,, \end{aligned} \tag{1.16}$$

it follows from equations (1.4) and (1.5) that

$$ \begin{aligned} R_{ipqj} - {}_0R_{ipqj} &= G_{im}\,R^m_{.pqj} - g_{im}\,{}_0R^m_{.pqj} \\ &= r_{ij}|_{pq} + r_{pq}|_{ij} - r_{iq}|_{jp} - r_{jp}|_{iq} + (g_{mn} + 2r_{mn})\,(\Lambda^m_{ij}\,\Lambda^n_{pq} - \Lambda^m_{iq}\,\Lambda^n_{jp}) \\ &= r_{ij}\|_{pq} + r_{pq}\|_{ij} - r_{iq}\|_{jp} - r_{jp}\|_{iq} - (G_{mn} - 2r_{mn})\,(\Lambda^m_{ij}\,\Lambda^n_{pq} - \Lambda^m_{iq}\,\Lambda^n_{jp}) \,, \end{aligned} \tag{1.17}$$

which vanish identically by equations (1.13) and (1.15). Thus one obtains the conditions of compatibility

$$ \left. \begin{aligned} {}_0S^{\lambda\mu} &= {}_0S^{\mu\lambda} = {}_0\varepsilon^{\lambda ip}\,{}_0\varepsilon^{\mu jq}\left\{r_{ij}|_{pq} + \frac{1}{2}\Big(g_{mn}+2r_{mn}\Big)\,\Lambda^m_{ij}\,\Lambda^n_{pq}\right\} = 0 \\ S^{\lambda\mu} &= S^{\mu\lambda} = \varepsilon^{\lambda ip}\,\varepsilon^{\mu jq}\left\{r_{ij}\|_{pq} - \frac{1}{2}\Big(G_{mn}-2r_{mn}\Big)\,\Lambda^m_{ij}\,\Lambda^n_{pq}\right\} = 0 \end{aligned} \right\} , \tag{1.18}$$

where ${}_0\varepsilon^{ijk}$ and ε^{ijk} are given by equation (1.19) i. e., the ε-system.[6),8)]

$$ \left. \begin{aligned} {}_0\varepsilon^{ijk} &= e^{ijk}/\sqrt{g} \qquad & {}_0\varepsilon_{ijk} &= e_{ijk}\sqrt{g} \\ \varepsilon^{ijk} &= e^{ijk}/\sqrt{G} \qquad & \varepsilon_{ijk} &= e_{ijk}\sqrt{G} \end{aligned} \right\} . \tag{1.19}$$

It is shown further that

$$ \left. \begin{aligned} &(g_{mn} + 2r_{mn})\,\Lambda^m_{ij} = (\delta^m_n + v^m|_n)\,v_m|_{ij} \\ &(g_{mn} + 2r_{mn})\,\Lambda^m_{ij}\,\Lambda^n_{pq} = v_m|_{ij}\,v^m|_{pq} \\ &(G_{mn} - 2r_{mn})\,\Lambda^m_{ij} = (\delta^m_n - V^m\|_n)\,V_m\|_{ij} \\ &(G_{mn} - 2r_{mn})\,\Lambda^m_{ij}\,\Lambda^n_{pq} = V_m\|_{ij}\,V^m\|_{pq} \end{aligned} \right\} . \tag{1.20}$$

The above results are useful for verifying the equations of compatibility (1.18) directly.

Rewriting equation (1.16) as follows,

$$ \begin{aligned} G_{ij} &= g_{ij} + 2r_{ij} = g_{mn}\,(\delta^m_i + v^m|_i)\,(\delta^n_j + v^n|_j) \\ g_{ij} &= G_{ij} - 2r_{ij} = G_{mn}\,(\delta^m_i - V^m\|_i)\,(\delta^n_j - V^n\|_j) \,, \end{aligned}$$

one has

$$ \begin{aligned} \det|\,\delta^m_i + v^m|_i\,| &= \sqrt{\frac{G}{g}} = \sqrt{I_3} \\ \det|\,\delta^m_i - V^m\|_i\,| &= \frac{1}{\sqrt{I_3}} \,. \end{aligned}$$

Equation (1.8) and the above lead to the expression

$$\left.\begin{aligned}\delta^i_j+v^i|_j&=\frac{\sqrt{I_3}}{2!}\varepsilon^{i\lambda\mu}\varepsilon_{jpq}\left(\delta^p_\lambda-V^p\|_\lambda\right)\left(\delta^q_\mu-V^q\|_\mu\right)\\ \delta^i_j-V^i\|_j&=\frac{1}{2!\sqrt{I_3}}\,{}_0\varepsilon^{i\lambda\mu}{}_0\varepsilon_{jpq}\left(\delta^p_\lambda+v^p|_\lambda\right)\left(\delta^q_\mu+v^q|_\mu\right)\end{aligned}\right\}. \tag{1.21}$$

From equation (1.21) and the relations

$$ {}_0\varepsilon^{mnk}{}_0\varepsilon_{ijk}=\varepsilon^{mnk}\varepsilon_{ijk}=e^{mnk}e_{ijk}=\delta^{mn}_{ij}=\delta^m_i\delta^n_j-\delta^m_j\delta^n_i\,, \tag{1.22}$$

it is further derived that

$$\left.\begin{aligned}{}_0\varepsilon^{pqm}\left(\delta^i_m+v^i|_m\right)&=I_3\,\varepsilon^{i\lambda\mu}\left(\delta^p_\lambda-V^p\|_\lambda\right)\left(\delta^q_\mu-V^q\|_\mu\right)\\ \varepsilon^{pqm}\left(\delta^i_m-V^i\|_m\right)&=\frac{1}{I_3}\,{}_0\varepsilon^{i\lambda\mu}\left(\delta^p_\lambda+v^p|_\lambda\right)\left(\delta^q_\mu+v^q|_\mu\right)\end{aligned}\right\}. \tag{1.23}$$

2. Equation of Equilibrium

Green considered four types of stress tensors as follows,[6),7)]

$$\left.\begin{aligned}\boldsymbol{t}d\Omega&=\tau^{ij}n_i\boldsymbol{G}_j d\Omega=s^{ij}\,{}_0n_i\boldsymbol{G}_j d\Omega_0\\ &=\pi^{ij}n_i\boldsymbol{g}_j d\Omega=t^{ij}\,{}_0n_i\boldsymbol{g}_j d\Omega_0\end{aligned}\right\}, \tag{2.1}$$

where $\boldsymbol{t}$ represents the stress vector, $d\Omega$ the area element of the deformed body, and $\boldsymbol{n}=n_m\boldsymbol{G}^m$ the outward unit normal vector of the deformed surface, while subscript "o" denotes the quantity of the undeformed state.

Using the relations

$$\frac{{}_0n_i\,d\Omega_0}{\sqrt{g}}=\frac{n_i d\Omega}{\sqrt{G}}=d\theta^j d\theta^k \quad (i\neq j\neq k\neq i)\,, \tag{2.2}$$

$$\boldsymbol{G}_i=\boldsymbol{g}_i+\boldsymbol{v}_{,i}=\left(\delta^m_i+v^m|_i\right)\boldsymbol{g}_m\,,\qquad \boldsymbol{G}^i=\left(\delta^i_m-V^i\|_m\right)\boldsymbol{g}^m$$

it is easily shown that

$$\left.\begin{aligned}s^{ij}&=\sqrt{I_3}\,\tau^{ij} & t^{ij}&=\sqrt{I_3}\,\pi^{ij}\\ \tau^{ij}&=\pi^{im}\left(\delta^j_m-V^j\|_m\right) & t^{ij}&=s^{im}\left(\delta^j_m+v^j|_m\right)\end{aligned}\right\}. \tag{2.3}$$

Representing the mass density by ρ, the volume of the region taken arbitrarily in the body by V, the surface area of the region by Ω, and the external force per unit mass by $\boldsymbol{K}=k^m\boldsymbol{g}_m=K^m\boldsymbol{G}_m$, equation (2,1) is substituted into the equation of the momentum conservation in integral form,

$$\int_V\rho\boldsymbol{K}dV+\int_\Omega\boldsymbol{t}d\Omega=0\,, \tag{2.4}$$

and that of the angular momentum

$$\int_V\boldsymbol{R}\times\rho\boldsymbol{K}dV+\int_\Omega\boldsymbol{R}\times\boldsymbol{t}d\Omega=0\,, \tag{2.5}$$

which may be expressed differently as below,

$$\int_V(\boldsymbol{R}+\boldsymbol{c})\times\rho\boldsymbol{K}dV+\int_\Omega(\boldsymbol{R}+\boldsymbol{c})\times\boldsymbol{t}d\Omega=0\,.$$

Making use of Gauss-Green's integral formula

$$\left.\begin{aligned}\int_{V_0}a^i|_i dV_0&=\int_{\Omega_0}a^i\,{}_0n_i d\Omega_0\\ \int_V A^i\|_i dV&=\int_\Omega A^i n_i d\Omega\end{aligned}\right\}, \tag{2.6}$$

and observing the mass conservation law

$$\rho_0 dV_0 = \rho dV$$

the equations of equilibrium in differential form are obtained as follows:

$$\left.\begin{array}{ll} \tau^{ij}\|_i + \rho K^j = 0 & \varepsilon_{ijk}\tau^{ij} = 0 \\ \{\pi^{im}(\delta^j_m - V^j\|_m)\}\|_i + \rho K^j = 0 & \varepsilon_{ijk}\pi^{im}(\delta^j_m - V^j\|_m) = 0 \\ \{s^{im}(\delta^j_m + v^j|_m)\}|_i + \rho_0 k^j = 0 & {}_0\varepsilon_{ijk}s^{ij} = 0 \\ t^{ij}|_i + \rho_0 k^j = 0 & {}_0\varepsilon_{ijk}t^{mi}(\delta^j_m + v^j|_m) = 0 \end{array}\right\} \tag{2.7}$$

3. Stress Function

When the body is free from the mass force, the equations of equilibrum are given by

$$t^{ij}|_j = 0\,, \tag{3.1}$$

$$ {}_0\varepsilon_{ijk}t^{mi}\,(\delta^j_m + v^j|_m) = 0\,. \tag{3.2}$$

If the condition (1.13) holds, from equation (3.1) t^{ij} can be expressed as

$$t^{ij} = {}_0\varepsilon^{\lambda\mu i}a^j_{.\mu}|_\lambda\,, \tag{3.3}$$

where $a^j_{.\mu}$ is an arbitrary function forming a tensor field. Substituting equation (3.3) into equation (3.2) and observing the relation

$$ {}_0\varepsilon^{\lambda\mu m}v^j|_{m\lambda} = 0\,,$$

one gets

$$\{{}_0\varepsilon_{ijk}\,{}_0\varepsilon^{\lambda\mu m}a^i_{\ \mu}\,(\delta^j_m + v^j|_m)\}|_\lambda = 0\,.$$

Hence

$$ {}_0\varepsilon_{ijk}\,{}_0\varepsilon_{\lambda\mu\nu}a^i_{.\mu}\,(\delta^j_\nu + v^j|_\nu) = {}_0\varepsilon^{\lambda\mu\nu}b_{k\mu}|_\nu\,, \tag{3.4}$$

where $b_{k\mu}$ is a tensor. So equation (1.22) proves that

$$ {}_0\varepsilon^{ijq}a^m_{.\mu}\,(\delta^n_\nu + v^n|_\nu) - {}_0\varepsilon^{\lambda\mu\nu}a^n_{.\mu}\,(\delta^m_\nu + v^m|_\nu) = {}_0\varepsilon^{mnk}\,{}_0\varepsilon^{\lambda\mu\nu}b_{k\mu}|_\nu\,.$$

Since equation (1.8) holds, it is deduced that

$$\begin{aligned} &{}_0\varepsilon^{\lambda\mu q}a^m_{.\mu}(\delta^p_m - V^p\|_m) - {}_0\varepsilon^{\lambda\mu p}a^m_{.\mu}(\delta^q_m - V^q\|_m) \\ &\quad = {}_0\varepsilon^{mnk}\,{}_0\varepsilon^{\lambda\mu\nu}\,(\delta^p_m - V^p\|_m)\,(\delta^q_n - V^q\|_n)\,b_{k\mu}|_\nu\,. \end{aligned}$$

Changing the index λ, p, q cyclically, one finds

$$\begin{aligned} 2{}_0\varepsilon^{\lambda\mu q}a^m_{.\mu}(\delta^p_m - V^p\|_m) &= \{{}_0\varepsilon^{mnk}\,{}_0\varepsilon^{\lambda\mu\nu}(\delta^p_m - V^p\|_m)\,(\delta^q_n - V^q\|_n) \\ &\quad + {}_0\varepsilon^{mnk}\,{}_0\varepsilon^{q\mu\nu}(\delta^\lambda_m - V^\lambda\|_m)\,(\delta^p_n - V^p\|_n) \\ &\quad - {}_0\varepsilon^{mnk}\,{}_0\varepsilon^{p\mu\nu}(\delta^q_m - V^q\|_m)\,(\delta^\lambda_n - V^\lambda\|_n)\}\,b_{k\mu}|_\nu\,. \end{aligned}$$

Again by equation (1.8), it is shown that

$$\begin{aligned} 2{}_0\varepsilon^{\lambda\mu q}a^j_{.\mu} &= \{{}_0\varepsilon^{jmk}\,{}_0\varepsilon^{\lambda\mu\nu}\,(\delta^q_m - V^q\|_m) \\ &\quad + {}_0\varepsilon^{mjk}\,{}_0\varepsilon^{q\mu\nu}\,(\delta^\lambda_m - V^\lambda\|_m) \\ &\quad - {}_0\varepsilon^{mnk}\,{}_0\varepsilon^{p\mu\nu}(\delta^q_m - V^q\|_m)\,(\delta^\lambda_n - V^\lambda\|_n)\,(\delta^j_p + v^j|_p)\}\,b_{k\mu}|_\nu\,. \end{aligned}$$

If equations (1.21) and (1.23) are used, the following equation can be derived,

$$2\sqrt{I_3}\,{}_0\varepsilon^{\lambda\mu i}\,a^j_{.\mu} = ({}_0\varepsilon^{\lambda pq}\,{}_0\varepsilon^{imn} + {}_0\varepsilon^{ipq}\,{}_0\varepsilon^{\lambda mn} + {}_0\varepsilon^{\lambda in}\,{}_0\varepsilon^{mpq})\,(\delta^j_m + v^j|_m)\,(\delta^k_n + v^k|_n)\,b_{kp}|_q\,. \tag{3.5}$$

From equations (1.8), (1.14), (2.3), (3.3) and (3.5), it follows that

$$
\begin{aligned}
s^{ij} &= t^{ir}\left(\delta^j_r - V^j\|_r\right) \\
&= {}_0\varepsilon^{\lambda\mu i}\, a^r_{.\mu}|_\lambda \left(\delta^j_r - V^j\|_r\right) \\
&= \frac{1}{2}\left({}_0\varepsilon^{\lambda pq}\,{}_0\varepsilon^{inm} + {}_0\varepsilon^{ipq}\,{}_0\varepsilon^{\lambda mn} + {}_0\varepsilon^{\lambda in}\,{}_0\varepsilon^{mpq}\right) \\
&\quad\times\left[\delta^j_m \left\{\frac{\left(\delta^k_n + v^k|_n\right) b_{kp}|_q}{\sqrt{I_3}}\right\}\Bigg|_\lambda + A^j_{m\lambda}\,\frac{\left(\delta^k_n + v^k|_n\right) b_{kp}|_q}{\sqrt{I_3}}\right].
\end{aligned}
$$

Here one puts

$$
b_{kp} = c_{k\nu}\left(\delta^\nu_p + v^\nu|_p\right). \tag{3.6}
$$

Accounting for equation (1.22) and the relations

$$
\begin{aligned}
&\frac{\partial}{\partial\theta^i}\left(\frac{1}{\sqrt{I_3}}\right) = \frac{1}{\sqrt{I_3}}\bigg|_i = \frac{1}{\sqrt{I_3}}\bigg\|_i = -\frac{1}{\sqrt{I_3}}\,A^t_{ti} \\
&{}_0\varepsilon^{\lambda pq}\, c_{k\nu}|_{p\lambda} = 0 \qquad \text{etc.}, \\
&\left(\delta^k_t + v^k|_t\right) A^t_{n\lambda} - \left(\delta^k_r + v^k|_r\right) A^t_{t\lambda} = \left(\delta^k_r + v^k|_r\right) A^t_{s\lambda}\,\delta^{rs}_{tn} \\
&\qquad = {}_0\varepsilon^{rsl}\,{}_0\varepsilon_{tnl}\left(\delta^k_r + v^k|_r\right) A^t_{s\lambda},
\end{aligned}
$$

it is found after some calculation that

$$
\begin{aligned}
2\sqrt{I_3}\, s^{ij} &= \left({}_0\varepsilon^{ipq}\,{}_0\varepsilon^{\lambda jn} + {}_0\varepsilon^{\lambda in}\,{}_0\varepsilon^{jpq}\right)\left(\delta^k_n + v^k|_n\right)\left(\delta^\nu_p + v^\nu|_p\right)\left(c_{k\nu}|_{q\lambda} - A^m_{q\lambda} c_{k\nu}|_m\right) \\
&= {}_0\varepsilon^{ipq}\,{}_0\varepsilon^{\lambda jn}\left(\delta^k_n + v^k|_n\right)\left(\delta^\nu_p + v^\nu|_p\right) \\
&\quad\times\left[\left(c_{k\nu}|_{q\lambda} + c_{\nu k}|_{\lambda q}\right) - A^m_{q\lambda}\left(c_{k\nu} + c_{\nu k}\right)|_m\right].
\end{aligned}
$$

Putting as follows in the above relation

$$
f_{k\nu} = f_{\nu k} = \frac{1}{2}\left(c_{k\nu} + c_{\nu k}\right), \tag{3.7}
$$

one finally gets such a solution as

$$
s^{ij} = \frac{1}{\sqrt{I_3}}\,{}_0\varepsilon^{ipq}\,{}_0\varepsilon^{jrs}\left(\delta^\lambda_p + v^\lambda|_p\right)\left(\delta^\mu_r + v^\mu|_r\right)\left(f_{\lambda\mu}|_{qs} - A^m_{qs} f_{\lambda\mu}|_m\right). \tag{3.8}
$$

According to equation (2.3), one also finds

$$
t^{ij} = \frac{1}{\sqrt{I_3}}\,{}_0\varepsilon^{ipq}\,{}_0\varepsilon^{krs}\left(\delta^j_k + v^j|_k\right)\left(\delta^\lambda_p + v^\lambda|_p\right)\left(\delta^\mu_r + v^\mu|_r\right)\left(f_{\lambda\mu}|_{qs} - A^m_{qs} f_{\lambda\mu}|_m\right). \tag{3.9}
$$

The procedure employed above in terms of the metric tensor of original state for finite deformation is regarded as an extension of the method first developed by Dorn and Schild[4] for the case of infinitesimal deformation. It is needless to say that their procedure is directly applicable to the case of finite deformation, provided that metric tensors in the state after deformation are used.

Equation (3.8) can be demonstrated differently as follows. From the equations

$$
\tau^{ij}\|_i = 0, \qquad \varepsilon_{ijk}\tau^{ij} = 0,
$$

one obtains[4]

$$
\tau^{ij} = \varepsilon^{ipq}\,\varepsilon^{jrs} F_{pr}\|_{qs},
$$

where F_{pr} is a symmetrical tensor.

Equations (1.19) and (2.3) immediately give the expression

$$
\begin{aligned}
s^{ij} &= \sqrt{I_3}\,\tau^{ij} = \sqrt{I_3}\,\varepsilon^{ipq}\,\varepsilon^{jrs} F_{pr}\|_{qs} \\
&= \frac{1}{\sqrt{I_3}}\,{}_0\varepsilon^{ipq}\,{}_0\varepsilon^{jrs} F_{pr}\|_{qs}.
\end{aligned}
$$

Moreover equations (1.6), (1.13), (1.15) and the relations

$$F_{pr}\|_{qs}-(F_{pr}\|_{q})|_{s}=-\Lambda^{n}_{ps}F_{nr}\|_{q}-\Lambda^{n}_{rs}F_{pn}\|_{q}-\Lambda^{n}_{qs}F_{pr}\|_{n}\,,$$

$$F_{pr}\|_{q}-F_{pr}|_{q}=-\Lambda^{m}_{pq}F_{mr}-\Lambda^{m}_{rq}F_{pm}\,,$$

prove that

$${}_0\varepsilon^{ipq}\,{}_0\varepsilon^{jrs}F_{pr}\|_{qs}={}_0\varepsilon^{ipq}\,{}_0\varepsilon^{jrs}\{F_{pr}|_{qs}+\Lambda^{m}_{pr}(F_{mq}|_{s}+F_{ms}|_{q}-F_{qs}|_{m}-\Lambda^{n}_{qs}F_{mn})\}\,.$$

If one define

$$F_{pr}=(\delta^{\lambda}_{p}+v^{\lambda}|_{p})(\delta^{\mu}_{r}+v^{\mu}|_{r})h_{\lambda\mu}\,,\qquad h_{\lambda\mu}=h_{\mu\lambda}\,, \tag{3.10}$$

and perform some calculation, it is derived that

$${}_0\varepsilon^{ipq}\,{}_0\varepsilon^{jrs}F_{pr}\|_{qs}={}_0\varepsilon^{ipq}\,{}_0\varepsilon^{jrs}(\delta^{\lambda}_{p}+v^{\lambda}|_{p})(\delta^{\mu}_{r}+v^{\mu}|_{r})(h_{\lambda\mu}|_{qs}-\Lambda^{m}_{qs}h_{\lambda\mu}|_{m})\,.$$

Accordingly, one has

$$s^{ij}=\frac{1}{\sqrt{I_3}}\,{}_0\varepsilon^{ipq}\,{}_0\varepsilon^{jrs}(\delta^{\lambda}_{p}+v^{\lambda}|_{p})(\delta^{\mu}_{r}+v^{\mu}|_{r})(h_{\lambda\mu}|_{qs}-\Lambda^{m}_{qs}h_{\lambda\mu}|_{m})\,,$$

which is the same as equation (3.8).

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