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THEORY OF DIFFUSION IN TURBULENCE*

By Kazunori KITAJIMA

Summary

Generalized definitions of Lagrangian correlations in turbulence were introduced, by means of which transfer equations of heat or momentum in trubulence were derived in the form of integro-differential equations. The equation of heat in particular provides a generalized equation of Fokker-Planck in the theory of "Random Walk" and contains the relation presented by G. I. Taylor (1921) as a relation derivable in a special case.

Then a theory determining the correlations in steady process was developed by introducing the approximation called as multiple Markov process approximation. On the other hand, the conception of self-preserving decay model was introduced for homogeneous shear flows. Then based on these considerations a theory of nearly isotropic shear flows was proposed.

§ 1. Introduction

Since the first study of Osborne Reynolds[1] characteristic mechanisms in fully developed turbulence have been pursued on the two lines, the one is the mechanism of transformation of energy from eddies to heat, and the other is that of transference of heat or momentum caused by eddies. The statistical theory introduced by G. I. Taylor[2] and Th. von Kármán[3], and directed to the former mechanism, has been developed so far by Kolmogoroff[4], Onsager[5] and Weizsäcker[6], that the reasonable explanations have been presented for the local structure of the energy spectrum of eddies.

While, as for the latter, the mechanism of transference, in which anisotropic eddies of larger scale not included in the above theories play an important role, no reasonable theory has yet been developed beyond the stage of "Mixing length" theories.

Prandtl[7] and Taylor[8] had introduced the conception of "Mixing length" which is analogous to the mean free path in the kinetic theory of gas. In these theories, however, mixing length itself must be determined based on the empirical ground. Gebelein[9] and Frenkiel[10] had proposed a transfer theory based on the stochastic theory, but the limitation of "Markov process" used in these theories could not be applied with success to the wide ranged spacial and temporal correlations of turbulent velocities which are particularly characteristic in turbulence. Corrsin[11] and Batchelor[12] had developed a correlation theory which is based on

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Eulerian view point of fluid flow and derived some usefull results.

However, an important step to the more reasonable theory had been suggested by G. I. Taylor[13] (1921). He had introduced the conception of Lagrangion correlation of turbulent velocities and applied it to a problem of diffusion in isotropic turbulence.

In this paper, extending the line of thought presented by Taylor, we develope a general theory of transfer in turbulence.

§ 2. Derivation of general transfer equations

§ 2. 1. Preliminaly formulations

From Lagrangian view point of fluid flow, a state of turbulent flow may be defined as a statistical state of a dynamical system constituting of large number of identical particles, whose motions are described by Navier-Stokes' equations of motion in their Lagrangian form.

In order to describe the statistical behaviour of a particle the formality used in the classical statistical mechanies has particular merits[14]. So that we shall now derive the transfer equations presented by 0. Reynolds by means of the formality used by Kirkwood[15], and Born and Green[16] in their theory of general irreversible process.

Now we shall consider a dynamical system composed of N identical particles. Denote space and velocity co-ordinates of the particles by (x_i $i=1\cdots 6N$) and time by t. The equations of motion of N particles are provided by

$$\frac{d}{dt}x_i=u_i, \qquad \cdots\cdots\cdots(1)$$

where u_i is unique function of $(x_j,\ t)$.

The statistical states of the system are microscopicaly specified by the probability density ρ in the phase space (x_i), and the time change of which is determined uniquely by the continuity equation of probability density in the phase space,

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x_i}(\rho u_i)=0. \qquad \cdots\cdots\cdots(2)$$

The characteristic lines of the partial differential equation are described by the trajectories of the motion of N particles determined by the equations (1). Then the Largragian coordinates $(a_i,\ s)$ in the phase space are defined by means of the trejectories of the N particles,

$$s=t, \quad a=x_i(a_j,\ s)_{S=0}.$$

From the definition we may obtain the relation

$$\frac{\partial}{\partial s}=\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x_i}\ . \qquad \cdots\cdots\cdots(3)$$

In virtue of the Lagrangian co-ordinates the continuity equation (2) is re-expressed in the form

$$\frac{\partial}{\partial s}(\sigma) = 0 \ , \qquad \cdots\cdots\cdots(2)'$$

where $\sigma = \rho \dfrac{\partial(x_1 \cdots x_{6N})}{\partial(a_1 \cdots a_{6N})}$, and $\dfrac{\partial(x_1 \cdots x_{6N})}{\partial(a_1 \cdots a_{6N})}$, is the Jacobian of the transformation of variables.

Then we shall define a physical quantity φ which is defined as an arbitarary function of (x_i, t). We obtain by means of (2)'

$$\frac{\partial}{\partial s}(\varphi\sigma) = \sigma\dot{\varphi} \ , \quad \dot{\varphi} = \frac{\partial}{\partial s}\varphi . \qquad \cdots\cdots\cdots(4)$$

Using (3), (4) is re-expressed as

$$\frac{\partial}{\partial t}(\rho\varphi) + \frac{\partial}{\partial x_i}(\rho\varphi u_i) = \rho\dot{\varphi} \ . \qquad \cdots\cdots\cdots(4)'$$

Then we shall now introduce the weight function θ which is defined as a phase function invariable along trajectories, for which we obtain

$$\frac{\partial}{\partial s}(\sigma\theta) = 0 \ , \quad \frac{\partial\theta}{\partial s} = 0 \ , \qquad \cdots\cdots\cdots(5)$$

$$\frac{\partial}{\partial t}(\rho\theta) + \frac{\partial}{\partial x_i}(\rho\theta u_i) = 0 \ . \qquad \cdots\cdots\cdots(5)'$$

Macroscopic state of the system may be defined statistically by means of small number of mean values or correlations, time change of which are described by transfer equations. We consider the subspace x_i $i=1\cdots n$ composed of, for instance, space co-ordinates of a particle $(n=3)$, then averaging (2)' and (4)' over the remaining co-ordinates $(x_j \ j=n+1\cdots\cdots 6N)$, we obtain the continuity equation and transfer equations of physical quantity φ,

$$\frac{\partial}{\partial t}\bar{\rho} + \frac{\partial}{\partial x_i}(\bar{\rho}\,\bar{u}_i) = 0 \ , \qquad \cdots\cdots\cdots(6)$$

$$\frac{\partial}{\partial t}\bar{\rho}\,\bar{\varphi} + \frac{\partial}{\partial x_i}\{\bar{\rho}\,\bar{\varphi}\,\bar{u}_i + \bar{\rho}\,\overline{(u_i-\bar{u}_i)(\varphi-\bar{\varphi})}\} = \bar{\rho}\,\bar{\dot{\varphi}}, \qquad \cdots\cdots\cdots(7)$$

where mean density and mean values are defined as

$$\bar{\rho} = \int\cdot\int \rho \prod_{j=n+1}^{6N} dx_j \ ,$$

$$\bar{\rho}\,\bar{\varphi} = \int\cdot\int \rho\varphi \prod_{j=n+1}^{6N} dx_j \ .$$

Then we shall consider the problem of diffusion of a particle in the space (x_i). In the theory of "Random Walk" we concern with the conditional probability $\psi(x_i, t/x_{i,0}, 0)\prod_{i=1}^{n} dx_i$ of a particle started from the point $x_{i,0}$ at time $t=0$, and staying in the space range (x_i, x_i+dx_i) at time $t=t$. Selecting θ such as

$$\theta_{t=0} = \prod_{i=0}^{n} \delta\,(x_i - x_{i,0}) \ ,$$

where δ is the Dirac's delta function, the probability density ψ is expressed by

$$\psi = \frac{1}{\bar\rho(x_{i,0}, 0)} \bar\rho(x_i, t)\,\bar\theta(x_i, t\,|\,x_{i,0}, 0)\ , \qquad \cdots\cdots\cdots(8)$$

or re-defined by means of Lagrangian co-ordinates as

$$= \frac{1}{\bar\rho(x_{i,0},\ 0)} \int\cdot\int \theta\,\sigma\,\vartheta \prod_{i=1}^{6N} da_i\ ,$$

where ϑ is the weight function defined as

$$\vartheta_{t=t} = \prod_{i=1}^{n} \delta\,(x_i - x_i)\ .$$

Using (2)′ conservation of total probability is easily verified. Finally averaging the equation (5)′ the continuity equation of the partial probability density ψ is provided by the diffusion equation of a weight function,

$$\frac{\partial}{\partial t}\bar\rho\,\bar\theta + \frac{\partial}{\partial x_i}\{\bar\rho\,\bar\theta\,\bar u_i + \bar\rho\,\overline{(u_i-\bar u_i)\,(\theta-\bar\theta)}\,\} = 0\ . \qquad \cdots\cdots\cdots(9)$$

§ 2. 2. Re-expression of transfer terms

The terms $\bar\rho\overline{(u_i-\bar u_i)(\theta-\bar\theta)}$ and $\bar\rho\,\overline{(u_i-\bar u_i)(\varphi-\bar\varphi)}$ are the transfer terms, and have been interpretted by analogy with the kinetic theory of gas. A portion of fluid mass started from a layer and carring say mean quantity of heat of that layer, proceeds some distance L across mean stream and mixes with surrounding fluid, thus transfers mean heat from a layer to a layer.

In order to examine precisely the process of mixing, we shall at first re-express the transfer term by means of Lagrangian co-ordinates,

$$\bar\rho\,\overline{(u_i-\bar u_i)(\theta-\bar\theta)} = \int\cdot\int (u_i-\bar u_i)\,(\theta-\bar\theta)\,\sigma\vartheta \prod_{i=1}^{6N} da_i.$$

Then tracing the term $(\theta-\bar\theta)$ along the trajectory of a particle back to past, re-express the term in the form of time integral,

$$(\theta-\bar\theta)_{x,t} = \int_0^t \frac{\partial}{\partial s}(\theta-\bar\theta)ds + (\theta-\bar\theta)_{S=0}. \cdots(10)$$

Since $\frac{\partial}{\partial s}\theta = 0$ and $(\theta-\bar\theta)_{S=0} = 0$ by definitions, and $\frac{\partial\bar\theta}{\partial s}$ is re-expressed using (3) as

$$\frac{\partial}{\partial s}\bar\theta = \frac{\partial}{\partial \tau}\bar\theta + u_i\frac{\partial}{\partial \xi_i}\bar\theta,$$

the transfer term is re-expressed by

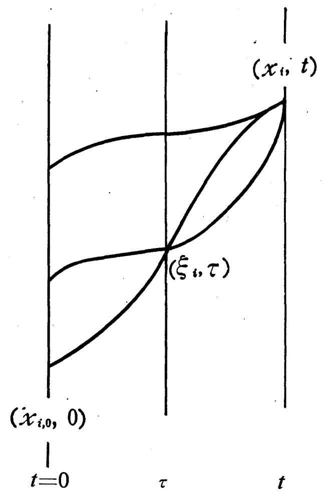

Fig. (1).

$$-\int_0^t \int \cdot \int \left(U_i \frac{\partial \bar{\theta}}{\partial \tau} + R_{ij} \frac{\partial \bar{\theta}}{\partial \xi_j} \right) D \prod_{j=1}^{n} d\xi_j dt, \qquad \cdots\cdots\cdots(11)$$

where R_{ij} is the correlation between the compornent of velocity $(u_i - \bar{u}_i)$ of a particle at the fixed point (x_k, t) and the compornent of velocity (u_j) of the same particle at the other point (ξ_k, τ), U_i is the partial mean of the compornent of velocity $(u_i - \bar{u}_i)$ which a particle starded from a point (ξ_k, τ) has at the point (x_k, t), and D is the joint probability density of the same particle at the two points (ξ_k, τ) and $(x\ , t)$, defined as follows;

$$U_i = U_i\,(\xi_k, \tau\,;\, x_k, t) = \frac{1}{D} \int (u_i - \bar{u}_i)_{s=t}\, \vartheta^* \sigma \vartheta \prod_{i=1}^{6N} da_i$$

$$R_{ij} = R_{ij}(\xi_k, \tau\,;\, x_k, t) = \frac{1}{D} \int (u_i - \bar{u}_i)_{s=t} (u_k)_{s=\tau}\, \vartheta^* \sigma \vartheta \prod_{i=1}^{6N} da_i$$

$$D\ = D(\xi_k, \tau\,;\, x_k, t) = \int \vartheta^* \sigma \vartheta \prod_{i=1}^{6N} da_i$$

$$\vartheta^*_{t=\tau} = \prod_{i=1}^{n} \delta(x_i - \xi_i)\ .$$

Thus the transfer equation of $\bar{\theta}$ is re-expressed in the form

$$\frac{\partial}{\partial t} \bar{\rho}\, \bar{\theta} + \frac{\partial}{\partial x_i} \left[\bar{\rho}\, \bar{\theta}\, \bar{u}_i - \int_0^t \int \cdot \int \left\{ U_i \frac{\partial \bar{\theta}}{\partial \tau} + R_{ij} \frac{\partial \bar{\theta}}{\partial \xi_j} \right\} D \prod_{k=1}^{n} dx_k\ d\tau \right] = 0, \quad \cdots\cdots\cdots(12)$$

or re-writing by means of (8), we obtain the continuing equation of ψ:

$$\frac{\partial}{\partial t} \psi + \frac{\partial}{\partial x_i} \left[\psi \bar{u}_i - \int_0^t \int \cdot \int \left\{ U_i \frac{\partial \psi}{\partial \tau} + R_{ij} \frac{\partial \psi}{\partial \xi_j} \right. \right.$$
$$\left. \left. - \frac{\psi}{\bar{\rho}} \left(U_i \frac{\partial \bar{\rho}}{\partial \tau} + R_{ij} \frac{\partial \bar{\rho}}{\partial \xi_j} \right) \right\} \frac{D}{\bar{\rho}} \prod_{k=1}^{n} d\xi_k\, d\tau \right] = 0\ . \qquad \cdots\cdots\cdots(13)$$

The same procedure can be applied for the transfer term of a physical quantity φ. However in this case we must take into account the time change of φ along a trajectory by (4)

$$\frac{\partial}{\partial s}(\varphi - \bar{\varphi}) = -\left(\frac{\partial \bar{\varphi}}{\partial \tau} + u_i \frac{\partial \bar{\varphi}}{\partial \xi_i} \right) + \dot{\varphi}\ ,$$

and replace the limit of integration by $-\infty$.

Thus we obtain the result:

$$\frac{\partial}{\partial t} \bar{\rho}\, \bar{\varphi} + \frac{\partial}{\partial x_i} \left[\bar{\rho}\, \bar{\varphi}\, \bar{u}_i - \int_{-\infty}^t \int \cdot \int \left\{ U_i \frac{\partial \bar{\varphi}}{\partial \tau} + R_{ij} \frac{\partial \bar{\varphi}}{\partial \xi_j} - \Phi_i \right\} D \prod_{k=1}^{n} d\xi_k\, d\tau \right] = \bar{\rho}\, \bar{\dot{\varphi}}, \quad \cdots(14)$$

where Φ_i is the correlation between the compornent of velocity $(u_i - \bar{u}_i)$ and $\dot{\varphi}$ of the same particle

$$\Phi_i = \Phi_i(\xi_k, \tau\,;\, x_k, t) = \frac{1}{D} \int (u_i - \bar{u}_i)_{s=t} (\dot{\varphi})_{s=\tau}\, \vartheta^* \sigma \vartheta \prod_{i=1}^{6N} da_i.$$

We have derived the above equations for the dynamical system composed of

N identical particles, but we can prove that the same equations hold for the dynamical system of continuum such as described by Navier-Stokes' equations of motion, and also for the stochastic process under the limitation that the sample processes have the contineous velocities.

§ 3. Examinations of the diffusion equations for particular cases

§ 3. 1. Isotropic turbulence

Omitting the detailed explanstions for these formulations, however, we shall then examine the diffusion equation (13) for particular cases. At first, we shall consider the case of isotropic and homogeneous turbulence, and take x_i $i=1,2,3$ the space co-ordinates of a fluid particle. In this case $\bar{\rho}=\text{const.}$, $\bar{u}_i=0$, and the correlations R_{ik} and U_i degenerates into the form

$$U_i=e\,\frac{r_i}{r},\quad R_{ij}=(f-g)\frac{r_i r_j}{r^2}+\delta_{ij}g,$$

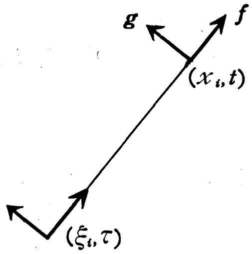

Fig. (2).

where e, f, g is the scaler functions defined as, Fig. (2),

$$e=\frac{\overline{(u_i r_i)_{x,t}}^{\xi,\tau}}{r},\quad f=\overline{(u_i r_i)_{\xi,t}\,(u_i r_i)_{x,t}}/r^2,$$

$$g=\overline{(u_i s_i)_{\xi,\tau}\,(u_i s_i)_{xt,}}/s^2,\quad r_i=x_i-\xi_i,\quad (r_i s_i)=0.$$

Then, multiplyin g x_i^n to the equation (13) and integrating with respect to the space co-ordinates over whole space, we obtain the expresstions for the time change of the moments. For instances

$$\frac{d\overline{x_i^n}}{dt}=\int x_i^n\frac{\partial\psi}{\partial t}\prod_{j=1}^{3}dx_j=-\int x_i^n\frac{\partial}{\partial x_j}\Big[\int_0^t\int\cdot\int\Big\{U_j\frac{\partial\psi}{\partial\tau}+R_{jk}\frac{\partial\psi}{\partial\xi_k}\Big\}D\prod_{k=1}^{3}d\xi_k d\tau\Big]\prod_{j=1}^{3}dx_j,$$

$$\frac{d\bar{x}_i}{dt}=0,$$

$$\frac{d\overline{x_i^2}}{dt}=2\int_0^t R_{11}(\tau)d\tau,\quad R_{11}(\tau)=\frac{1}{3}\int(f+2g)\,D4\pi r^2 dr, \qquad \cdots\cdots\cdots(15)$$

....

The equation (15) is no more than the equation derived by G. I. Taylor(13).

§ 3. 2. Fokker-Planck's equation in velocity space

Then we shall consider a steady process in the space (x_i, u_i, $i=1,2,3$) composed of the space and vecocity coordinates of a particle. In this case $\bar{\rho}(x,u)$ is no more constant. By means of the similar procedure stated above we can derive the approximate expressions for the moments

$$A_i=\frac{\overline{\Delta u_i}}{\Delta t}=\bar{\dot{u}}_i-\Big(\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial u_j}\Big)D_{ij}$$

$$D_{ii}=\frac{\overline{\Delta u_i^2}}{2\Delta t}=\frac{1}{\Delta t}\int_0^{\Delta t}\int_0^{\tau}\overline{\dot{u}_i\,\dot{u}_i(\tau')}\,d\tau' d\tau\ .$$

Using these moments and after some approximations, we can derive formally from (13) the equation for ψ similar to Fokker-Planck's one.

$$\frac{\Delta\psi}{\Delta t}+\frac{\partial}{\partial x_i}\left[A_i\psi - D_{ij}\frac{\partial\psi}{\partial u_j}\right]=0. \qquad \cdots\cdots\cdots(16)$$

The similar expressions have been presented by Kirkwood[15] and M. S. Green[7] with regard to their theory of general irreversible process. And the expressions for the moments were considered to be particularly important, because they relate the microscopic and dynamical behaviour of the system to macroscopic and statistical one through the equation (16). However we must point out a fact which was miss-interpretted in their theories, that is, the values of moments, and consequently the degree of approximation of the equation to the original process, are in general dependent closely on the choice of Δt.

General characters of the arguments can be illustrated by the Fig. (3), where curves 1 and 2 are the correlations of velocities $\overline{uu(\tau)}$ and accelalations $\overline{\dot{u}\dot{u}(\tau)}$ respectively, and the curve 3 is the moment D derived above. The curve $\overline{\dot{u}\dot{u}(\tau)}$ must in general have the negative part as shown in the figure, because in steady process the correlation $\overline{\dot{u}\dot{u}(\tau)}$ is related to the correlation $\overline{uu(\tau)}$ by $\overline{\dot{u}\dot{u}(\tau)} = -\frac{d^2}{d\tau^2}\overline{uu(\tau)}$, and the integral $\int_0^\infty \overline{\dot{u}\dot{u}(\tau)}\,d\tau = \frac{d}{d\tau}\overline{uu(\tau)}_{\tau=0} = 0$ must vanish. Correspondingly D must have a maximum and then decrease and vanish at infinity. While Kirkwood have assumed that D has the plateau value independent of Δt when Δt is sufficiently larger than the microscopic time scale.

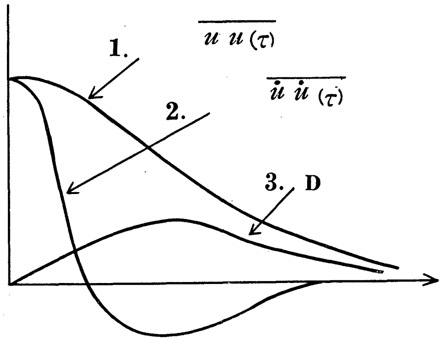

Fig. (3).

His argument is really correct, as we can prove it, only for such a case as the Brownian motion of a colloidal particle which has sufficiently larger mass than those of surrounding particles, but not for the systems such as composed of similar particles.

§ 3. 3. Multiple Markov process approximation

Then we shall suggest an alternative method determining the approximate process.

According to Doob's theorem[18] in the theory of stochastic process, it is well known that for any given stationary process there exsists a normal process which has the same correlation R with that process, and the normal process is determined uniquely provided that the correlation R is given. Based on the theorem we shall now find an appropriate normal Markov process which approximates the original process.

We shall now take a normal Markov process in the space $(x, \dot{x}, \ddot{x} \cdots x^{(n)})$,

$x^{(i)} = \frac{d^i}{dt^i} x$, which is defined by Fokker-Planck's equation,

$$\frac{\partial}{\partial t}\psi + \left[\frac{\partial}{\partial x}\dot{x} + \frac{\partial}{\partial \dot{x}}\ddot{x} + \cdots + \frac{\partial}{\partial x^{(n)}}\left(A\psi - D\frac{\partial}{\partial x^{(n)}}\right)\right]\psi = 0, \quad \cdots\cdots\cdots(17)$$

where A is a linear function of $x^{(i)}$, and D a constant. Then we shall determine the constants so that the moments $\overline{x^{(i)2}}$, the second moments of i-th derivatives, of the approximate process have the same values as those of the original process, and the correlation of the n-th derivatives $\overline{x^{(n)}x^{(n)}(\tau)}$ has the same value as the original one at the point τ^* where the correlation is reduced by half $\overline{x^{(n)}x^{(n)}(\tau^*)} = \overline{x^{(n)2}}/2$, Fig. (4). Since the original correlation $\overline{xx(\tau)}$ can be expanded into the series

$$\overline{xx(\tau)} = \overline{x^2} - \overline{\dot{x}^2}\frac{\tau^2}{2!} + \cdots (-1)^i\,\overline{x^{(i)2}}\,\frac{\tau^{2i}}{2i!} + \cdots,$$

we may have the closer approximation for $\overline{xx(\tau)}$, the larger the value of n. Thus the problem of determination of the process is reduced to that of the calculations of the second moments $\overline{x^{(i)2}}$ by means of the dynamical equations (1). We shall call the procedure stated above the multiple Markov process approximation.

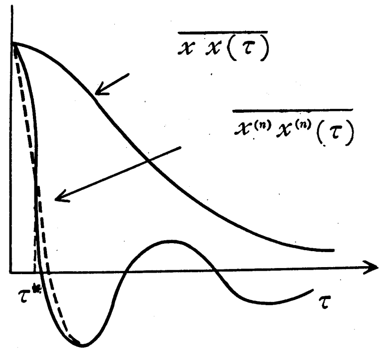

Fig. (4).

§ 4. Theory of nearly isotropic shear flow

Then we shall proceed to the problem of determination of transfer coefficients in nearly isotropic shear flows.

It is well known that the turbulence behind grid has the strong tendency to decay preserving its spectrum form(19). Extending the conception to the case of nearly isotropic shear flows, we shall introduce now a model in which turbulence and mean flow are assumed to decay preserving its spectrum form in company with relative magnitude of mean shear.

In our experimences, such a condition is known to be realized approximately in the free turbulences such as two or three dimensional wake and jet(25). In homogeneous shear flows, however, we may note such a condition will only be realized under the limitation that some appropriate devises are enabled to controll mean flow and largest eddies so as to maintain the similarity of spectrum form.

The mathematical treatment of the model thus introduced are considerably simplified, since it is reduced to that of steady flow by means of appropriate transformation of variables, and for which the theories stated above can be applied.

Abridging the detailed explanations for calculations, we shall now review our main results.

We shall consider a homogeneous shear flow, which has mean shear $\frac{dU}{dy}$. The non-dimensional parameter $\frac{L}{v'}\frac{dU}{dy}$ constructed from the root mean square of turbulent velocity v', and mean scale of eddies L, is considered to be appropriate characterizing the effect of mean shear to turbulent motions. Then we shall consider the case, $\frac{L}{v'}\frac{dU}{dy} \ll 1$, that is, the parameter is sufficiently smaller than 1, and the parameter is maintained to be constant during the period of decay in our model.

From the equations (12) and (14), the transfer terms of heat and momentum are in this case simplified as,

$$\overline{v(\theta-\bar{\theta})}=\epsilon_\theta\frac{d\bar{\theta}}{dy}\ ,\quad \epsilon_\theta=\int_{-\infty}^{0}\overline{v\,v\,(\tau)}\,d\tau$$

$$\overline{v(u-\bar{u})}=\epsilon_m\frac{dU}{dy}\ ,\quad \epsilon_m=\int_{-\infty}^{0}\left\{\overline{v\,v\,(\tau)}+\overline{v\frac{\partial p(\tau)}{\partial x}}\Big/\rho\,\frac{dU}{dy}\right\}d\tau\ . \qquad \cdots\cdots\cdots(18)$$

In these expressions the transfer coefficient of momentum ϵ_m differs from that of heat ϵ_θ by the term $\overline{v\frac{\partial p(\tau)}{\partial x}}$, which is the correlation between the compornent of velocity v and the pressure gradient $\frac{\partial p}{\partial x}$ of a same fluid particle.

On the other hand, for a given energy spectrum of isotropic turbulence, we can calculate by means of Navier-Stokes' equations of motion, and under the method of perturbation, the moments

$$\overline{\left(\frac{\partial p}{\partial y}\right)^2}\ ,\quad \overline{\left\{\frac{d}{dt}\left(\frac{\partial p}{\partial y}\right)\right\}^2}\ ,\quad \overline{v\frac{\partial p}{\partial x}}\ ,\quad \overline{\frac{\partial p}{\partial y}\left\{\frac{d}{dt}\left(\frac{\partial p}{\partial x}\right)\right\}}\ ,$$

from which we can obtain the correlations $\overline{vv(\tau)}$ and $\overline{v\frac{\partial p(\tau)}{\partial x}}$ by means of the theory stated above.

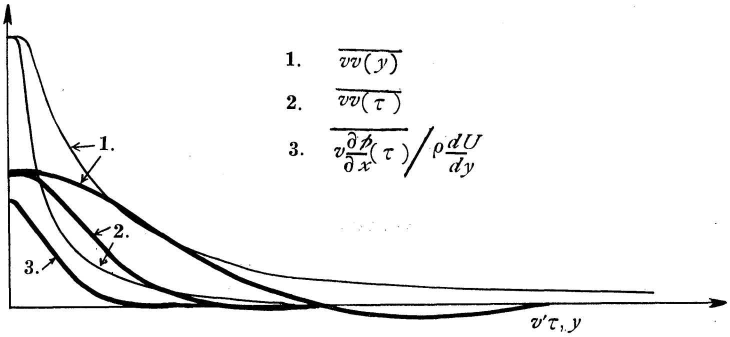

Fig. (5).

In Fig. (5), the thick lines are the calculated Eulerian correlation $\overline{vv(y)}$ and the Lagrangian correlations $\overline{vv(\tau)}$ and $\overline{v\frac{\partial p(\tau)}{\partial x}}$ respectively. In these calculations the form of spectrum is assumed to be a spherical shell in the wave number space. While the thin lines are the experimental ones in the turbulence behind grid taken from Taylor's paper[20].

Then, in order to analyse shear flows, we show the Fig. (6), where the abscissae is the parameter G/ε, where G and ε are the rate of generation and dissipation of turbulent energy respectively, and is related to the parameter

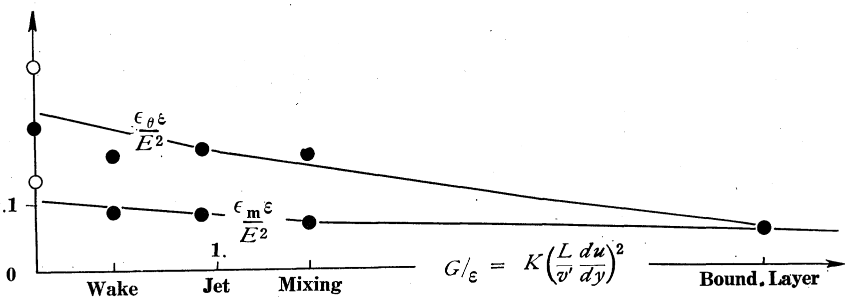

Fig. (6).

$\frac{L}{v'}\frac{dU}{dy}$ as $G/\varepsilon = k\left(\frac{L}{v'}\frac{dU}{dy}\right)^2$. And non-dimensional diffusion and momentum transfer coefficients $\frac{\epsilon_\theta \varepsilon}{E^2}$ and $\frac{\epsilon_m \varepsilon}{E^2}$, where E is the turbulent energy, are plotted against the parameter G/ε for shear flows. The closed circles are the representative experimental values of the two dimensional wake[21], jet[22], free mixing flow[23], and boundary layer on a flat plate[24] repectively, taken from the papers of Townsent, Tamaki, Liepmann and Laufer, and Schbauer and other available experiments[25]. These values are obtained averaging local values over y section. As is shown, the difference between transfer coefficients of heat and that of momentun is remarkable particularly in free turbulences. This fact was first pointed out by G. I. Taylor[26] in his vorticity transfer theory.

While theoretical values are these open circles. We shall notice here that the value of the transfer coefficient of momentum $\frac{\epsilon_m \varepsilon}{E^2}$ in particular can be derived without any other assumptions or approximations than that of self-preserving decay model, and takes the exact value 2/15 irrespective of particular form of energy spectrum assumed.

Taking into account the roughness of our approximations employed and the nature of scatterings of experimental values, the coincidence of theoretical values with experimental ones seems to be fairly well[27].

§ 5. Conclusion

In conclusion, we shall summarize our main results into the three articles.

1) At first, we introduced the generalized definitions of Lagrangian correlations, by means of which the correct mathematical expressions are obtained for the transfer terms connecting with the mean gradients of heat and momentum. Thus the transfer equations in turbulence presented by 0. Reynolds were replaced in the form of integro-differential equations. The equations of heat, in particular,

contains the relation provided by Taylor as a relation derivable in a special case.

2) Secondly, we discussed the relations between the continuity equation of probability density (13), and the equation of Fokker-Planck in the theory of "Random Walk", and also that of Kirkwood in the theory of irreversible process. Then, based on these considerations, we proposed an approximation theory, called as multiple Markov process approximation, for steady dynamical process.

3) Thirdly, we introduced the conception of self-preserving decay model for shear flows, which combining with the theories developed above enabled us to calculate the Lagrangian correlations and transfer coefficients for nearly isotropic shear flow, thus obtained the Fig. (4) and (5). In particular, the theory provided an explanation for the difference between transference of heat and that of momentum in free turbulences, which was the problem presented first by G. I. Taylor.

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NOTE

Supplementary Data for $\bar{A}_z$ and K_4

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The progressive wave height and added mass of two dimensional body heaving in a free surface were calculated for wide Lewis form section of H=4.0.

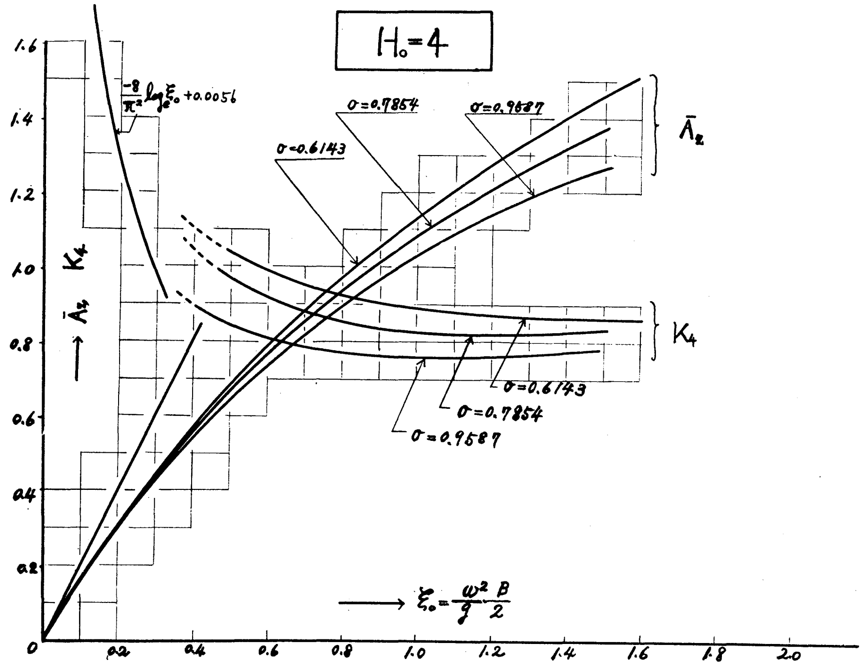
$\bar{A}_z$ and K_4 were shown in the Figure. The following nomenclature is used in this note.

$H_0 = \frac{B}{2T}$, B = breadth of the cylinder in a still water line

T = draught of the cylinder

$$\bar{A}_z = \frac{\text{amplitude of progressive wave}}{\text{amplitude of forced heaving}}$$

$$K_4 = \frac{\text{two dimensional added mass of cylinder}}{\frac{1}{2}\rho\pi\left(\frac{B}{2}\right)^2 \cdot C_0}$$



$$C_0 = \frac{\text{added mass of cylinder in case of } \omega \to \infty}{\frac{1}{2}\rho\pi\left(\frac{B}{2}\right)^2}$$

ρ = deusity of fluid

ω = circular frequency of heaving

$\sigma = \frac{A}{BT}$ = sectional area coefficient

A = immersed sectional area of the cylinder

For $H_0=4.0$, Co is given as follows:

σ	C_0
0.9587	1.1030
0.7854	1.0
0.6143	0.9249

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