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NOTE

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In the statistical theory of homogeneous turbulence, it is often assumed that fourth-order velocity correlations relate to second-order ones *as if* the joint-probabilities of velocity fluctuations were normally distributed.[1] This assumption leads at once to the hypothesis of zero fourth-order cumulants (04C, in short), saying

$$<u_i u_j' u_k'' u_l'''> = <u_i u_j'><u_k'' u_l'''> + <u_i u_k''><u_j' u_l'''> + <u_i u_l'''><u_j' u_k''>, \quad (1)$$

in general, where u_i, u_j', u_k'' and u_l''' are the ith, jth, kth and lth components of simultaneous velocities at the four points $\boldsymbol{x}$, $\boldsymbol{x}'=\boldsymbol{x}+\boldsymbol{r}$, $\boldsymbol{x}''=\boldsymbol{x}+\boldsymbol{r}'$ and $\boldsymbol{x}'''=\boldsymbol{x}+\boldsymbol{r}''$, respectively, and averages $<\ >$ are to be understood.[2] The relation(1), being based on the apparent randomness of turbulent motions, is essentially a kind of disorder hypothesis which has no direct relevance to the basic hydrodynamical equations. Thus, when all four points are close together (i.e., when the high wave-number regions in wave-number space are considered), for instance, the 04C hypothesis is expected to be considerably in error because of strong dynamical connections between u_i, u_j', u_k'' and u_l'''. This "ultra-violet inconsitency", so to speak, has been already demonstrated experimentally and discussed in some detail, e.g., [**1**, **9**].[3]

For large values of separations (i.e., for the lowest wave-number regions), on the other hand, the 04C relation (1) seems identically true, since the homogeneous turbulent motion in the large is likely to be random enough. However, it has been pointed out by Batchelor & Proudman [**2**] that there is present the pressure action of long-range character such as to ensure statistical connections between the velocities at widely separated points. In other words, the fluctuating velocities u_i, u_j',$\cdots$ may be subject to some intrinsic restrictions even when the separations are large, and if so, a new question will arise: to what extent is the relation (1) legitimate in the presence of long-range pressure forces ? A little considerations will be made below after the manner of Batchelor & Proudman, in order to throw some light on this problem.

For simplicity and clarity, let us consider the special case where $\boldsymbol{r}'=\boldsymbol{r}''=0$, that is $\boldsymbol{x}=\boldsymbol{x}''=\boldsymbol{x}'''$, in particular. Then, the relation (1) reads

$$<u_i u_k u_l u_j'> = <u_i u_j'><u_k u_l> + <u_i u_k><u_l u_j'> + <u_i u_l><u_k u_j'>, \quad (2)$$

the both-hand sides of which are spatially functions of $\boldsymbol{r}$ alone. We shall now assume after Batchelor & Proudman that all of the velocity cumulants are exponentially small for large values of $r=|\boldsymbol{r}|$, and hence their integral moments of any order are absolutely convergent, at some initial instant $t=t_0$ (the hypothesis of initially convergent integral

1) See [**1**] Chap. VIII. For more recent contributions and discussions reference should be made to [**5**, **6**, **7**, **8**] in particular.

2) The 04C hypothesis can be introduced for space-time correlations as well [**3**], but the present note is solely concerned with correlations taken at the same instant of time.

3) Recently Kraichnan [**4**] has raised another kind of objection to the 04C hypothesis, with which however we shall not be concerned here.

moments). This assumption is certainly consistent with the 04C hypothesis (2) for large values of r at that particular instant t_0. Batchelor & Proudman, however, has shown that as the effect of pressure forces which are not local but depend on the entire velocity field, the time derivatives of cumulants at t_0 are no longer exponentially small for large values of r unlike the cumulants themselves. For the double correlation, they have found that in general

$$\left[\frac{\partial}{\partial t}<u_i u_j'>\right]_0 \sim O(r^{-5}), \tag{3}$$

(the suffix 0 indicates the value at t_0) as $r\to\infty$.[4] When this result is applied to the right-hand side of the relation (2), it is seen that as $r\to\infty$,

$$\left[\frac{\partial}{\partial t}\left(<u_i u_j'><u_k u_l>+<u_i u_k><u_l u_j'>+<u_i u_l><u_k u_j'>\right)\right]_0 \sim O(r^{-5}). \tag{4}$$

Now, our task is to make a similar analysis for the left-hand side of the relation (2). From the Navier-Stokes equation at the points x and x', we have

$$\frac{\partial}{\partial t}<u_i u_k u_l u_j'>$$
$$=\frac{\partial}{\partial r_j}<u_i u_k u_l P'>-<\frac{\partial P}{\partial x_i}u_k u_l u_j'>-<\frac{\partial P}{\partial x_k}u_l u_i u_j'>-<\frac{\partial P}{\partial x_l}u_i u_k u_j'>+\Delta_{iklj}, \tag{5}$$

and by taking its divergence, it follows

$$\nabla^2<u_i u_k u_l P'> = -\frac{\partial^2}{\partial r_\alpha \partial r_\beta}<u_i u_k u_l u_\alpha' u_\beta'>, \tag{6}$$

where $P=p/\rho$ denotes the kinematic pressure and Δ_{iklj} represents the inertia and the viscous terms, α, β being dummy indices. According to our basic assumption, contributions from Δ_{iklj} are exponentially small quantities for large values of r at t_0, but this is not the case for the pressure terms. If we write $v(y)$ for the velocity at the "Quellpunkt" y, and abbreviate as $\overline{ab}=ab-<ab>$, the solution of the equation (6) is of the form

$$<u_i u_k u_l P'> = -\frac{1}{4\pi}\int\frac{\partial^2}{\partial y_\alpha \partial y_\beta}<u_i u_k u_l \overline{v_\alpha v_\beta}>\frac{dy}{|y-x'|}, \tag{7}$$

(the integration extends all over y-space), whose multiple-pole expansion at the instant t_0 is guaranteed in virtue of the hypothesis of initially convergent integral moments. In our case, the first two terms of expansion vanish by Green's theorem, and thus it proves to be

$$\left[\frac{\partial}{\partial r_j}<u_i u_k u_l P'>\right]_0 \sim \frac{1}{4\pi}\frac{\partial^3}{\partial r_j \partial r_\alpha \partial r_\beta}\left(\frac{1}{r}\right)\int<u_i u_k u_l \overline{v_\alpha v_\beta}>_0 dy \sim O(r^{-4}), \tag{8}$$

in general, when r is large. As to the remaining pressure terms, we have, for instance,

$$<\frac{\partial P}{\partial x_i}u_k u_l u_j'>_0 \sim <\frac{\partial P}{\partial x_i}u_j'>_0<u_k u_l>_0 \sim -\left[\frac{\partial}{\partial r_i}<Pu_j'>\right]_0<u_k u_l>_0 \sim O(r^{-5}), \tag{9}$$

as $r\to\infty$ (cf. [2] Eq. (4.3)),[5] and the like. Summarizing, we find that as $r\to\infty$,

$$\left[\frac{\partial}{\partial t}<u_i u_k u_l u_j'>\right]_0 \sim O(r^{-4}), \tag{10}$$

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4) Batchelor & Proudman have examined the behaviors of higher order time derivatives also. But in the present context, we have only to consider the first time derivative.

5) The hypothesis of initially convergent integral moments gives the general asymptotic relation

$$<\Pi(x)F(x)G(x')>_0 \sim <\Pi(x)F(x)>_0<G(x')>_0+<\Pi(x)G(x')>_0<F(x)>_0,$$

as $r\to\infty$, where $\Pi(x)$ stands for P or grad$P$ at the point x, and exponentially small quantities are omitted. (cf. [2] p. 382, footnote.) On putting $F(x)\equiv u_k u_l$, $G(x')\equiv u_j'$, we obtain (9), while $F(x)\equiv 1$ corresponds to the case of (8).

which is not in agreement with the prediction (4).

Such a discrepancy in the asymptotic order of the first time derivatives indicates a non-persistent character of the 04C relation (2) at the lowest wave-number regions; if the relation (2) holds for all values of r at some instant, it will be soon destroyed by the action of long-range pressure forces at least for large values of r. This state of affairs may be called the "infra-red inconsistency" of the 04C hypothesis.

To be worth noting in the present connection,

$$\left[\frac{\partial}{\partial t}\langle u_i u_k u_j'\rangle\right]_0 \sim O(r^{-4}) \tag{11}$$

has been obtained by Batchelor & Proudman [2] p. 382. The contrast between (3) and (10) or (11) comes from the fact that the integral of the type

$$\int \langle F(\mathbf{x})\overline{v_i v_j}\rangle_0 d\mathbf{y}$$

cancels, if the function $F(\mathbf{x})$ is solenoidal with respect to $\mathbf{x}$. Then the term like that appears in the expression (8) vanishes and the next term in the expansion must be taken. For $\langle u_i u_j'\rangle$, where $F(\mathbf{x})\equiv u_i$, this is exactly the case, but for $\langle u_i u_k u_j'\rangle$ as well as for $\langle u_i u_k u_l u_j'\rangle$, where $F(\mathbf{x})\equiv u_i u_k$ and $F(\mathbf{x})\equiv u_i u_k u_l$, respectively, the same is not. Considering these circumstances, it is not surprising that $\langle u_j u_k u_l u_j'\rangle$ resembles $\langle u_i u_k u_j'\rangle$ rather than $\langle u_i u_j'\rangle$ so far as its asymptotic behavior is concerned.

The foregoing is only a single example of demonstrating the infra-red inconsistency of the 04C hypothesis. For the other type of fourth-order correlations, e.g., $\langle u_i u_k u_l' u_j'\rangle$, the analysis seems to be more complicated. Furthermore, the present argument is based on the somewhat artificial hypothesis of initially convergent integral moments as Batchelor & Proudman's work was. Nevertheless, apart from detailed quantitative expressions for the time being, we are lead to the general expectation that the presence of the pressure action, in view of its long-range character, is not laways consistent with the 04C hypothesis which presupposes in effect the short-range correlation of turbulent fluid motions.

Of course, the role of infra-red part must be of little consequence for the over-all decay of energy-containing eddies, because except in the very later stage big eddies contain only a small portion of the total energy. By the same reason, direct experimental determination of the asymptotic properties of correlations or spectra is hardly possible at present. The point is that it seems questionable whether we may use the 04C hypothesis for dealing with *the large-scale structure* and related problems of homogeneous turbulence.

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