Laminar Boundary Layers on Wedges

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https://doi.org/10.5109/7165027

出版情報：Reports of Research Institute for Applied Mechanics. 10 (38), pp.45-51, 1962. 九州大学応用力学研究所

Reports of Research Institute for Applied Mechanics
Vol. X, No. 38, 1962

NOTE

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By Jun-ichi OKABE and Susumu INOUE

1. In a short note published about twenty years ago, I. TANI stated that the momentum thickness of the laminar boundary layer of two dimensions, throughout its whole region from the forward stagnation down to the separation, can be estimated approximately by means of the formula

$$\frac{\theta^2}{\nu}=\frac{0.44}{u_1{}^6}\int_0^x u_1{}^5\,dx, \tag{1.1}$$

where θ denotes the momentum thickness, u_1 the velocity of the flow just outside the boundary layer, and ν the kinematic viscosity of the fluid; x is the arc-length measured along the surface of the body and the point $x=0$ corresponds to the forward stagnation.[1] In deriving the formula, it is assumed that $u_1{}^3\theta$ would vanish at the forward stagnation, as is naturally expected from physical point of view.[2]

When, therefore, u_1 varies as x^n $(n>0)$, we may put

$$u_1=ax^n, \tag{1.2}$$

where a denotes some positive constant, and substituting (1.2) into (1.1), we have readily

$$\theta=\sqrt{\frac{0.44}{1+5n}\,\frac{\nu}{a}\,x^{1-n}}. \tag{1.3}$$

2. For example, if we consider the flow along a flat plate placed edgeways to the uniform stream U, we should write

$$a=U \quad \text{and} \quad n=0 \tag{2.1}$$

in equation (1.2). Then from (1.3) we obtain

$$\theta=0.663\sqrt{\nu x/U}, \tag{2.2}$$

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1) TANI, I., On a Simplified Calculation of the Laminar Separation Point (in Japanese), *Journal of the Aeronautical Research Institute*, Tokyo Imperial University, no. 199 (1941), pp. 62–67, esp. 63. It is of interest to note that A. WALZ proposed independently a similar expression in the same year:

$$\frac{\theta^2}{\nu}=\frac{0.470}{u_1{}^6}\int_0^x u_1{}^5\,dx, \tag{1.1$'$}$$

cf. SCHLICHTING, H., *Boundary Layer Theory* (translated by KESTIN, J.) (1955), XII, **b**, p. 213, eqn. (12.37). The formula obtained later by B. THWAITES reads

$$\frac{\theta^2}{\nu}=\frac{0.45}{u_1{}^6}\int_0^x u_1{}^5\,dx, \tag{1.1$''$}$$

cf. Approximate Calculation of the Laminar Boundary Layer, *The Aeronautical Quarterly*, vol. I (1949), pp. 245–280.

2) Cf. eqn. (3) of TANI's paper.

where the exact value on the basis of the BLASIUS' solution is found to be

$$\theta = 0.664\sqrt{\nu x/U}.^{1)} \tag{2.3}$$

In the next place, let us take the case exemplified by

$$n = 1, \tag{2.4}$$

namely

$$u_1 = ax. \tag{2.5}$$

By virtue of the relation (1.3), we can readily arrive at the result

$$\theta = 0.271\sqrt{\nu/a}\ . \tag{2.6}$$

The velocity distribution given by (2.5) corresponds to the flow field produced when the fluid impinges perpendicularly upon a flat plate with infinite breadth and divides into two streams.[2] Starting from the velocity distribution,

$$u_1 = ax, \quad v_1 = -ay, \tag{2.7}$$

of the potential flow outside the boundary layer,[3] the equations of STOKES and NAVIER were solved exactly for the flow adjacent to the wall, and the computed velocity profile is shown numerically in Table 5.1 of SCHLICHTING's book (p. 73, *op. cit.*). From this table we are enabled to calculate the momentum thickness by numerical integration, and we obtain accordingly[4]

$$\theta = 0.292\sqrt{\nu/a}\ . \tag{2.8}$$

The deviation of the approximate value (2.6) from the exact one (2.8) is found to be about 8 %.

On the other hand, when use is made of the formulas (1.1)′ (WALZ) and (1.1)″ (THWAITES), cf. footnote on p. 13, in place of (1.1), then the numerical coefficient in (2.2) is found to be 0.686 and 0.671, and also (2.6) becomes 0.280 $(\nu/a)^{1/2}$ and 0.274 $(\nu/a)^{1/2}$, respectively. Comparing with the exact values given by (2.3) and (2.8), we know readily that in the first example, namely a flat plate placed *edgeways* to the uniform stream, (1.1) yields the best result while (1.1)′ gives the poorest agreement. In the second example in which a plate is placed *perpendicularly* to the stream, however, the largest discrepancy from the exact value is produced by (1.1), when the smallest by (1.1)′. From these two examples, therefore, we may suppose that, generally speaking, the three formulas above-mentioned might have almost the same degree of approximation.

3. More generally, consider the flow impinging upon the wedge whose walls are inclined to the dividing stream-line by the angle $\alpha\pi$ $(0<\alpha<1)$; see FIGURE 1. By the well-known procedure of conformal mapping, we can arrive at the result[5]

$$u_1 = ax^n, \tag{3.1}$$

where

$$n = (1-\alpha)/\alpha. \tag{3.2}$$

From (1.3), therefore, we obtain

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1) SCHLICHTING, H., *ibid.*, VII, **f**. p. 110, eqn. (7.42).

2) SCHLICHTING, H., *ibid.*, V. **b**, **8**, p. 70 *et seq.*

3) x and y are taken along, and perpendicularly to, the plate respectively; u_1 and v_1 denote the components of the velocity in x-, and y-, directions.

4) SCHLICHTING, H., *ibid.*, XII, **c**, **2**, Table 12.3, p. 214. It should be added incidentally that the approximate method due to VON KÁRMÁN and POHLHAUSEN gives the result

$$\theta = 0.278\sqrt{\nu/a}\ ;$$

cf. the same table.

5) LAMB, H., *Hydrodynamics* (1959), IV, **63**, eqn. (4), p. 69.

$$\theta = \sqrt{\frac{0.44\alpha}{5-4\alpha}\frac{\nu}{a}}\,x^{(2\alpha-1)/\alpha}, \tag{3.3}$$

or more briefly, θ is found to vary as $x^{(2\alpha-1)/2\alpha}$.

For the values of n, such that

$$n=0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ \text{and}\ 1.0, \tag{3.4}$$

we have from (3.2)

$$\alpha=1\ (\alpha\pi=180^\circ),\ 0.833\ (150^\circ),\ 0.714\ (128.6^\circ),\ 0.625\ (112.5^\circ),\ 0.556\ (100^\circ),\ \text{and}\ 0.5\ (90^\circ), \tag{3.5}$$

and

$$(2\alpha-1)/2\alpha=0.5,\ 0.4,\ 0.3,\ 0.2,\ 0.1,\ \text{and}\ 0, \tag{3.6}$$

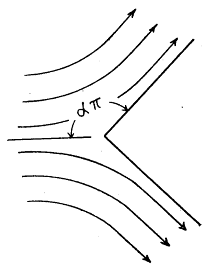


FIG. 1.

respectively. In FIGURE 2, the general behaviors of the functions x^n are shown for the values of x and n, ranging from zero to one, and 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, and 1.0, respectively.

On the other hand, however, for the values of n greater than one, namely for α less than 1/2 (the concave wedge, in other words), $(2\alpha-1)/2\alpha$ which is the power of x in

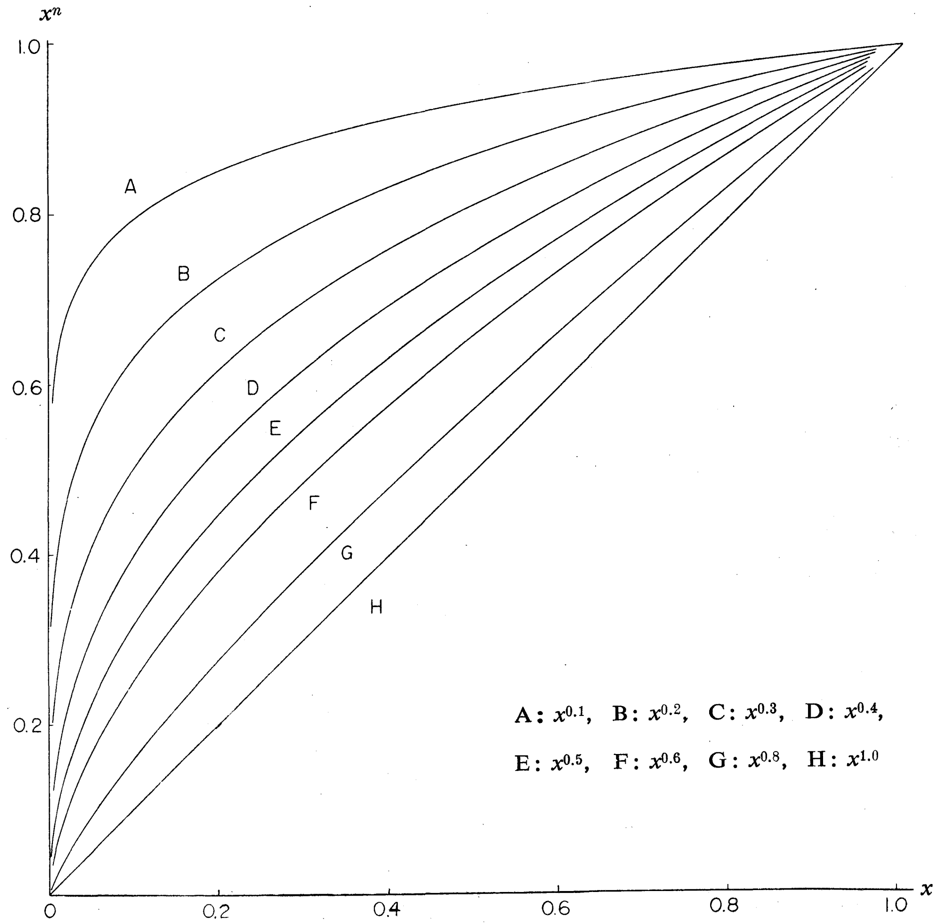
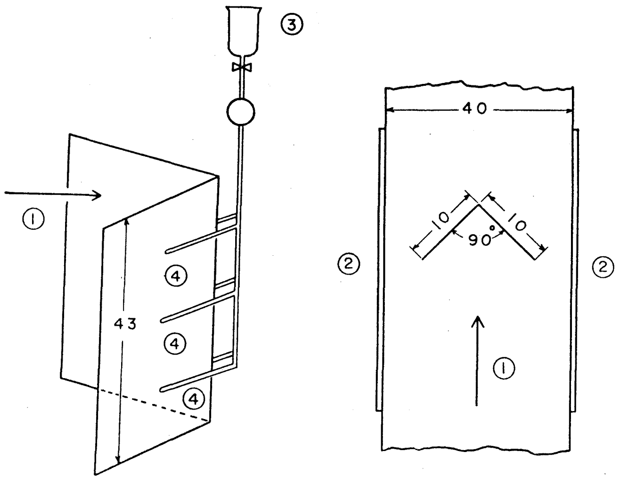


FIG. 2.

(3.2) becomes negative, and the momentum thickness increases without bounds at $x=0$, the stagnation point, mathematically.[1]

Thus, in contrast with the ordinary dead water behind a bluff body in a stream, the forward dead water region, so to speak, is formed in front of a concave body.

4. With a view to examining more closely the state of affairs which take place in front of a concave wedge, observations were made using a water channel (40 cm wide, 40 cm deep, and 600 cm long). A steel plate (20 cm×43 cm) was bent into a right angle, and was put vertically into the stream with its concave side facing upstream; see FIGURE 3.



Side view of the wedge General arrangement

FIG. 3.

① Direction of stream, ② Glass windows,
③ Reservoir of red ink, ④ Small holes (3 on each side),
Unit in cm.

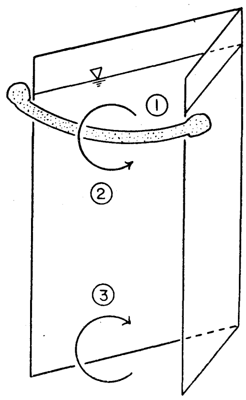


FIG. 4.

① Cavity, ② Surface vortex,
③ Bottom vortex.

When the stream velocity was below 10 cm/sec, the stream-lines flowing round the wedge were found almost horizontal except the weak horizontal vortex formed in the neighborhood of the bottom of the channel; see FIGURE 4.[2] As the speed was raised, however, another horizontal vortex became remarkable which connected both of the points of intersection of the edges of the angle and the surface of water. The faster flowed the water, the more vigorous became the surface vortex, and when the speed reached about 30 cm/sec, a cavity was formed at its core; see FIGURE 4. Both of these vortices, however, are not inherent in our problem, because they were due to the wall effects in the experiment: the lower vortex was caused by the bottom of the channel, while the upper one was generated by the combined effect of the free surface of water with the vertical edges of the angle. Except these two vortices the region embraced within the angle was found free from all vortices and of course far less turbulent than the wake behind the wedge. It should be

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1) From (3.1), (3.2) and (3.3), $u_1^3\theta$ varies as $x^{5n/2+1/2}$ and vanishes at $x=0$.

2) In this note, a vortex is termed horizontal or vertical according as its axis lies horizontally or vertically, respectively.

mentioned finally that in the region no vertical vortices were observed in our experiment.[1)]

The writers are very grateful to Miss S. HOSHINO and Mr. N. FUKAMACHI for their assistances in the numerical calculations and the experiment.

(Received Oct. 5, 1962)

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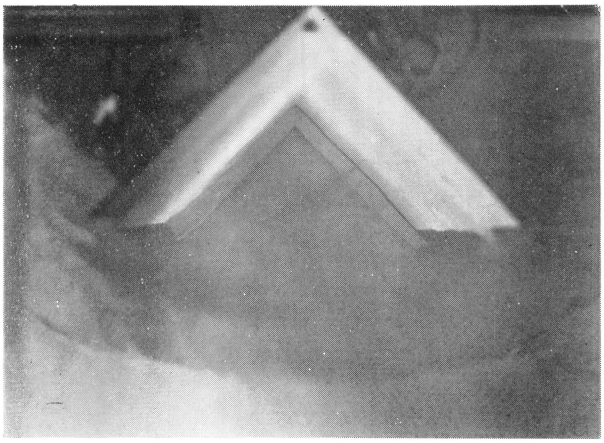
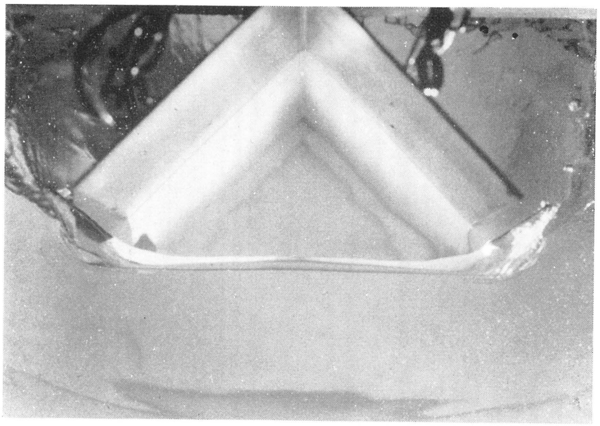
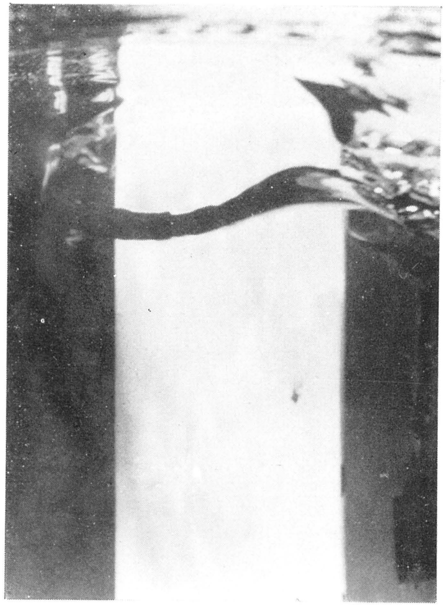
1) Because of the finite length (10 cm) of the sides of the angle (see FIGURE 3), the flow field realized experimentally was only a rough approximation of the dead water appearing in front of a cancave wedge with an infinite breadth. The form of the stagnation region, therefore, was observed so different from that predicted by the theory; cf. photograph no. 1.



No. 1.



No. 2.



No. 3.

Notes for the photographs

1) Photograph no. 1 (a bird's-eye view) shows the boundary of the forward stagnation region. The velocity of the stream pointing upward in the picture is about 10 cm/sec. The region was visualized by emitting red ink from the small holes in the plate.

2) Photographs nos. 2 (a bird's-eye view, the stream pointing upward) and 3 (a side view through the glass window of the water channel, the stream pointing to the right) show the cavity formed at the core of the surface vortex. In both of these pictures, the velocity of the stream is about 30 cm/sec.