STUDIES ON THERMO-PLASTIC WORKING OF STEEL 1: Thermo-Plastic Elongation and Contraction of Strip

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STUDIES ON THERMO-PLASTIC WORKING OF STEEL [1]
(Thermo-Plastic Elongation and Contraction of Strip)

By Toshiro SUHARA

Introduction

When metalic plates or bars heated non-uniformly above certain temperatures are cooled, the inelastic strains produce in the materials causing the permanent deformations of plates or bars with residual stresses in them. Although these phenomena are usually undesirable in welding or heat-treating the materials, yet these inelastic strains may be utilized in order to bend plates and bars into required shapes or to set adequate residual stresses in the materials.

The inelastic strains consist of two parts, namely the plastic strain and the visco-elastic strain. When the above processes are performed in a short time and metals are not so highly heated, the visco-elastic strains are very small during operations. The above thermal processes are generally called the "Thermo-Plastic Working" and are distinguished from the usual mechanical plastic working.

The thermo-plastic working may be divided into two kind according to its object. The one is the forming of materials such as bending of plates or bars and reforming of the residual deformations of welded structures, and the other is the setting or the removing the residual stresses in materials in order to give higher strength to the structures.

(1) Forming and reforming of materials

One example of forming of materials by thermo-plastic working is the so-called method of linear heating which was put into practise for the first time in 1954 at Ishikawajima Shipbuilding Co. Ltd. By means of this method the man hour of plate bending work was successfully reduced by several tens percent compared with the usual mechanical operations such as the cold-working by hydraulic presses or rollers. In this method, the plate is heated by oxy-acetylene torch which runs on the plate surface along a number of previously planned lines and then is cooled by spray water. It is therefore rather difficult to bring about such a large deformation as in the case of mechanical working, but it is much effective in the work to give comparatively small curvature to the elements of structures such as ships, buildings and so forth, and much hope is placed on it for the improvement of working efficiency and for the further development of automatization.

On the other hand, in order to reform the residual deformations of welded structures, the heating and cooling processes (for example, spot-heating and water-cooling) have been widely used.

(2) Setting and removing the residual stresses in materials

It has been usually believed that the residual stresses exert important influences upon the strength of materials. Sometimes they decrease the buckling and fatigue-strength of the elements of structures, but if they are set suitably in the materials, they increase considerably the strength of structures. The thermo-plastic working is one of the effective method for these purposes. For example, the process of spot-heating and water-cooling operated on the adequate region near the notches of elements of structures produce residual stresses in their roots and they increase considerably the fatigue-strength of structures for the corresponding applied loads(1).

On the other hand, the low temperature stress relieving process which is used to remove the residual stresses near the welded joints of plates and bars is an another example of the thermo-plastic working.

On account of the complexity of thermo-plastic phenomena and the difficulty of theoretical analysis, the most of these operations have not been carried out on the basis of precise scientific researches, but by the workman's experiences and intuitions. In order to establish a more rational working method, it seems to be necessary to make further synthetic research on these phenomena. With the above object, the author analyzed, from the thermo-elastic and plastic point of view, various phenomena accompanying to the elongation, contraction and bending of steel strip which were the fundamental problems of the working, and verified the theory by several experiments. Consequently, he could clarify the deformation mechanism of material in the process of working, and introduce some important rules for practical working.

In Chapter 1, statements are given on the object of studies as well as fundamental hypotheses and equations, and in Chapter 2, theoretical calculations of stresses and deformations of strip under the typical heating-cooling processes, are carried out. In the next report the author will show the effect of heating-cooling hysteresis upon the elongation and contraction of strip and will give the considerations on the method of giving the greatest residual deformation. The analysis of the phenomena about the thermo-plastic bending of strip will be given in the further reports.

The author offers his acknowledgement to Professors Y. Watanabe and T. Isibasi of Kyūshū University for their valuable advices during the course of this work.

Chapter 1. Assumptions and Fundamental Equations

§ 1. Assumptions

As stated in the introduction, since the thermo-plastic problem is too complicated to make the exact theoretical investigation, the following assumptions and simplifications of problems are made.

1) The thickness of strip is much smaller compared with its breadth, and the temperature is assumed to be constant in the direction of plate thickness.

2) A Cartain part of the breadth of the strip is heated uniformly, the other part being kept at constant room temperature, while its lengthwise temperature is

supporsed not to vary throughout the length of the strip. The stresses will be therefore in discontinuous distributions on the boundary lines between heated and unheated regions.

3) The length of strip is sufficiently large compared with its breadth, and each transverse section remains on the plane through the process of deformation.

4) It is assumed that there is no external constraint against the thermal deformation of strip.

5) The elastic constants and thermal expansion coefficients do not change during processes.

6) Two types of the temperature dependence of yield stress are assumed as shown in Fig. 1.1, that is the rectangular type and the trapezoidal type. In the

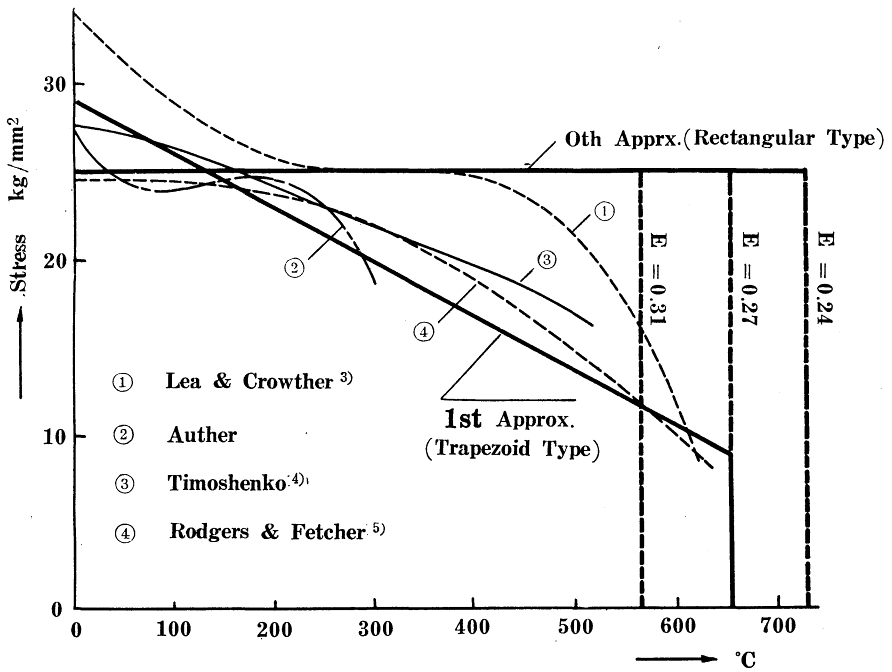


Fig. 1. 1. Temperature dependence of tensile yield stress of mild steel.

rectangular type, the yield stress depends on the temperature such that it is constant up to critical temperature T_c' and is equal to zero at higher than T_c'. In the trapezoidal type, it decreases in proportion to the increase of temperature up to critical temperature T_c and is equal to zero at higher than that. It is seen from Fig. 1. 1 that the rectangular approximation is deteriorated in the case above 300°C, but it makes the analysis much simpler than the other.

7) We will use Praudtl-Reuss' stress-strain equations and von Mises' yield criterion as the plastic laws of materials. When trapezoidal approximation of the temperature dependence of yield stress is used, the stress-strain relations of Prandtl-Reuss' materials agrees well with experimental results as shown in Fig. 1. 2.

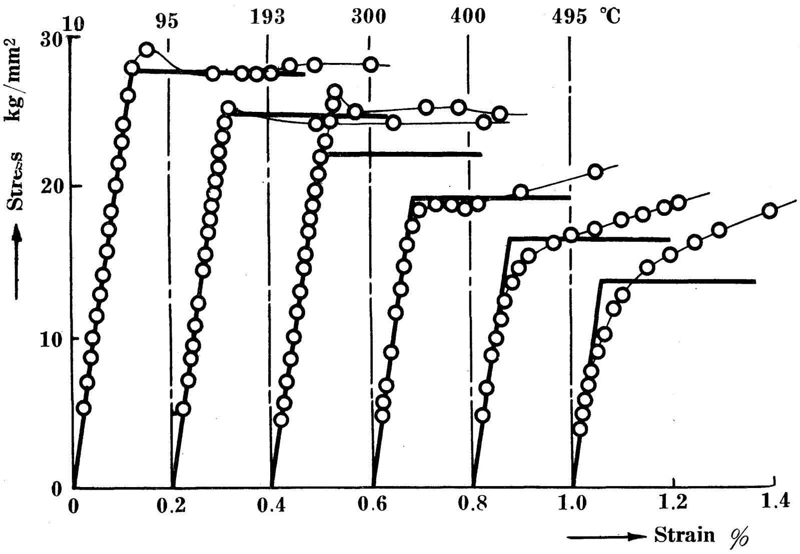


Fig. 1. 2. High temperature tension test of killed steel. (Strain velocity . 0.008 %/sec.)

§ 2. Fundamental equations

As mentioned before, yield stress σ_e at temperature T°C is expressed in the case of trapezoidal type as follows,

$$\sigma_e = \sigma_0 (1 - kT), \tag{1.1}$$

where $\sigma_0 \doteqdot 29\,\text{kg/mm}^2$, $k = 1/932$°C.

Putting now σ_{ij}, ε_{ij} $(i, j = 1, 2, 3)$ as the components of stress and strain in Cartesian coordinates x, y, z, and s and e as mean stress and mean strain, we can express as

$$s = \frac{1}{3}\sigma_{ii}, \qquad e = \frac{1}{3}\varepsilon_{ii}, \tag{1.2}$$

and deviatric stress and strain s_{ij}, e_{ij} will be expressed as

$$s_{ij} = \sigma_{ij} - \delta_{ij}s, \qquad e_{ij} = \varepsilon_{ij} - \delta_{ij}e, \tag{1.3}$$

where δ_{ij} is the Kronecker's delta.

Material is considered as a perfectly elasto-plastic body, and is assumed to satisfy von Mises' yield condition of

$$\left.\begin{aligned} &\bar{\sigma} = \sigma_e \equiv \sigma_0(1-kT) \equiv E\varepsilon_0(1-kT) \\ &\bar{\sigma} = \left(\frac{3}{2} s_{ij}\ s_{ij}\right)^{1/2} \end{aligned}\right\} \tag{1.4}$$

and

$$\varepsilon_{ij} = \varepsilon_{ij}{}^{e} + \delta_{ij}\alpha T + \varepsilon_{ij}{}^{p}(+\varepsilon_{ij}{}^{0}), \tag{1.5}$$

where ε_{ij}^e and ε_{ij}^p are elastic and plastic strains respectively and αT is thermal strain.

Then

$$\left.\begin{aligned} \dot{\varepsilon}_{ij} &= \dot{\varepsilon}_{ij}^e + \delta_{ij}\alpha\dot{T} + \dot{\varepsilon}_{ij}^p \\ \dot{\varepsilon}_{ij} &= \frac{\dot{s}_{ij}}{2G} + \delta_{ij}\frac{\dot{s}}{3K} \\ \dot{\varepsilon}_{ij}^p &= \frac{3\dot{W}_p}{2\sigma_e^2} s_{ij} \end{aligned}\right\} \tag{1.6}$$

is Prandtl-Reuss' equation with thermal strain added. In this equation K, G, is the bulk and shear modulus of elasticity, and α coefficient of themal expansion and ($\dot{}$) denotes the increments of stress, strain and temperature and $\dot{W}_p$ the increment of plastic work.

$$\dot{W}_p = \begin{cases} 0, \text{ provided } \bar{\sigma} < \sigma_e(T), \text{ or } \bar{\sigma} = \sigma_e(T) \text{ and also } \dot{\bar{\sigma}} < \dfrac{d\sigma_e}{dT}\dot{T}, \\ s_{pq}\,\dot{e}_{pq}^p, \text{ provided } \bar{\sigma} = \sigma_e(T) \text{ and also } \dot{\bar{\sigma}} = \dfrac{d\sigma_e}{dT}\dot{T}, \end{cases} \tag{1.7}$$

where $\dot{e}_{pq}^p$ is the increment of plastic strain.

In the next place, strain increment $\dot{\varepsilon}_{ij}$ is induced from velocity $\dot{u}_i$ by

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left\{\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right\}, \tag{1.8}$$

and stress and stress increment satisfy the following equilibrium equations.

$$\frac{\partial}{\partial x_i}(\sigma_{ij}) = 0, \qquad \frac{\partial}{\partial x_i}(\dot{\sigma}_{ij}) = 0. \tag{1.9}$$

Chapter 2. Thermo-plastic elongation and contraction of steel strip

§ 1. Fundamental Equations

The studies on this chapter provide us with the primary problems in thermo-plastic working. Here the theoretical analysis was made under simple assumptions to clarify the characteristics of the deformation mechanism.

1) Assuming the maximum temperature of heating lower than 500°C, a trapezoidal type will be used for temperature dependency of yield stress, (Fig. 1.1).

2) Separating the strip into two parts ① and ② with unit length and cross-section b and ($1-b$) respectively, as shown in Fig. 2.1, strip ① is heated uniformly up to given temperature and then is cooled to room temperature. Then the whole system either stretches length wise or contracts. Strip ② is assumed to be kept at room temperature throughout the process.

3) It will be assumed that the stress in the normal direction to the length of strip is zero and then the cross-section of the whole system is maintained on the plane throughout heating-cooling pro-

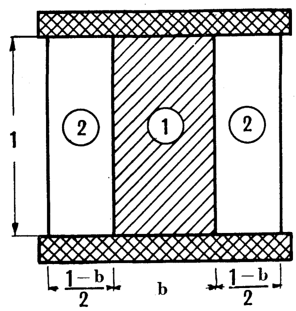


Fig. 2. 1.

cess. It is assumed moreover that buckling does not occur.

Axial stress and strain of strips ① and ② are denoted as σ_1, σ_2; ε_1, ε_2 respectively. Stress deviation is

$$\left.\begin{aligned} s_{x1}&=\frac{1}{3}\sigma_1,\ s_{y1}=s_{z1}=-\frac{1}{3}\sigma_1 \\ s_{x2}&=\frac{2}{3}\sigma_2,\ s_{y2}=s_{z2}=-\frac{1}{3}\sigma_2\ . \end{aligned}\right\} \tag{2.1}$$

Now, consideration will be limited only to x-axis component and suffix x will be hereafter dropped.

Total axial strain ε^t is

$$\begin{aligned} \varepsilon_1{}^t&=\varepsilon_1{}^e+\varepsilon_1{}^p+\varepsilon_1{}^0+\alpha T \\ \varepsilon_2{}^t&=\varepsilon_2{}^e+\varepsilon_2{}^p+\varepsilon_2{}^0, \end{aligned}$$

where, ε^e, ε^p, ε^0 are respectively elastic strain, plastic strain and initial strain, while αT is thermal strain. And since $\varepsilon_1{}^t=\varepsilon_2{}^t$, we have

$$\varepsilon_1{}^e+\varepsilon_1{}^p+\varepsilon_1{}^0+\alpha T=\varepsilon_2{}^e+\varepsilon_2{}^p+\varepsilon_2{}^0 \tag{2.2}$$

$$\dot{\varepsilon}_1{}^e+\dot{\varepsilon}_1{}^p+\alpha\dot{T}=\dot{\varepsilon}_2{}^e+\dot{\varepsilon}_2{}^p\ , \tag{2.3}$$

and

$$\varepsilon_1{}^e=\sigma_1/E,\ \varepsilon_2{}^e=\sigma_2/E,\ \dot{\varepsilon}_1{}^e=\dot{\sigma}_1/E,\ \dot{\varepsilon}_2{}^e=\dot{\sigma}_2/E, \tag{2.4}$$

and in the case of strip ①,

$$\left.\begin{aligned} &\dot{\varepsilon}_1{}^p=\frac{\dot{W}_{p1}}{\{\sigma_e(T)\}^2}\sigma_1, \\ &\text{where}\quad \dot{W}_{p1}=\begin{cases} 0, \text{ provided } \bar{\sigma}_1<\sigma_e(T), \text{ or } \bar{\sigma}_1=\sigma_e(T) \text{ and also } \dot{\bar{\sigma}}_1<\dfrac{d\sigma_e}{dT}\dot{T}, \\ \sigma_1\dot{\varepsilon}_1{}^p, \text{ provided } \bar{\sigma}_1=\sigma_e(T) \text{ and also } \dot{\bar{\sigma}}_1=\dfrac{d\sigma_e}{dT}\dot{T}, \end{cases} \end{aligned}\right\} \tag{2.5}$$

and
$$\bar{\sigma}_1=|\sigma_1|,\quad \dot{\bar{\sigma}}_1=\frac{\sigma_1\dot{\sigma}_1}{|\sigma_1|}\ .$$

In the case of strip ②

$$\left.\begin{aligned} &\dot{\varepsilon}_2{}^p=\frac{\dot{W}_{p2}}{\sigma_0{}^2}\sigma_2\ , \\ &\text{where}\quad \dot{W}_{p2}=\begin{cases} 0, \text{ provided } \bar{\sigma}_2<\sigma_0, \text{ or } \bar{\sigma}_2=\sigma_0 \text{ and also } \dot{\bar{\sigma}}_2<0, \\ \sigma_2\dot{\varepsilon}_2{}^p, \text{ provided } \bar{\sigma}_2=\sigma_0 \text{ and also } \dot{\bar{\sigma}}_2=0, \end{cases} \\ &\text{and}\quad \bar{\sigma}_2=|\sigma_2|,\ \dot{\bar{\sigma}}_2=\frac{\sigma_2\dot{\sigma}_2}{|\sigma_2|}\ . \end{aligned}\right\} \tag{2.6}$$

From equilibrium condition of force

$$b\sigma_1+(1-b)\sigma_2=0. \tag{2.7}$$

Also from the equilibrium condition of the increment of force.

$$b\dot{\sigma}_1+(1-b)\dot{\sigma}_2=0. \tag{2.8}$$

The solution will be obtained by using the foregoing (2.2)~(2.8).

§ 2. Characteristics of Solution

In evaluating the solution of the preceding section, it is necessary to calculate the deformation hysteresis by pursuing the process of rise and fall of temperature, but here the condition of existence of plastic solution will be sought using equations (2.3)~(2.8). This condition will give the relation that holds generally among b, T and $\dot{T}$, independently of the initial strain.

In equation (2.5) and (2.8), $\dot{W}_{p1}$ and $\dot{W}_{p2}$ should be positive so that strips ① and ② may be in loading condition. When they are zero, the strips are in neutral condition, but when they are negative, the strips cannot be in plastic state. In the case of $\dot{W}_{p1}>0$ and $\dot{W}_{p2}>0$, σ_1 and $\dot{\varepsilon}_1{}^p$ and also σ_2 and $\dot{\varepsilon}_2{}^p$ should be the same signs. From this condition and from (2.3), (2.4), (2.7) and (2.8), the condition for strip ① or ② to be in plastic state can be obtained.

[A] The case of $\sigma_1<0$ and $\sigma_2>0$.

Putting first $\sigma_1=-\sigma_e(T)\equiv-\sigma_0(1-kT)$, $\sigma_2=\sigma_0$ and substituting into (2.7), then

$$\theta_b = 2 - \frac{1}{b}, \tag{2.9}$$

where $\theta=kT$. This is expressed with curve on b–θ plane as shown in Fig. 2.2. In domain I in the figure, we have $\sigma_2<\sigma_0$, that is, in this domain, there occurs no yielding of ②. And in domain II, we have $\sigma_1>-\sigma_0(1-\theta)$, that is, there occur no yielding of ①. The solutions statisfying (2.3)~(2.8) in each of the three cases of (1), $\theta=2-1/b$, (θ_b-line); (2), $\theta>2-1/b$ (Domain I); (3), $\theta<2-1/b$, (Domain II), are as follows, provided that $T<T_C$, i.e., $\theta<0.7$. (the proof is given in appendix)

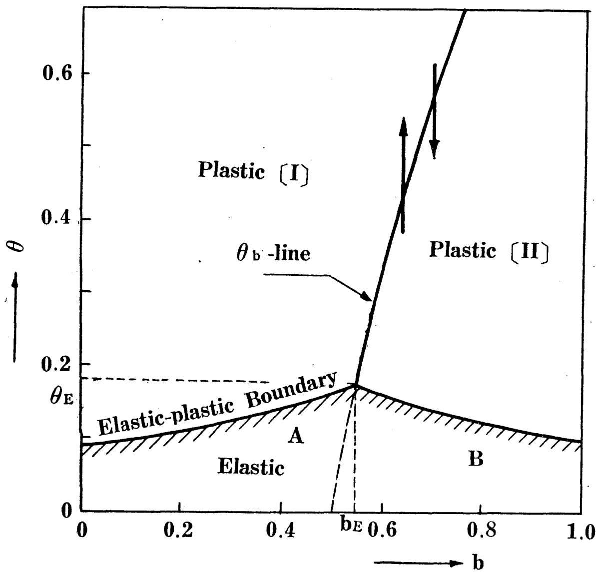


Fig. 2. 2. θ_b-line and elastic-plastic boundary lines.

(1) $\sigma_1=-\sigma_0(1-\theta)$, $\sigma_2=\sigma_0$, (θ_b-line).

When $\dot{\theta}>0$, strip ① is in loading state, and strip ② is in unloading state.
When $\dot{\theta}<0$, strip ① and ② are both in unloading state.

(2) $\sigma_1=-\sigma_0(1-\theta)$, $0<\sigma_2<\sigma_0$, (Domain I).

When $\dot{\theta}>0$, strip ① is in loading state, and strip ② is in elastic state.
When $\dot{\theta}<0$, strip ① is in unloading state, and strip ② is in elastic state.

(3) $0>\sigma_1>-\sigma_0(1-\theta)$, $\sigma_2=\sigma_0$, (Domain II).

When $\dot{\theta}>0$, strip ① is in elastic state, and strip ② is in loading state.
When $\dot{\theta}<0$, strip ① is in elastic state, and strip ② is in unloading state.

[B] The case of $\sigma_1>0$, and $\sigma_2<0$.

Similarly as before, in the case of $\sigma_1=\sigma_0(1-\theta)$, $\sigma_2=-\sigma_0$, the relation between b and θ is $\theta_b=2-1/b$. In domain I of Fig. 2.2, yielding of strip ② does not occur, and in domain II, yielding of strip ① does not occur. As in the case of [A], evaluating the solution on each of the three cases of $\theta=2-1/b$, $\theta>2-1/b$ and $\theta<2-1/b$, within the range of $T<T_c$, i.e., $\theta<0.7$, we get,

(1) $\sigma_1=\sigma_0(1-\theta)$, $\sigma_2=-\sigma_0$, (θ_b-line).

When $\theta>0$, strip ① and ② are both in unloading state.
When $\dot{\theta}<0$, strip ① is in unloading state, and strip ② is in loading state.

(2) $\sigma_1=\sigma_0(1-\theta)$, $0>\sigma_2>-\sigma_0$, (Domain I)·

When $\dot{\theta}>0$, strip ① is in unloading state, and strip ② is in elastic state.
When $\dot{\theta}<0$, strip ① is in loading state, and strip ② is in elastic state.

(3) $\sigma_0(1-\theta)>\sigma_1>0$, $\sigma_2=-\sigma_0$, (Domain II).

When $\dot{\theta}>0$, strip ① is in etastic state and strip ② is in unloading state.
When $\dot{\theta}<0$, strip ① is in elastic state, and strip ② is in loading state.

§ 3. Heating temperature that gives the maximum contraction or elongation of strip after cooling

1. Maximum contraction

Temperature θ of strip ① is limited to between 0 and 0.7. In order to produce residual strain, compressive yield must be occur in ①, for which it is necessary to heat at least up to the interior of Domain I, but within the range of $0<b<1/2$ in Domain I, it never occurs that θ_b-line is crossed vertically in heating-cooling as evident from Fig. 2.2. Domain I can be therefore divided into two parts of $0<b<1/2$, and $1/2<b<1$.

1) In the case of $0<b<1/2$.

In this range strip ① can yield when temperature rises as proved in [A], (2) of the preceding section, but strip ② is elastic. Consider the case that the temperature is raised up to a certain level θ_m making strip ① yield by compressive stress and then is lowered to θ^* $(\theta_m>\theta^*)$, making strip ① yield by tensile stress.

(a). In the case of $\theta^*>0$: If the temperature of strip ① is reduced further to $\theta=0$ from θ^*, strip ① remains in loading condition. This fact is obvious from [B], (2) of the preceding section. The stress σ_1 of ① after cooling is therefore equal to σ_0. Strip ② is, however, invariably elastic throughout the process. Accordingly, plastic strain of strip ① after cooling and residual strain of the whole system are constant independently of the value θ^*.

(b). In the case of $\theta^*<0$: In this case, since stress of strip ① does not reach to yield value at cooling, residual strain is as a matter of course smaller than that of $\theta^*\geqq 0$.

(c). If heating temperature θ_m is chosen just to make θ^* zero, this is the heating temperature giving the maximum residual strain, and its magnitude does not change even if heating temperature is higher than θ_m.

Since $\sigma_1=-\sigma_0(1-\theta_m)$, we have $\sigma_2=\sigma_0(1-\theta_m)b/(1-b)$ from (2.7). Substituting this into (2.2) then

$$\frac{\varepsilon_1{}^p}{\varepsilon_0}=\frac{1}{1-b}[1-\{1+A(1-b)\}\theta_m]+\frac{\varepsilon_2{}^0}{\varepsilon_0}, \tag{a}$$

where $\varepsilon_0=\sigma_0/E$, $A=\alpha/(\varepsilon_0 k)$, and $k=\theta/T$. Since strip ① is in unloading condition by lowering temperature from θ_m to θ^* and $\varepsilon_1{}^p$ of the above equation remains as initial strain in this cooling process, it can be rewritten as $\varepsilon_1{}^0$.

Assuming that strip ① has reached to tensile yield value at θ^*,

$$\sigma_1=\sigma_0\ (1-\theta^*),\ \sigma_2=-\frac{b}{1-b}\sigma_0(1-\theta^*).$$

Therefore, from (2.2)

$$\frac{\varepsilon_1{}^0}{\varepsilon_0}=-\frac{1}{1-b}[1-\{1-A(1-b)\}\theta^*]+\frac{\varepsilon_2{}^0}{\varepsilon_0}. \tag{b}$$

Since the left side of (a) and (b) is equivalent, we have

$$\theta_m=\frac{2-\{1-A(1-b)\}\theta^*}{1+A(1-b)}. \tag{c}$$

Putting $\theta^*=0$ from the foregoing discussion,

$$\theta_m=\frac{2}{1+A(1-b)},\ \left(A=\frac{\alpha}{\varepsilon_0 k}\right). \tag{2.10}$$

Namely the maximum contraction of whole system is given when it is heated up to the temperature given by (2.10), and even if heated higher than that the magntiude of contraction does not change. ⓐ line of Fig. 2.3 represents (2.10).

2) In the case of $1/2<b<1$.

When the strip ① is heated, strips ① and ② deform elastically at the beginning. At higher temperature, it occurs that strip ① yields first or that strip ② yields first according to the value of b. These two elastic-plastic boundary lines in $b-\theta$ plane are obvious to cross at a point on θ_b-line. Coordinates at this point are denoted as $(b_E,\ \theta_E)$, (Fig. 2.2). Namely if temperature rises up to θ_E with heating breadth b_E, stresses of strip ① and ② reach to yield value simultaneously.

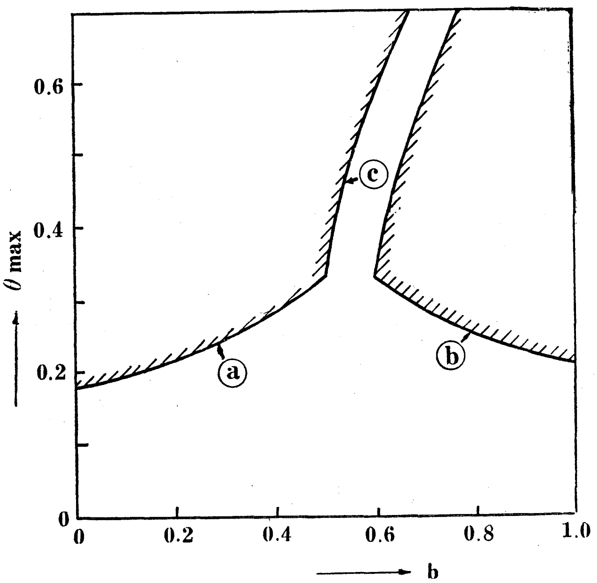


Fig. 2. 3. Heating temperature that gives the maximum elongation or contraction.
When value of heating breadth b is given, the heating temperature θ_m on ⓐ and ⓒ line in this figure give the maximum contraction, that on ⓑ line the maximum elongation and that in shaded region the constant residual deformation.

Heating process can be devided into two cases within the range of $1/2<b$, the one is that strip ① is elastic at the beginning of heating and is then compressive plastic, and the other is that strip ② is elastic, tensile plastic and unloading successively. In the latter case, unloading of strip ② and compressive loading of strip ① take place simultaneously when temperature has reached to θ_b-line.

Consider that strip ① is heated up to temperature θ_m to cause compressive yield and lower it to θ^* (it is assumed here that $\theta^* \geqslant \theta_b$) making its stress to reach tensile yield value, (a), (b) and (c) equations mentioned before hold as they are. It must be noted here that $\varepsilon_2{}^0$ in equations (a) and (b) is equal to the plastic strain of strip ② at the moment it passed θ_b during temperature rise in the case of $b_E<b$.

There occurs no unloading of strip ① as proved in [B], (2), until it reaches to θ_b by lowering temperature further. On the other hand in the case of $b_E>b$, strip ② is elastic throughout the whole process up to the present stage, whereas in the case of $b_E<b$, it is elastic only after passing θ_b when temperature rise. Plastic strain of strip ① when θ reaches to θ_b by lowering temperature is therfore constant and does not depend on the value of θ_m, and it is to be determined from $\varepsilon_2{}^0$.

In the case of $\theta^*=\theta_b$, we get from (c),

$$\left.\begin{aligned} \theta_m &= \frac{2-\{1-A(1-b)\}\theta_b}{1+A(1-b)} \\ \text{where } \theta_b &= 2-\frac{1}{b} \text{ and } b>\frac{1}{2}\,. \end{aligned}\right\} \qquad (2.11)$$

Lowering temperature further strip ① becomes in unloading and ② becomes in loading (from [B], (1)), and they remain in the same states until $\theta=0$. Namely, after cooling the stress of strip ② is $\sigma_2=-\sigma_0$ and in strip ① plastic strain is retained as it was at the moment it reached to θ_b during temperature fall. Since this plastic strain is determined only by $\varepsilon_2{}^0$ as mentioned above, the residual strain after cooling is constant even if θ_m takes a value larger than (2.11).

In the case of $\theta^*<\theta_b$, however, there occurs no unloading of strip ① in the process of cooling, and strip ② does not yield. Then the residual strain of whole system is smaller than that in the case of $\theta^*>\theta_b$.

That is, heating temperature for giving the maximum contraction after cooling is given by (2.11), and the residual strain after cooling is constant even if it is heated higher than this temperature. © line in Fig. 2.3 represents (2.11).

2. Maximum elongation

To produce residual elongation of the whole system we must produce tensile plastic strain in strip ②. When strip ① is heated and cooled within Domain II of Fig. 2.2, we can see from [A], (3) and [B], (3) of preceding section that the plastic strain is produced only in strip ②, which is tensile strain at heating, and strip ① is always elastic. Considering now that strip ① is heated up to θ_m and tensile plastic strain produces in strip ②, we have $\sigma_2=\sigma_0$, $\sigma_1=-\sigma_0\ (1-b)/b$. Substituting this into (2.2).

$$\frac{\varepsilon_2{}^p}{\varepsilon_0}=-\frac{1}{b}+A\theta_m+\frac{\varepsilon_1{}^0}{\varepsilon_0} \tag{a'}$$

Here $\varepsilon_1{}^0$ is the initial strain of strip ①. Further we consider that the stress of strip ② reaches the compressive yield value at the moment when temperature of strip ① has been lowered from θ_m to θ^*, $(\theta^*>0)$. Then $\sigma_2=-\sigma_0$, $\sigma_1=\sigma_0\ (1-b)/b$. Substituting similarly into (2.2).

$$\frac{\varepsilon_2{}^0}{\varepsilon_0}=\frac{1}{b}+A\theta^*+\frac{\varepsilon_1{}^0}{\varepsilon_0}\ . \tag{b'}$$

Since $\varepsilon_2{}^0$ is equivalent to $\varepsilon_2{}^p$ of (a'), we have

$$\theta_m-\theta^*=\frac{2}{Ab}\ . \tag{c'}$$

Lowering temperature further from θ^* to zero, stress of strip ② after cooling is $-\sigma_0$, since no unloading of strip ② occurs in this interval ([B], (3)). Final strain is then tensile strain and is determined by $\varepsilon_1{}^0$.

When $\theta^*=0$, (c') gives

$$\theta_m=\frac{2}{Ab}\ ,\ (\theta<2-\frac{1}{b}\ ,\ \text{Domain II}). \tag{2.12}$$

That is, the maximum residual elongation is produced when strip ① is heated up to temperature given in (2.12), and the magnitude of elongation is not change even if it is heated higher than that. ⓑ line of Fig. 2.3, represents (2.12).

The equations (2.10), (2.11) and (2.12) generally hold independently of initial strain and are the relation to be considered first to analize the phenomena. However, since these results are obtained under the assumption that the material is perfectly elasto-plastic, the residual deformation is, in actual heating-cooling processing, to be increased when the strip is heated up to higher temperature than that given in (2.10)~(2.12), but in that case the rate of increase is reduced, and it is not advisable from the processing efficiency to heat up to much higher temperature than that.

§ 4. Plastic phase diagram of strip without initial strain and its residual deformation

In $b-\theta$ plane, the two boundary lines where strip ① and ② yields at rise of temperature cross on θ_b-line, as shown in Fig. 2.2. Similarly boundary lines where the stress of strip ① or ② reaches the yield value of opposite sign at fall of temperature cross on θ_b-line. Changing the maximum heating temperature, these elastic-plastic boundary lines at cooling change their position vertically along θ_b-line (Fig. 2.4). By dividing these boundary lines into the case that they are in upper side of the elastic-plastic boundary lines at heating and the case that

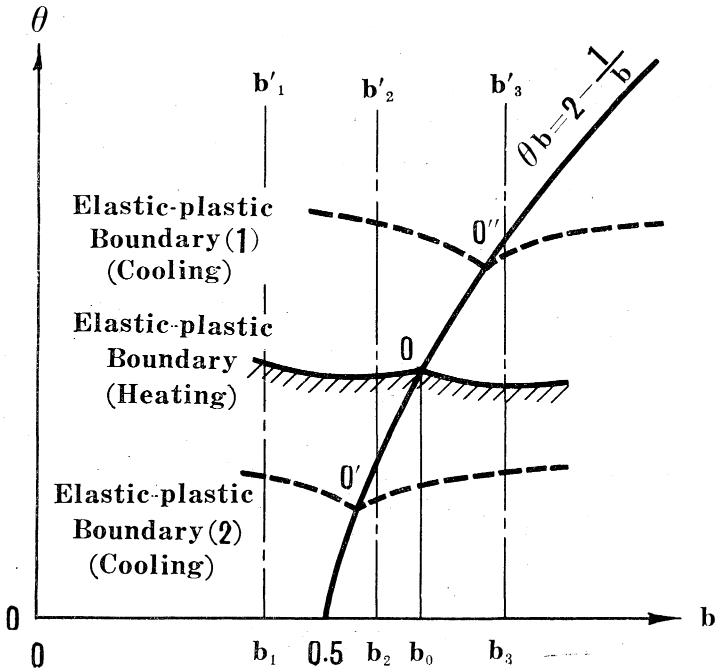
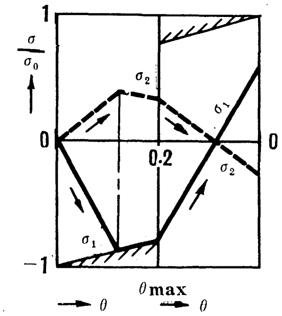


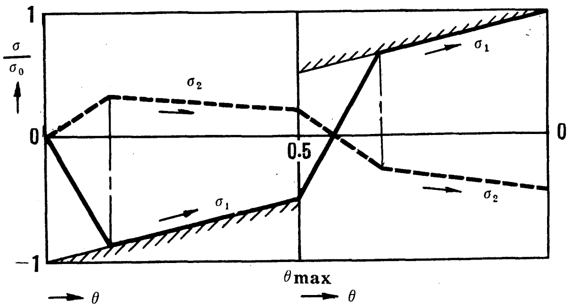
Fig. 2. 4.

they are in lower side of these lines, we analyze the changes of states of strips ① and ②. Now letting temperature rise and fall along the three vertical lines of $b_1 - b_1'$, $b_2 - b_2'$, $b_3 - b_3'$ of Fig. 2.4, the states of strips change at the moment they cross vertically the elastic-plastic boundary lines and θ_b-line.

From the above consideration and using (1.1), (2.2), (2.4) and (2.7), we can

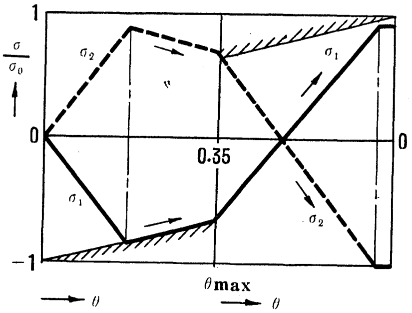


Heating Cooling
[I e] $b = 0.30$, $\theta_m = 0.20$

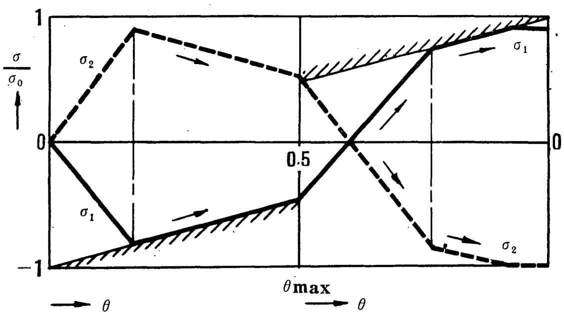


Heating Cooling
[I 1] $b = 0.30$, $\theta_m = 0.50$

Fig. 2. 5 (a).

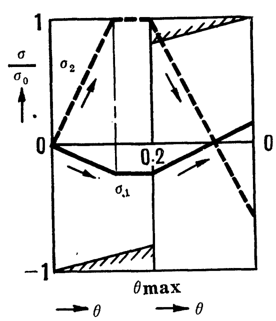


Heating Cooling
[I 2] $b = 0.52$, $\theta_m = 0.35$

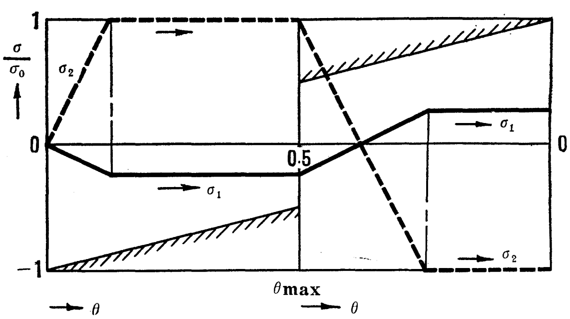


Heating Cooling
[I 3] $b = 0.52$, $\theta_m = 0.50$

Fig. 2. 5 (b).



Heating Cooling
[II e] $b = 0.80$, $\theta_m = 0.20$



Heating Cooling
[II 2] $b = 0.80$, $\theta_m = 0.50$

Fig. 2. 5 (c).

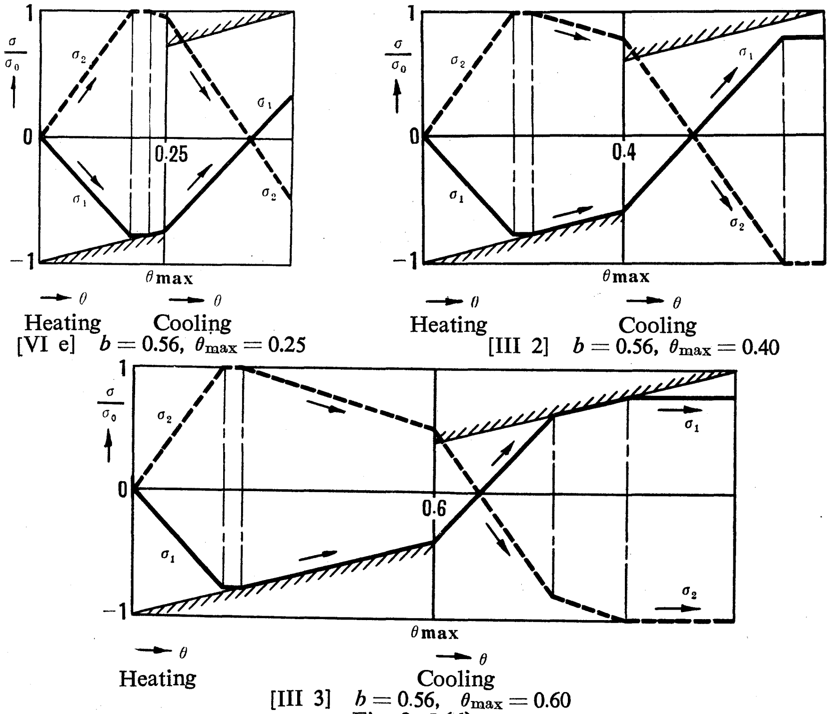
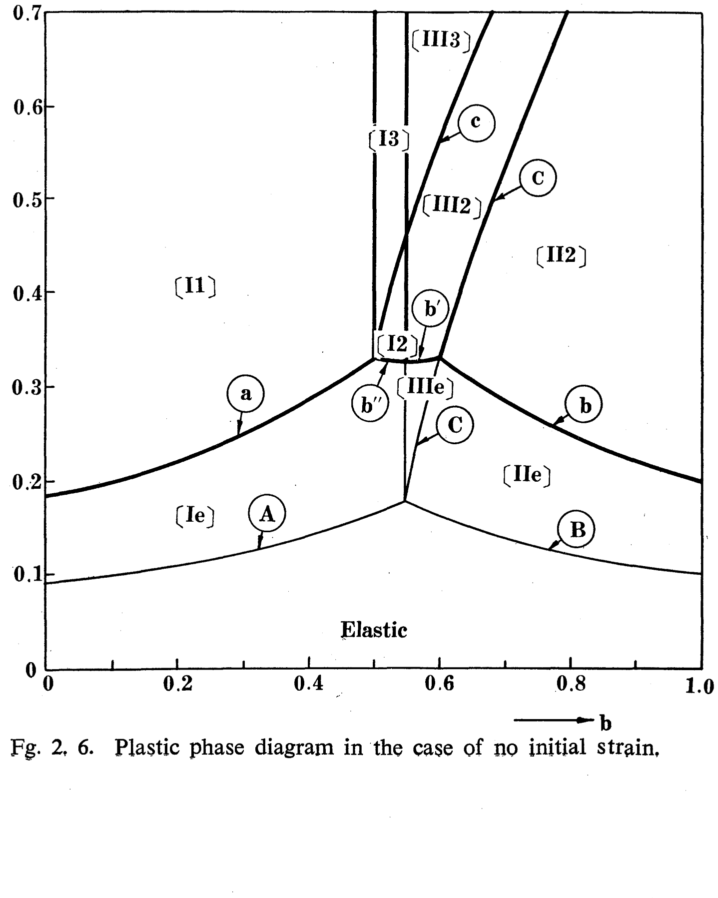


Fig. 2. 5 (d).



Fg. 2. 6. Plastic phase diagram in the case of no initial strain.

Table 1.

Domain	cf. Fig. 24	Extent of Domain	Change of State		Residual Strain after Cooling $\frac{\varepsilon^t}{\varepsilon_0}$
			Heating	Cooling	
[Ie]	$b_1 \underset{\leftarrow}{\Rightarrow} b_1'$ $b_2 \underset{\leftarrow}{\Rightarrow} b_2'$	$\frac{2}{1+A(1-b)} \geq \theta_m \geq \frac{1}{1+A(1-b)}$ $\left(\frac{1}{2} \geq b\right)$ $\frac{1}{b\{1+A(1-b)\}} \geq \theta_m \geq \frac{1}{1+A(1-b)}$ $\left(\frac{A+1}{2A} \geq b \geq \frac{1}{2}\right)$	① $E^- \Rightarrow \sigma_e^-$ ② $E^+ \Rightarrow E^+$	$\rightarrow U^- \rightarrow U^+$ $\rightarrow E^+ \rightarrow E^-$	$\frac{b}{1-b}\,[1-\{1+A(1-b)\}\theta_m]$
[I1]	$b_1 \underset{\leftarrow}{\Rightarrow} b_1'$	$1 \geq \theta_m \geq \frac{2}{1+A(1-b)}$ $\left(\frac{1}{2} \geq b\right)$	① $E^- \Rightarrow \sigma_e^-$ ② $E^+ \Rightarrow E^+$	$\rightarrow U^- \rightarrow U^+ \rightarrow \sigma_0^+$ $\rightarrow E^+ \rightarrow E^- \rightarrow E^-$	$-\frac{b}{1-b}$
[I2]	$b_2 \underset{\leftarrow}{\Rightarrow} b_2'$	$\frac{1-A(1-b)(1-2b)}{b\{1+A(1-b)\}} \geq \theta_m \geq \frac{1}{b\{1+A(1-b)\}}$ $\left(\frac{A+1}{2} \geq b \geq \frac{1}{2}\right)$	① $E^- \Rightarrow \sigma_e^-$ ② $E^+ \Rightarrow E^+$	$\rightarrow U^- \rightarrow U^+ \rightarrow U^+$ $\rightarrow E^+ \rightarrow E^- \rightarrow \sigma_0^-$	$\frac{1}{1-b}\,[(\frac{1}{b}-1+b)$ $-\{1+A(1-b)\}\theta_m]$
[I3]	$b_2 \underset{\leftarrow}{\Rightarrow} b_2'$	$1 \geq \theta_m \geq \frac{1-A(1-b)(1-2b)}{b\{1+A(1-b)\}}$ $\left(\frac{A+1}{2A} \geq b \geq \frac{1}{2}\right)$	① $E^- \Rightarrow \sigma_e^-$ ② $E^+ \Rightarrow E^+$	$\rightarrow U^- \rightarrow U^+ \rightarrow \sigma_e^+ \rightarrow U^+$ $\rightarrow E^+ \rightarrow E^- \rightarrow E^- \rightarrow \sigma_0^-$	$-1-A\left(2-\frac{1}{b}\right)$
[IIe]	$b_3 \underset{\leftarrow}{\Rightarrow} b_3'$	$2-\frac{1}{b} \geq \theta_m \geq \frac{1}{Ab}$ $\left(1 \geq b \geq \frac{A+2}{2A}\right)$ $\frac{2}{Ab} \geq \theta_m \geq \frac{1}{Ab}$ $\left(1 \geq b \geq \frac{A+2}{2A}\right)$	① $E^- \Rightarrow E^-$ ② $E^+ \Rightarrow \sigma_0^+$	$\rightarrow E^- \rightarrow E^+$ $\rightarrow U^+ \rightarrow U^-$	$1-\frac{1}{b}+A(1-b)\theta_m$

(Continued on the next page)

Table 1, (Continued).

Domain	cf. Fig. 24	Extent of Domain	Change of State: Heating	Change of State: Cooting	Residual Strain after Cooling $\frac{\varepsilon^t}{\varepsilon_0}$
[II2]	$b_3 \overset{\Rightarrow}{\leftarrow} b_3'$	$2-\frac{1}{b}\geq\theta_m\geq\frac{2}{Ab}$ $\left(1\geq b\geq\frac{A+2}{2A}\right)$	① $E^-\Rightarrow E$ ② $E^+\Rightarrow\sigma_0^+$	$\rightarrow E^-\rightarrow E^+\rightarrow E^+$ $\rightarrow U^+\rightarrow U^-\rightarrow\sigma_0^-$	$-1+\frac{1}{b}$
[IIIe]	$b_3 \overset{\Rightarrow}{\leftarrow} b_3'$	$\frac{1}{b\{1+A(1-b)\}}\geq\theta_m\geq 2-\frac{1}{b}$ $\left(\frac{A+2}{2A}\geq b\geq\frac{A+1}{2A}\right)$	① $E^-\Rightarrow E^-\Rightarrow\sigma_e^-$ ② $E^+\Rightarrow\sigma_0^+\Rightarrow U^+$	$\rightarrow U^-\rightarrow U^+$ $\rightarrow U^+\rightarrow U^-$	$\frac{b}{1-b}\left[\frac{1}{b}\left\{1+b-\frac{1}{b}+A(1-b)\left(2-\frac{1}{b}\right)\right\}-\{1+A(1-b)\}\theta_m\right]$
[III2]	$b_3 \overset{\Rightarrow}{\leftarrow} b_3'$	$\frac{1-A(1-b)(1-2b)}{b\{1+A(1-b)\}}\geq\theta_m\geq\frac{1}{b\{1+A(1-b)\}}$ $\left(\frac{A+2}{2A}\geq b\geq\frac{A+1}{2A}\right)$ $\frac{1-A(1-b)(1-2b)}{b\{1+A(1-b)\}}\geq\theta_m\geq 2-\frac{1}{b}$ $\left(1\geq b\geq\frac{A+2}{2A}\right)$	① $E^-\Rightarrow E^-\Rightarrow\sigma_e^-$ ② $E^+\Rightarrow\sigma_0^+=U^+$	$\rightarrow U^-\rightarrow U^+\rightarrow U^+$ $\rightarrow U^+\rightarrow U^-\rightarrow\sigma_0^-$	$\frac{1}{1-b}[b+A(1-b)(2-\frac{1}{b})$ $-\{1+A(1-b)\}\theta_m]$
[III3]	$b_3 \overset{\Rightarrow}{\leftarrow} b_3'$	$1\geq\theta_m\geq\frac{1-A(1-b)(1-2b)}{b\{1+A(1-b)\}}$ $\left(\frac{A-1}{A}\geq b\geq\frac{A+1}{2A}\right)$	① $E^-\Rightarrow E^-\Rightarrow\sigma_e^-$ ② $E^+\Rightarrow\sigma_0^+\Rightarrow U^+$	$\rightarrow U^-\rightarrow U^+\rightarrow\sigma_e^+\rightarrow U^+$ $\rightarrow U^+\rightarrow U^-\rightarrow U^-\rightarrow\sigma_0^-$	$-\frac{1+b}{b}$

E^+ and E^- indicate elastic states of tensile and compressive stresses respectively, $\sigma_e{}^+$ and $\sigma_e{}^-$ plastic states at T°C, and U^+ and U^- unloading states. $\Rightarrow$ indicate temperature increase of strip ① and $\rightarrow$ temperature decrease of strip ①. $A=\alpha/(\varepsilon_0 k)=10$, α is coefficient of thermal expansion, $\varepsilon_0=\sigma_0/E$, and σ_0 is yield stress at room temperature. Yield stress at T°C is denoted by $\sigma_e=\sigma_0\ (1-kT)$, and $\theta_m=kT_m$ where T_m is the maximum heating temperature.

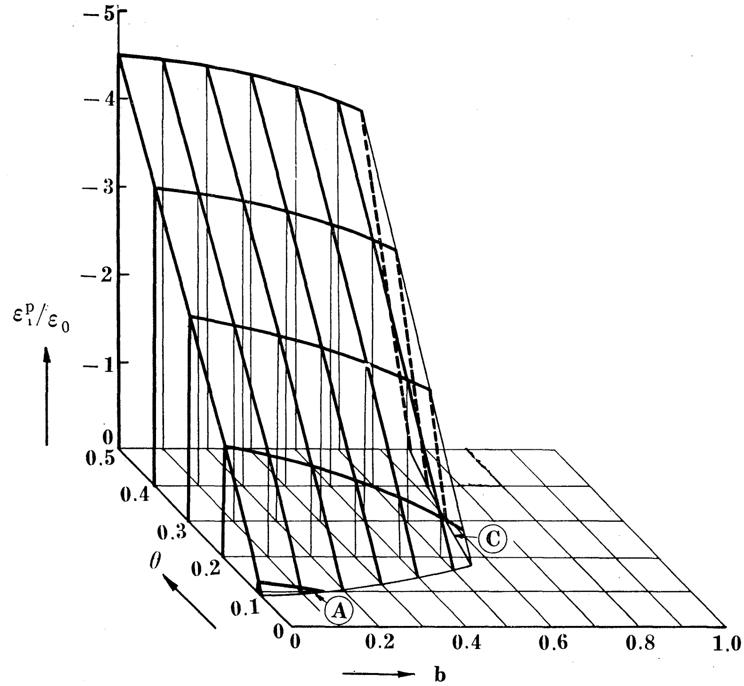


Fig. 2. 7 (a). Plastic strain of strip ① during heating.

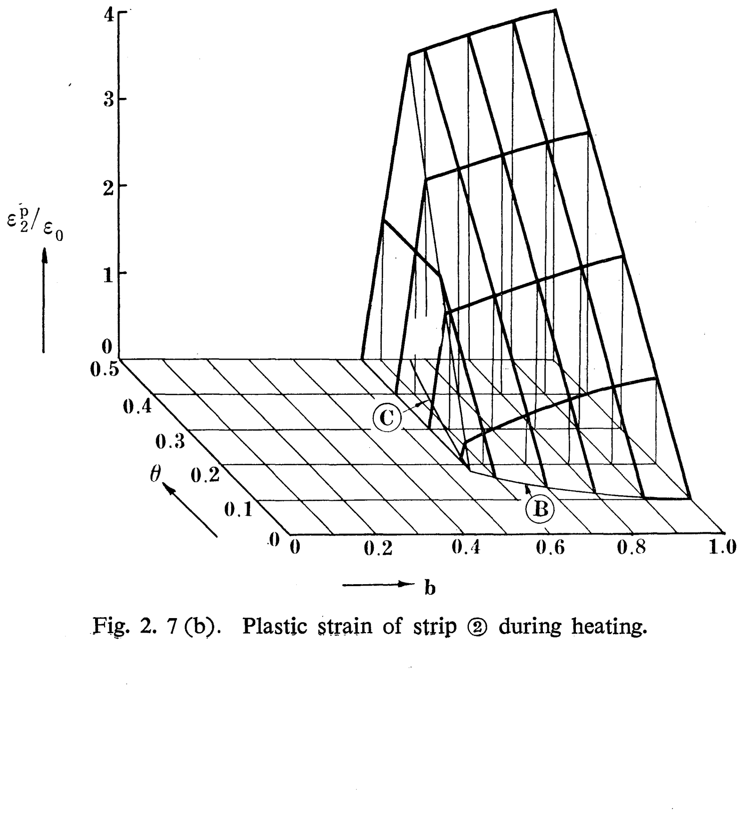


Fig. 2. 7 (b). Plastic strain of strip ② during heating.

evaluate the stresses of strips ① and ② and the results are shown in Table 1. It is seen from the Table that there are nine stress states. The first column of Table 1 gives the classification of each state, the second column the explanation of heating-cooling process in Fig. 2.4, the third column the range of each of them, the forth column the order of the state changes and the fifth column the residual deformations of the whole system after cooling. Fig. 2.5, (a), (b), (c) and (d)

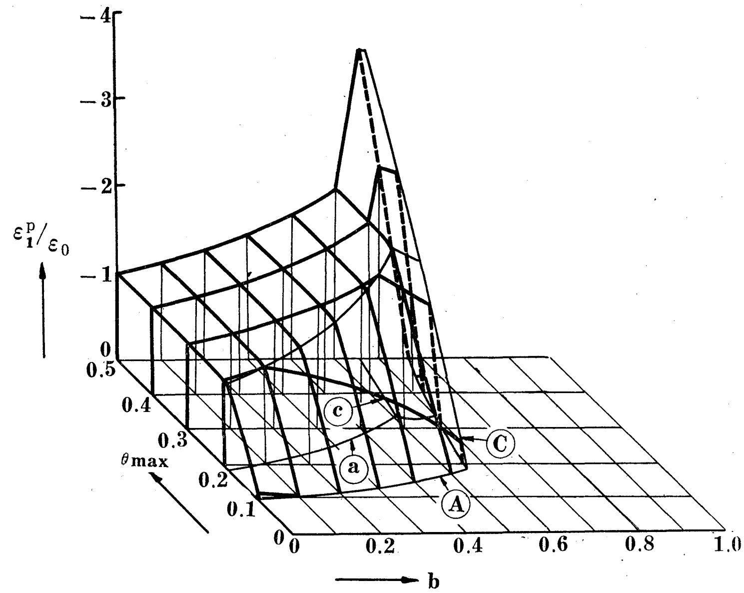


Fig. 2. 8 (a). Residual plastic strain of strip ① after cooling.

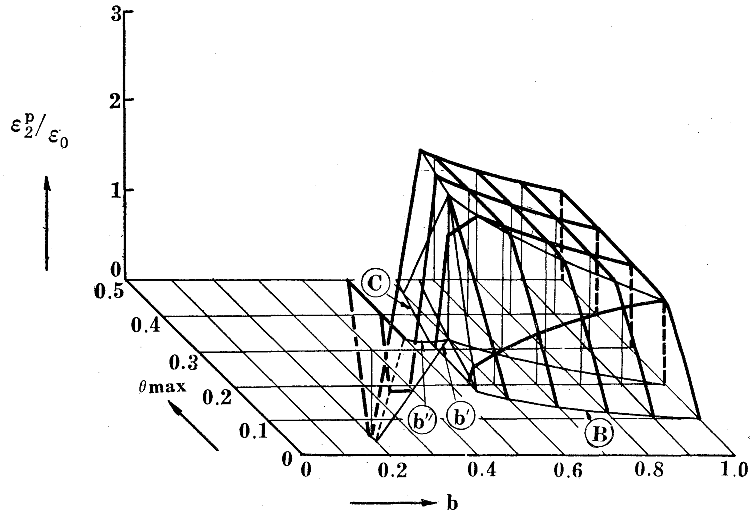


Fig. 2. 8 (b). Residual plastic strain of strip ② after cooling.

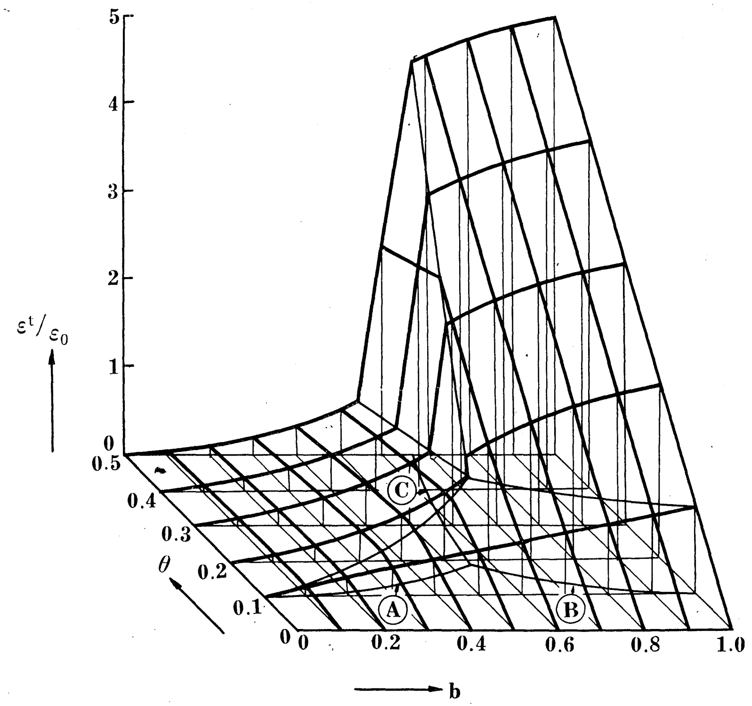


Fig. 2. 9 (a). Total strain of whole system during heating.

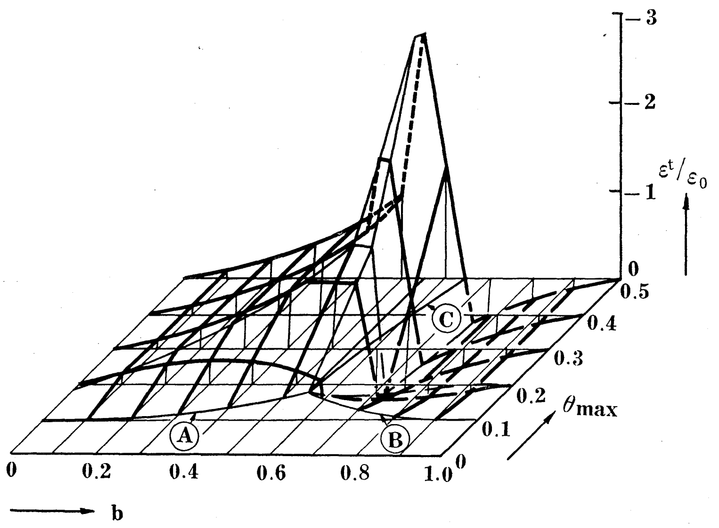


Fig. 2. 9 (b). Residual strain of whole system after cooling.

give the stress change of each state. Taking now the maximum heating temperature as ordinates of $b-\theta$ plane, the nine states given above can be represented as shown in Fig. 2.6, in which ⓐ, ⓑ and ⓒ lines are those already given in Fig. 2.3. Fig. 2.6 is the plastic phase diagram of the strip. Fig. 2.7 (a) and (b) show plastic strains of strips ① and ② during heating, while Fig. 2.8 (a) and (b) show the residual strain of strip ① and ② after cooling. Fig. 2.9 (a) shows the total strain of the whole system during heating and Fig. 2.9 (b) residual strain of the whole system after cooling. In Fig. 2.9 (b), the condition on the maximum residual strain referred to in section 3 is shown clearly.

Summary

Analysis has be made in this report on the mechanism of thermo-plastic elongation and contraction of strip by heating-cooling process. It is assumed that yield stress decreases in proportion to increase of temperature and the other physical and mechanical constants do not vary throughout the process, and that stress-strain relation of material satisfies Prandtl-Reuse′ equation. Next assumption is that heating is carried out symmetrically to the axis of strip, that temperature at heating part is uniform and that unheated part is constantly maintained at zero. Under these assumption, the author evaluate first the characteristics of solution of fundamental equations in the case that heating temperature is below 500∼600°C and then obtain the heating temperature θ_m which gives the maximum elongation or contraction of strip after cooling. The temperature which give the maximum contraction of strip after cooling is independently of the initial strain of strips,

$$\theta_m=\begin{cases} \dfrac{2}{1+A(1-b)}\,, & \left(0\leq b\leq \dfrac{1}{2}\right), \\ \dfrac{2(1-\theta_b)}{1+A(1-b)}+\theta_b, & \left(\dfrac{1}{2}\leq b\right). \end{cases}$$

The compressive residual strain in this case is, if there is no initial strain,

$$\frac{\varepsilon^t}{\varepsilon_0}=\begin{cases} -\dfrac{b}{1-b}\,, & \left(0\leq b\leq \dfrac{1}{2}\right), \\ -1-A\left(2-\dfrac{1}{b}\right), & \left(\dfrac{1}{2}\leq b\right), \end{cases}$$

where, $\theta=kT$, $A=\alpha/(\varepsilon_0 k)$, $\theta_b=2-1/b$ and $\theta_m<0.7$. And α is the coefficient of thermal expansion, $\varepsilon_0=\sigma_0/E$, σ_0 yield stress at room temperature, T temperature of strip and $\sigma_e(T)=\sigma_0(1-kT)$. And the heating temperature giving the maximum elongation of strip after cooling is

$$\theta_m=\frac{2}{Ab}\,,\quad \left(\frac{1}{2}\leq b\leq 1\right),$$

and the compressive residual strain in this case is, if there is no initial strain,

$$\frac{\varepsilon^t}{\varepsilon_0}=-1+\frac{1}{b}\,,\quad \left(\frac{1}{2}\leq b\leq 1\right),$$

where, $\theta_m < 2 - 1/b$ must be satisfied.

If material is not perfectly plastic, the residual strain is larger than that given above when heated higher than θ_m, but it is not advisable, from the view point of processing efficiency, because the rate of increasing residual strain is then deteriorated suddenly.

The stresses and strains of strip are evaluated in the cases where heating temperature is lower than 500°C, and it is confirmed theoretically that nine different elastic and plastic states occur during heating-cooling processing of strip. These states are represented by the corresponding nine domains of plastic phase diagram.

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Appendix

(Proof of the theorems in chapter 2, Section 2)

[A] The case of $\sigma_1 < 0$ and $\sigma_2 > 0$.

In this case, we can put $\dot{\bar{\sigma}}_1 = -\dot{\sigma}_1$, $\dot{\bar{\sigma}}_2 = \dot{\sigma}$ in (2.5) and (2.6).

1. $\sigma_1 = -\sigma_e(T) = -\sigma_0(1-\theta)$, $\sigma_2 = \sigma_0$, (θ_b-line).

(a) When $\dot{\theta} > 0$,

: If $\dot{\bar{\sigma}}_1 = \frac{d\sigma_e}{dT}\dot{T}$, then $\dot{\sigma}_1 = \sigma_0\dot{\theta} > 0$. From (2.8), $\dot{\sigma}_2 = -\frac{b}{1-b}\sigma_0\dot{\theta} < 0$. Then it will be shown that strip ② is in unloading state. Now, $\dot{\varepsilon}_1{}^p = -\frac{\sigma_0\dot{\theta}}{E(1-b)} - \dot{\theta} < 0$ from (2.3) and (2.4), then $\dot{W}_{p1} = \sigma_1\dot{\varepsilon}_1{}^p > 0$. From this, it is evident that strip ① is in loading state.

: If $\dot{\bar{\sigma}}_1 < \frac{d\sigma_e}{dT}\dot{T}$, then $\dot{\sigma}_1 > \sigma_0\dot{\theta} > 0$ and $\dot{\sigma}_2 < 0$. From (2.5) and (2.6), strips ① and ② are both in unloading state and $\dot{\varepsilon}_1{}^p = \dot{\varepsilon}_p = 0$. Then $\dot{\bar{\sigma}}_1 \nless \frac{d\sigma_e}{dT}\dot{T}$. This conflicts with the assumption.

: If $\dot{\bar{\sigma}}_2 = 0$, then $\dot{\sigma}_1 = \dot{\sigma}_2 = -\dot{\bar{\sigma}}_1 = 0$ and $\dot{\bar{\sigma}}_1 > \frac{d\sigma_e}{dT}\dot{T}$. This relation conflict with the assumption that materials do not exhibit work-hardening. Then $\dot{\bar{\sigma}} \neq 0$.

: If $\dot{\bar{\sigma}}_2 < 0$, then $\dot{\sigma}_2 < 0$, $\dot{\sigma}_1 > 0$ and $\dot{\bar{\sigma}}_1 < 0$. Moreover, $\dot{\bar{\sigma}}_1 \leqslant \frac{d\sigma_e}{dT}\dot{T} < 0$ must be satisfied. But we have already referred to this relation.

Namely, when $\sigma_1=-\sigma_0(1-\theta)$, $\sigma_2=\sigma_0$ and $\dot{\theta}>0$, strip ① is in loading state and strip ② is in unloading states.

(b) When $\dot{\theta}<0$,

: If $\dot{\sigma}_1=\frac{d\sigma_e}{dT}\dot{T}$, then $\dot{\sigma}_1=\sigma_0\dot{\theta}<0$. From (2.8), $\dot{\sigma}_2>0$. This conflicts with the assumption that materials do not exhibit work-hardening.

: If $\dot{\sigma}_1<\frac{d\sigma_e}{dT}\dot{T}$, then $\dot{\sigma}_1>\sigma_0\dot{\theta}$. In the case of $0>\dot{\sigma}_1>\sigma\dot{\theta}$, we get $\dot{\sigma}_2>0$ and this is inconsistent. But in the case of $\dot{\sigma}_1>0>\sigma_0\dot{\theta}$, we get $\dot{\sigma}<0$ and $\dot{\sigma}_1<\frac{d\sigma_e}{dT}\dot{T}$ and $\dot{\sigma}_2<0$. It will be shown from these inequalities that strips ① and ② are both in unloading states.

: If $\dot{\sigma}_2=0$, then $\dot{\sigma}_1=\dot{\sigma}_2=0$ and $\dot{\sigma}_1<\frac{d\sigma_e}{dT}\dot{T}$. Then strip ① is in unloading state and $\dot{\varepsilon}^p{}_1=0$. From this and equation (2.3), we get $\dot{\varepsilon}_2{}^p=\alpha\dot{T}=A\varepsilon_0\dot{\theta}<0$ and hence $\dot{W}_{p2}=\sigma_2\ \dot{\varepsilon}_2{}^p<0$. It will be shown from this that strip ② is in unloading state.

: If $\dot{\sigma}_2<0$, then inequalities $\dot{\sigma}_2<0$, $\dot{\sigma}_1>0$ and $\dot{\sigma}_1\leqslant\frac{d\sigma_e}{dT}\dot{T}<0$ must be satisfied. But we have already referred to this relations.

Namely, when $\sigma_1=-\sigma_0(1-\theta)$, $\sigma_2=\sigma_0$ and $\dot{\theta}<0$, strip ① and ② are both in unloading states.

2. $\sigma_1=-\sigma_0(1-\theta)$, $0<\sigma_2<\sigma_0$, (Domain I).

From (2.6), $\dot{\sigma}_2=\dot{\sigma}_2<0$ and $\dot{\varepsilon}_2{}^p=0$. Then strip ② is in elastic state. From (2.3) and (2.4), we get

$$\dot{\varepsilon}_1{}^p=-\frac{1}{1-b}\frac{\dot{\sigma}_1}{E}-\alpha\dot{T}. \tag{1}$$

(a) When $\dot{\theta}>0$,

: If $\dot{\sigma}_1=\frac{d\sigma_e}{dT}\dot{T}$, then $\dot{\sigma}_1=\sigma_0\dot{\theta}>0$ and from (1), we get $\dot{\varepsilon}_1{}^p=-\frac{1}{1-b}\frac{\dot{\sigma}_1}{E}-A\varepsilon_0\dot{\theta}<0$. Then $\dot{W}_{p1}>0$ and strip ① is in loading state.

: If $\dot{\sigma}_1<\frac{d\sigma_e}{dT}\dot{T}$, then $\dot{\sigma}_1>\sigma_0\dot{\theta}>0$ and $\dot{\varepsilon}_1{}^p=0$ from (2.5). This dose not satisfy (2.3). Then $\dot{\sigma}_1\nless\frac{d\sigma^e}{dT}\dot{T}$.

(b) When $\dot{\theta}<0$,

: If $\dot{\sigma}_1=\frac{d\sigma_e}{dT}\dot{T}$, then $\dot{\sigma}_1=\sigma_0\dot{\theta}<0$ and $\dot{\sigma}_2<0$. This does not satisfy (2.8).

: If $\dot{\sigma}_1<\frac{d\sigma_e}{dT}\dot{T}$ then $\dot{\sigma}_1>\sigma_0\dot{\theta}$ and also $\dot{\varepsilon}_1{}^p=0$ from (2.5). Then $\dot{\sigma}_1=-A\sigma_0(1-b)\dot{\theta}$. from (1) and $-A\sigma_0(1-b)\dot{\theta}>\sigma_0\dot{\theta}$ is satisfied.

Namely, when $\sigma_1=-\sigma_0(1-\theta)$, $0<\sigma_2<\sigma_0$ in Domain I and when $\dot{\theta}>0$, strip ① is in loading state and strip ② is in elastic state. But when $\dot{\theta}<0$, strip ① is in unloading state and strip ② is in elastic state.

3. $0>\sigma_1>-\sigma_0(1-\theta)$, $\sigma_2=\sigma_0$, (Domain II).

From (3.5), $\dot{\varepsilon}_1{}^p=0$ and strip ① is in elastic state.

From (2.3) and (2.4), we get

$$\dot{\varepsilon}_1{}^p = -\frac{1}{b}\frac{\dot{\sigma}_2}{E} + A\varepsilon_0\dot{\theta}. \tag{2}$$

(a) When $\dot{\theta}>0$,

: If $\dot{\sigma}_2=0$, then $\dot{\sigma}_1=\dot{\sigma}_2=0$, and from (2), $\dot{\varepsilon}_2{}^p=A\varepsilon_0\dot{\theta}>0$. Then $\dot{W}_{p2}>0$ and strip ② is in loading state.

: If $\dot{\sigma}_2<0$ then $\dot{\sigma}_1>0$, $\dot{\sigma}_2<0$ and from (2.6) $\dot{\varepsilon}_2{}^p=0$. Substituting this into (2), we get $\dot{\sigma}_2=Ab\sigma_0\dot{\theta}>0$. This conflicts with the assumption and $\dot{\sigma}_2\not<0$ must be satisfied.

(b) When $\dot{\theta}<0$,

: If $\dot{\sigma}_2=0$, then $\dot{\sigma}_1=\dot{\sigma}_2=0$. From (2), $\dot{\varepsilon}_2{}^p=A\varepsilon_0\dot{\theta}<0$. Then $\dot{W}_{p2}<0$ and strip ② is in unloading state and this conflicts with $\dot{\sigma}_2=0$.

: If $\dot{\sigma}_2<0$, then $\dot{\sigma}_1>0$, $\dot{\sigma}_2<0$ and from (2.6) $\dot{\varepsilon}_2{}^p=0$. Substituting this into (2), we get $\dot{\sigma}_2=Ab\sigma_0\dot{\theta}<0$. Then strip ② is in unloading state.

Namely, when $0>\sigma_1>-\sigma_0(1-\theta)$, $\sigma_2=\sigma_0$ in Domain II and when $\dot{\theta}>0$, then strip ① is in elastic state and strip ② is in loading state. But when $\dot{\theta}<0$, then strip ① is in elastic state and strip ② is in unloading state.

[B] In the case of $\sigma_1>0$, $\sigma_2<0$, we can proof the theorems in the same way as in [A].

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