SOME CONSIDERATIONS ON THE FOUR-POINT DYNAMICAL EQUATIONS OF HOMOGENEOUS TURBULENCE

OHUJI, Michio
Research Institute for Applied Mechanics, Kyushu University

https://doi.org/10.5109/7164837

出版情報：Reports of Research Institute for Applied Mechanics. 10 (38), pp.33-43, 1962. 九州大学応用力学研究所
バージョン：

Reports of Research Institute for Applied Mechanics
Vol. X, No. 38, 1962

SOME CONSIDERATIONS ON THE FOUR-POINT DYNAMICAL EQUATIONS OF HOMOGENEOUS TURBULENCE*

By

Michio OHJI

By introducing the quadruple and quintuple cumulants of the corresponding correlations, the four-point dynamical equations are reconsidered in the corrected form. It is shown that Deissler's approximation of neglecting quintuple terms is closely related to the "quasi-normal" relation which has been often assumed in the statistical theory of homogeneous turbulence. A doubt is cast on the validity of the cumulant discard approximation for intense turbulence, and by the way the apparent inconsistency of the quasi-normal relation at the lowest wave-numbers is interpreted.

1. Introduction. Most of the efforts devoted to the statistical theory of homogeneous turbulence have been concentrated, in effect, on the problem of finding how to close the infinite hierarchy of coupled equations into a determinate set containing some finite number of unknowns. The intuitive theories of Obukhoff, Heisenberg, Kovasznay, e.g. [**1**] Chap. VI, or the two-equation theories of Proudman and Reid [**7**], Tatsumi [**8**, **9**] are typical examples. A recent contribution to this "closure problem" has been made by Deissler [**3**, **4**]. The principle of his method, which may be called Deissler's approximation (DA, hereafter), is very simple: namely, to retain only the velocity products up to a certain order n by ignoring the higher-order products altogether. The n-point dynamical equation, being thereby freed from the inertia terms, will close by itself, and consequently the whole set of lower-order equations is made determinate in principle as an initial value problem. Inasmuch as the procedure is clearly the natural extension of the well-established theory of the final period of decay (which corresponds to the case of $n=2$), we may suppose that DA will be valid for *weak* turbulence in particular, although no explicit proof has been given.

Along these lines of thought, Deissler has carried out detailed calculations first for the case of $n=3$ [**3**] and then for the case of $n=4$ [**4**]. In the course of analysis, however, he seems to have overlooked the fact that the quadruple and quintuple correlations have no finite Fourier transforms (or spectra) in general [**1**] p. 21. Accordingly, the four-point spectral equation derived by him is incomplete. The device of avoiding such a difficulty is to introduce, in place of those correlations themselves, the corresponding "cumulants" whose Fourier transforms are secured to be definitely convergent [**1**] *loc. cit.* In this paper we shall show to what extent Deissler's formulation of the four-point analysis [**4**] should be modi-

* A short account of this paper has already appeared under the slightly different title [**6**].

fied, and some considerations will be made on a few related problems.

2. Four-point correlation and cumulant equations. The four-point correlation equation of homogeneous incompressible turbulence was correctly derived by Deissler [4]. It can be written as

$$\frac{\partial}{\partial t}\langle u_iu_j'u_k''u_l'''\rangle=2\nu\mathfrak{D}^{(3)}\langle u_iu_j'u_k''u_l'''\rangle+\left(\frac{\partial}{\partial r_m}+\frac{\partial}{\partial r_m'}+\frac{\partial}{\partial r_m''}\right)\langle u_iu_mu_j'u_k''u_l'''\rangle$$

$$-\frac{\partial}{\partial r_m}\langle u_iu_j'u_m'u_k''u_l'''\rangle-\frac{\partial}{\partial r_m'}\langle u_iu_j'u_k''u_m''u_l'''\rangle-\frac{\partial}{\partial r_m''}\langle u_iu_j'u_k''u_l'''u_m'''\rangle$$

$$+\left(\frac{\partial}{\partial r_i}+\frac{\partial}{\partial r_i'}+\frac{\partial}{\partial r_i''}\right)\langle Pu_j'u_k''u_l'''\rangle-\frac{\partial}{\partial r_j}\langle u_iP'u_k''u_l'''\rangle-\frac{\partial}{\partial r_k'}\langle u_iu_j'P''u_l'''\rangle$$

$$-\frac{\partial}{\partial r_l''}\langle u_iu_j'u_k''P'''\rangle, \tag{2.1}$$

where $\langle\quad\rangle$ denotes statistical average, $P=p/\rho$ is the kinematic pressure, and

$$\mathfrak{D}^{(3)}=\frac{\partial^2}{\partial r_m\partial r_m}+\frac{\partial^2}{\partial r_m\partial r_m'}+\frac{\partial^2}{\partial r_m\partial r_m''}+\frac{\partial^2}{\partial r_m'\partial r_m'}+\frac{\partial^2}{\partial r_m'\partial r_m''}+\frac{\partial^2}{\partial r_m''\partial r_m''}, \qquad (2.2)^{1)}$$

with otherwise customary symbols. For derivation see Deissler [4] p. 177.

We define the quadruple and quintuple cumulants in the following way:

$$\langle u_iu_j'u_k''u_l'''\rangle^*=\langle u_iu_j'u_k''u_l'''\rangle-\langle u_iu_j'\rangle\langle u_k''u_l'''\rangle-\langle u_iu_k''\rangle\langle u_j'u_l'''\rangle$$
$$-\langle u_iu_l'''\rangle\langle u_j'u_k''\rangle, \tag{2.3a}$$

$$\langle u_iu_mu_j'u_k''u'''\rangle^{**}=\langle u_iu_mu_j'u_k''u_l'''\rangle-\langle u_iu_m\rangle\langle u_j'u_k''u_l'''\rangle$$
$$-\langle u_iu_mu_j'\rangle\langle u_k''u_l'''\rangle-\langle u_iu_mu_k''\rangle\langle u_j'u_l'''\rangle$$
$$-\langle u_iu_mu_l'''\rangle\langle u_j'u_k''\rangle, \tag{2.4a}$$

$$\langle Pu_j'u_k''u_l'''\rangle^*=\langle Pu_j'u_k''u_l'''\rangle-\langle Pu_j'\rangle\langle u_k''u_l'''\rangle-\langle Pu_k''\rangle\langle u_j'u_l'''\rangle$$
$$-\langle Pu_l'''\rangle\langle u_j'u_k''\rangle. \tag{2.5a}$$

It is readily seen that these cumulants, unlike the correlations themselves, vanish whenever any two of the points are separated widely enough. Thus, we are safe to introduce their Fourier transforms such that[2)]

$$\langle u_iu_j'u_k''u_l'''\rangle^*=\mathfrak{F}[\Phi_{ijkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')], \tag{2.3b}$$

$$\langle u_iu_mu_j'u_k''u_l'''\rangle^{**}=\mathfrak{F}[\Psi_{(im)jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')], \tag{2.4b}$$

$$\langle Pu_j'u_k''u_l'''\rangle^*=\mathfrak{F}[\Phi_{0jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')], \tag{2.5b}$$

where $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}'$ and $\boldsymbol{\kappa}''$ are three independent wave-number vectors, and the nine-dimensional Fourier integral

$$\mathfrak{F}[\ldots]=\iiint(\ldots)\exp[-\iota(\boldsymbol{\kappa}.\boldsymbol{r}+\boldsymbol{\kappa}'.\boldsymbol{r}'+\boldsymbol{\kappa}''.\boldsymbol{r}'')]\,d\boldsymbol{\kappa}\,d\boldsymbol{\kappa}'\,d\boldsymbol{\kappa}'' \tag{2.6}$$

---

1) The superscript (3) indicates the number of independent variable vectors. Cf. (A3) and (A6) in Appendix A.

2) The present symbols for Fourier transforms are different from those used in the abridged report [6].

is to be understood.

Note that in our notation,* refers to the n-point n-order cumulant (the non-degenerate case) and ** to the n-point $n+1$-order cumulant (the simply-degenerate case). In wave-number space, the two cases are distinguished by the symbols Φ and Ψ, respectively. Further, the tensor indices which correspond to an identical point are put in parentheses and the index 0 indicates that the kinematic pressure should be taken at the corresponding point. These conventions will apply similarly to the later cases too.

Now, using (2.3a)$\sim$(2.5a) together with the two-point equation, we can express Eq. (2.1) in terms of the cumulants as

$$\begin{aligned}\frac{\partial}{\partial t}\langle u_i u_j' u_k'' u_l'''\rangle^* &= 2\nu\mathfrak{D}^{(3)}\langle u_i u_j' u_k'' u_l'''\rangle^* + \left(\frac{\partial}{\partial r_m}+\frac{\partial}{\partial r_m'}+\frac{\partial}{\partial r_m''}\right)\langle u_i u_m u_j' u_k'' u_l'''\rangle^{**}\\ &-\frac{\partial}{\partial r_m}\langle u_i u_j' u_m' u_k'' u_l'''\rangle^{**}-\frac{\partial}{\partial r_m'}\langle u_i u_j' u_k'' u_m'' u_l'''\rangle^{**}-\frac{\partial}{\partial r_m''}\langle u_i u_j' u_k'' u_l''' u_m'''\rangle^{**}\\ &+\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial r'}+\frac{\partial}{\partial r_i''}\right)\langle P u_j' u_k'' u_l'''\rangle^*-\frac{\partial}{\partial r_j}\langle u_i P' u_k'' u_l'''\rangle^*-\frac{\partial}{\partial r_k'}\langle u_i u_j' P'' u_l'''\rangle^*\\ &-\frac{\partial}{\partial r_l''}\langle u_i u_j' u_k'' P'''\rangle^*, \end{aligned}\tag{2.7}$$

(see Appendix B for proof). It is rather remarkable that the cumulant equation (2.7) is formally same as Eq. (2.1) except for that the correlation tensors in the latter are replaced by the corresponding cumulant tensors.

3. **Four-point spectral equations.** To turn to wave-number space, let us take the Fourier transform of Eq. (2.7). Because of the apparent similarity between Eqs. (2.1) and (2.7), we can follow Deissler [4] p. 178 without repeating his manipulations, so long as the spectral tensors are re-interpreted in the sense of (2.3b)$\sim$(2.5b). Thus, refering to Eq. (D10)[3], we have at once

$$\begin{aligned}\left(\frac{\partial}{\partial t}+2\nu D^{(3)}\right)\Phi_{ijkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'') &= -\iota\,[\sigma_m''\Psi_{(im)jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')+\kappa_m\Psi_{(jm)kli}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\boldsymbol{\sigma}'')\\ &+\kappa_m'\Psi_{(km)lij}(\boldsymbol{\kappa}'',\boldsymbol{o}'',\boldsymbol{\kappa})+\kappa_m''\Psi_{(lm)ijk}(\boldsymbol{\sigma}'',\boldsymbol{\kappa},\boldsymbol{\kappa}')+\sigma_i''\Phi_{0jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')\\ &+\kappa_j\Phi_{0kli}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\boldsymbol{\sigma}'')+\kappa_k'\Phi_{0lij}(\boldsymbol{\kappa}'',\boldsymbol{\sigma}'',\boldsymbol{\kappa})+\kappa_l''\Phi_{0ijk}(\boldsymbol{\sigma}'',\boldsymbol{\kappa},\boldsymbol{\kappa}')], \end{aligned}\tag{3.1}$$

with the abbreviations

$$D^{(3)}=\kappa^2+\kappa_m\kappa_m'+\kappa_m\kappa_m''+\kappa'^2+\kappa_m'\kappa_m''+\kappa''^2, \tag{3.2}$$

and

$$\boldsymbol{o}''=-(\boldsymbol{\kappa}+\boldsymbol{\kappa}'+\boldsymbol{\kappa}''). \tag{3.3}$$

The pressure terms in Eq. (3.1), however, can be transformed by virtue of the incompressiblity condition. That is, the continuity relation yields

$$\sigma_m''\sigma_m''\Phi_{0jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')=-\sigma_n''\sigma_m''\Psi_{(nm)jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}''), \tag{3.4}$$

3) (D10) denotes Eq. (10) of Deissler's paper [4] and so on.

cf. (D12), hence it turns out that

$$\sigma_m''\Psi_{(im)jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')+\sigma_i''\Phi_{0jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')=\sigma_m''\Big(\delta_{in}-\frac{\sigma_i''\sigma_n''}{\sigma''^2}\Big)\Psi_{(nm)jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}''),$$

and the like for the other terms on the right-hand side of Eq. (3.1). In consequence, the four-point spectral equation (3.1) reduces to

$$\begin{aligned}\Big(\frac{\partial}{\partial t}+2\nu D^{(3)}\Big)\Phi_{ijkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}\ \boldsymbol{\kappa}'')&=-\iota[\sigma_m''\Delta_{in}(\boldsymbol{\sigma}'')\Psi_{(nm)jkl}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'')\\&+\kappa_m\Delta_{jn}(\boldsymbol{\kappa})\Psi_{(nm)kli}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\boldsymbol{\sigma}'')+\kappa_m'\Delta_{kn}(\boldsymbol{\kappa}')\,\Psi_{(nm)lij}(\boldsymbol{\kappa}'',\boldsymbol{\sigma}'',\boldsymbol{\kappa})\\&+\kappa_m''\Delta_{ln}(\boldsymbol{\kappa}'')\Psi_{(nm)ijk}(\boldsymbol{o}'',\boldsymbol{\kappa},\boldsymbol{\kappa}')]=-\iota\textstyle\sum_{\mathrm{C}}^{(4)}\sigma_m''\Delta_{in}(\boldsymbol{\sigma}'')\Psi_{(nm)\,jkl}\,(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}''),\end{aligned}\tag{3.5}$$

where

$$\Delta_{ij}(\boldsymbol{x})=\delta_{ij}-\frac{x_i\cdot x_j}{x^2}\ ,$$

for any vector $\boldsymbol{x}$, and the summation $\sum_{\mathrm{C}}^{(4)}$ runs over all simultaneous cyclic changes of (i,j,k,l) and $(\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}'',\boldsymbol{\sigma}'')$[4].

Before entering into further considerations on the above-obtained equation, we shall add a few kinematical definitions and relations of the two- and three-point cases.

(i) Second order:

$$\langle u_iu_j'\rangle^*=\langle u_iu_j'\rangle=\int\Phi_{ij}(\boldsymbol{\kappa})\exp(-\iota\boldsymbol{\kappa}.\boldsymbol{r})d\boldsymbol{\kappa}.\tag{3.6}$$

(ii) Third order:

$$\langle u_iu_j'u_k''\rangle^*=\langle u_iu_j'u_k''\rangle=\iint\Phi_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\exp[-\iota(\boldsymbol{\kappa}.\boldsymbol{r}+\boldsymbol{\kappa}'.\boldsymbol{r}')]\,d\boldsymbol{\kappa}d\boldsymbol{\kappa}';\tag{3.7}$$

third-order degenerate:

$$\langle u_iu_ku_j'\rangle^{**}=\langle u_iu_ku_j'\rangle=\int\Psi_{(ik)j}(\boldsymbol{\kappa})\exp(-\iota\boldsymbol{\kappa}.\boldsymbol{r})d\boldsymbol{\kappa},\tag{3.8}$$

where

$$\Psi_{(ik)j}\,(\boldsymbol{\kappa})=\int\Phi_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}')d\boldsymbol{\kappa}'.\tag{3.9}$$

(iii) Fourth-order degenerate:

$$\begin{aligned}\langle u_iu_lu_j'u_k''\rangle^{**}&=\langle u_iu_lu_j'u_k''\rangle-\langle u_iu_l\rangle\langle u_j'u_k''\rangle\\&=\langle u_iu_lu_j'u_k''\rangle^*+\langle u_iu_j'\rangle\langle u_lu_k''\rangle+\langle u_iu_k''\rangle\langle u_lu_j'\rangle\ (\text{cf.}(2.3\text{a})),\\&=\iint\Psi_{(il)jk}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\exp[-\iota(\boldsymbol{\kappa},\boldsymbol{r}+\boldsymbol{\kappa}'.\boldsymbol{r}')]\,d\boldsymbol{\kappa}d\boldsymbol{\kappa}'\ ,\end{aligned}\tag{3.10}$$

4) The superscript (4) indicates the number of terms under the summation. Cf. (A2) and (A5) in Appendix A. See also the derivations of Proudman and Reid [7] and of Tatsumi [8] for the three-point equation.

which has the following Fourier transform:

$$\Psi_{(il)jk}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') = \int \Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')d\boldsymbol{\kappa}'' + \Phi_{ij}(\boldsymbol{\kappa})\Phi_{lk}(\boldsymbol{\kappa}') + \Phi_{ik}(\boldsymbol{\kappa}')\Phi_{lj}(\boldsymbol{\kappa}), \tag{3.11}$$

in striking contrast to (D18).

The dynamical equations governing these lower-order spectra are reproduced in the Appendix A.

4. Four-point Deissler's approximation. Now, following the principle of DA, as was suggested in the introductory section, we neglect the quintuple velocity products. Eq. (3.5) then becomes simply

$$\left(\frac{\partial}{\partial t} + 2\nu D^{(3)}\right)\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'') = 0, \tag{4.1}$$

the solution of which is

$$\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'') = [\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1 \exp[-2\nu(t-t_1)D^{(3)}], \tag{4.2}$$

where $[\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1$ denotes the prescribed initial value at $t=t_1$. Thus, with the aid of the relation (3.11), we are lead to the following conclusions which are essentially different from Deissler's [4]:

(1) If $[\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}' \boldsymbol{\kappa}'')]_1=0$, then $\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')=0$ identically and hence we have, from (3.11),

$$\Psi_{(il)jk}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') = \Phi_{ij}(\boldsymbol{\kappa})\Phi_{lk}(\boldsymbol{\kappa}') + \Phi_{ik}(\boldsymbol{\kappa}')\Phi_{lj}(\boldsymbol{\kappa}), \tag{4.3}$$

or

$$\langle u_i u_l u_j' u_k'' \rangle = \langle u_i u_l \rangle \langle u_j' u_k'' \rangle + \langle u_i u_j' \rangle \langle u_l u_k'' \rangle + \langle u_i u_k'' \rangle \langle u_l u_j' \rangle, \tag{4.3'}$$

which is nothing but the "zero fourth-order cumulant" or "quasi-normal" relation (QNR, in short) as it is called. Namely, once the QNR is set up at some instant, it will persist afterwards, provided that the quintuple velocity products are neglegibly small. The QNR was originally postulated from the purely statistical point of view [**1**] Chap. VIII, but it is now obtained as a *solution* of the four-point DA for the said initial condition. In such a case, the complete solution of the problem will be determined by solving the two equations (A1) and (A3) with (3.9) and (4.3) just as in the theories of Proudman and Reid [**7**] and of Tatsumi [**8, 9**].

(2) If $[\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1 \neq 0$, there remains, on the right-hand side of (3.11), the integral

$$\int \Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')d\boldsymbol{\kappa}'' = \exp[-2\nu(t-t_1)(\kappa^2 + \kappa_m\kappa_m' + \kappa'^2)]$$

$$\times \int [\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1 \exp[-2\nu(t-t_1)(\kappa_m\kappa_m'' + \kappa_m'\kappa_m'' + \kappa''^2)]\, d\boldsymbol{\kappa}', \tag{4.4}$$

where $[\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1$ must really depend on $\boldsymbol{\kappa}''$, because of the solenoidal condition $\partial \langle u_i u_j' u_k'' u_l''' \rangle / \partial r_l''=0$ or $\kappa_l''\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')=0$. However, so long as $[\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1$ is analytic everywhere in $\boldsymbol{\kappa}''$-space, the argument analogous to the theory of final period of decay [1] § 5.4 is applicable to the asymptotic evaluation of the integral.

The essential point is that the exponential factor in the integrand of (4.4) decreases rapidly as the quadratic form $\kappa_m\kappa_m''+\kappa_m'\kappa_m''+\kappa''^2$ ($=Q$, say) increases. Expanding $[\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1$ in the Taylor series near the point $\boldsymbol{\kappa}''=-(\boldsymbol{\kappa}+\boldsymbol{\kappa}')/2$, at which Q becomes minimum for $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ fixed, and integrating termwise, we obtain

$$\int [\Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')]_1 \exp[-2\nu(t-t_1)Q]\, d\boldsymbol{\kappa}''$$

$$=\left(\frac{\pi}{2\nu}\right)^{3/2} A_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}')\, [(t-t_1)^{-3/2}+O(t^{-5/2})] \exp\left[2\nu(t-t_1)\left(\frac{\boldsymbol{\kappa}+\boldsymbol{\kappa}'}{2}\right)^2\right],$$

and therefore

$$\int \Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')d\boldsymbol{\kappa}'' \sim A_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}')\left[\frac{\pi}{2\nu(t-t_1)}\right]^{3/2}$$

$$\times\exp\left[-2\nu(t-t_1)\left(\frac{3}{4}\kappa^2+\frac{1}{2}\kappa_m\kappa_m'+\frac{3}{4}\kappa'^2\right)\right] \to 0, \tag{4.5}$$

for $t\to\infty$, where $A_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}')\equiv\left[\Phi_{ijkl}\left(\boldsymbol{\kappa}, \boldsymbol{\kappa}', -\frac{\boldsymbol{\kappa}+\boldsymbol{\kappa}'}{2}\right)\right]_1$.

It would be premature to conclude from this that there is a true tendency to the QNR in general, inasmuch as the asymptotic behavior of the second-order tensors $\Phi_{ij}(\boldsymbol{\kappa})$ etc. is not known here. But an interesting prediction seems possible on the flatness factor of the one-point probability distribution of the velocity fluctuations. Namely, when all four points coincide, (3.10) becomes

$$\langle u_iu_ju_ku_l\rangle^{**}=\iiint \Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')d\boldsymbol{\kappa}d\boldsymbol{\kappa}'d\boldsymbol{\kappa}''+\langle u_iu_j\rangle\langle u_ku_l\rangle+\langle u_iu_k\rangle\langle u_ju_l\rangle$$

$$=\langle u_iu_ju_ku_l\rangle-\langle u_iu_l\rangle\langle u_ju_k\rangle. \tag{4.6}$$

We use (4.5) on assuming $A_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}')$ is analytic in $\boldsymbol{\kappa}'$-space, and repeat the similar procedure as before to obtain

$$\iiint \Phi_{ijkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'')\, d\boldsymbol{\kappa}\, d\boldsymbol{\kappa}'\, d\boldsymbol{\kappa}''$$

$$\sim\left[\frac{\pi}{2\nu(t-t_1)}\right]^3 \int B_{ijkl}(\boldsymbol{\kappa})\exp\left[-\frac{4}{3}\nu(t-t_1)\kappa^2\right] d\boldsymbol{\kappa}, \tag{4.7}$$

for $t\to\infty$, where $(3/4)^{3/2}B_{ijkl}(\boldsymbol{\kappa})\equiv A_{ijkl}(\boldsymbol{\kappa}, -\boldsymbol{\kappa}/3)\equiv[\Phi_{ijkl}(\boldsymbol{\kappa}, -\boldsymbol{\kappa}/3, -\boldsymbol{\kappa}/3)]_1$. Considering that the solenoidal condition requires $B_{ijkl}(0)=0$, the integral on the right-hand side of (4.7) is found to be of the order of $(t-t_1)^{-5/2}$ asymptotically, so long as $B_{ijkl}(\boldsymbol{\kappa})$ is an analytic function of $\boldsymbol{\kappa}$. By contracting the indices in the relation (4.6), we find

$$\langle u^4\rangle=3\langle u^2\rangle^2+O(t^{-11/2}), \tag{4.8}$$

with an appropriate choice of the time origin. It is known experimentally that the observed decay of turbulent energy $\langle u^2\rangle$ lies between the linear decay (in the initial period) and the $-5/2$ power decay (in the final period), i.e., $O(t^{-2}) > \langle u^2 \rangle^2 > O(t^{-5})$, cf. [1] Chap. VII. Hence we may suppose that $\langle u^4 \rangle / \langle u^2 \rangle^2 \to 3$, which shows the approach of the flatness factor to the normal value 3. No further detailed comparison with experiments, however, is intended here.

5. Discussion. In the preceding section, we have seen how DA is closely related to the QNR. However, it should be noted that Eq. (4.1) on which the whole arguments depend does not necessarily require the neglect of the quintuple velocity products themselves. More generally, we have only to assume

$$\sum_{C}^{(4)} \sigma_m'' \Delta_{in}(\boldsymbol{\sigma}'') \Psi_{(nm)jkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'') = 0, \tag{5.1}$$

to obtain the same conclusions as before. The necessary and sufficient condition for the persistence of the QNR (§4. (1)) is also given by (5.1). From physical point of view, the most probable solution satisfying this condition is likely to be

$$\Psi_{(im)jkl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'') \equiv 0 \rightleftarrows \langle u_i u_m u_j' u_k'' u_l''' \rangle^{**} \equiv 0, \tag{5.2}$$

with which we shall be particularly concerned for a moment[5].

Consider now the five-point velocity cumulant defined as

$$\begin{aligned}
\langle u_i u_j' u_k'' u_l''' u_m'''' \rangle^* &= \langle u_i u_j' u_k'' u_l''' u_m'''' \rangle - \langle u_i u_j' \rangle \langle u_k'' u_l''' u_m'''' \rangle \\
&- \langle u_i u_k'' \rangle \langle u_j' u_l''' u_m'''' \rangle - \langle u_i u_l''' \rangle \langle u_j' u_k'' u_m'''' \rangle - \langle u_i u_m'''' \rangle \langle u_j' u_k'' u_l''' \rangle \\
&- \langle u_j' u_k'' \rangle \langle u_i u_l''' u_m'''' \rangle - \langle u_j u_l''' \rangle \langle u_i u_k'' u_m'''' \rangle - \langle u_j' u_m'''' \rangle \langle u_i u_k'' u_l''' \rangle \\
&- \langle u_k'' u_l''' \rangle \langle u_i u_j' u_m'''' \rangle - \langle u_k'' u_m'''' \rangle \langle u_i u_j' u_k'' \rangle - \langle u_l''' u_m'''' \rangle \langle u_i u_j' u_k'' \rangle .
\end{aligned} \tag{5.3}$$

Comparing this with (2.4), we have

$$\begin{aligned}
\langle u_i u_m u_j' u_k'' u_l''' \rangle^{**} &= \langle u_i u_m u_j' u_k'' u_l''' \rangle^* + [\langle u_i u_j' \rangle \langle u_m u_k'' u_l''' \rangle \\
&+ \langle u_i u_k'' \rangle \langle u_m u_j' u_l''' \rangle + \langle u_i u_l''' \rangle \langle u_m u_j' u_k'' \rangle + \langle u_i u_k'' u_l''' \rangle \langle u_m u_j' \rangle \\
&+ \langle u_i u_j' u_l''' \rangle \langle u_m u_k'' \rangle + \langle u_i u_j' u_k'' \rangle \langle u_m u_l''' \rangle],
\end{aligned} \tag{5.4}$$

which is the four-point analogue of the three-point relation (3.10). Thus, the condition (5.2) for the QNR is equivalent to assume

$$\langle u_i u_m u_j' u_k'' u_l''' \rangle^* = -\,[\text{the same terms as in (5.4)}], \tag{5.5a}$$

or, in the spectral form[6],

$$\begin{aligned}
\int \Phi_{ijklm}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'', \boldsymbol{\kappa}''') d\boldsymbol{\kappa}''' &= -\,[\Phi_{ij}(\boldsymbol{\kappa}) \Phi_{mkl}(\boldsymbol{\kappa}', \boldsymbol{\kappa}'') + \Phi_{ik}(\boldsymbol{\kappa}') \Phi_{mjl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}'') \\
&+ \Phi_{il}(\boldsymbol{\kappa}'') \Phi_{mjk}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') + \Phi_{ikl}(\boldsymbol{\kappa}', \boldsymbol{\kappa}'') \Phi_{mj}(\boldsymbol{\kappa}) + \Phi_{ijl}(\boldsymbol{\kappa}, \boldsymbol{\kappa}'') \Phi_{mk}(\boldsymbol{\kappa}') + \Phi_{ijk}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') \Phi_{ml}(\boldsymbol{\kappa}'')].
\end{aligned} \tag{5.5b}$$

Since the right-hand side of this expression does not vanish in general, unless the

---

5) The solution of (5.1) other than (5.2), though kinematically possible, represents a very stringent restriction on the dynamical interactions between the wave-numbers $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}'$, $\boldsymbol{\kappa}''$, $\boldsymbol{\sigma}''$, and seems almost unreal.

6) $\Phi_{ijklm}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'', \boldsymbol{\kappa}''')$ is defined by an obvious extension of (2.3a) as the twelve-dimensional Fourier transform of $\langle u_i u_j' u_k'' u_l''' u_m'''' \rangle^*$.

quintuple velocity products are small enough, it follows

$$\Phi_{ijklm}(\kappa, \kappa', \kappa'', \kappa''')\neq 0 \rightleftarrows <u_i u_j' u_k'' u_l''' u_m''''>^* \neq 0, \tag{5.6}$$

namely, the fifth-order cumulant is not identically zero, which implies the non-zero higher order cumulants in turn. The full probability distribution of the velocity field in this case is thus expected to be of such a special form that only the fourth-order cumulant is *exceptionally* zero. Under these circumstances, it seems open to question, at least conceptually, whether the QNR is applicable to highly intense turbulence as well. The author feels that the essentially similar state of affairs would occur in the general cumulant discard approximation, too[7].

In his previous paper [5], the author has thrown a doubt on the validity of the QNR at the lowest wave-number regions in particular, taking account of the long-range pressure action first pointed out by Batchelor and Proudman [2]. According to the foregoing considerations, however, we may regard it as a mere special manifestation of the more general fact that the QNR cannot persist in the presence of the inertia terms in the four-point dynamical equation. When the quintuple velocity products are taken into account, the QNR will be dynamically inconsistent not only for the largest but also for all wave-numbers, so long as the relation (5.1) is not satisfied. On the contrary, when the quintuple velocity products are so small as to permit DA, the pressure action has no significant effect on the QNR, and the above-mentioned inconsistency reduces to a trivial one.

In conclusion, we are lead to an anticipation that the assumption of the QNR would be the weak turbulence approximation by nature, as DA is. Further discussion together with possible generalizations will be published in future.

6. **Acknowledgements.** The author is indebted to Dr. K. Kitajima for his stimulating discussion. This work was partly sponsored by Scientific Research Funds from the Ministry of Education.

Appendix A

According to the present system of notation, the lower-order dynamical equations of homogeneous turbulence are written as follows:

(i) Second-order [1] § 5.2:

$$\frac{\partial}{\partial t}\langle u_i u_j'\rangle = 2\nu\mathfrak{D}^{(1)}\langle u_i u_j'\rangle + \frac{\partial}{\partial r_m}\langle u_i u_m u_j'\rangle - \frac{\partial}{\partial r_m}\langle u_i u_j' u_m'\rangle$$

$$+\frac{\partial}{\partial r_i}<Pu_j'> - \frac{\partial}{\partial r_j}<u_i P'>, \tag{A1}$$

or

---

[7] As the QNR corresponds to the fourth-order cumulant discard, the term "zero fourth-order cumulant" (04C) seems preferable rather than the term "quasi-normal" for specifying the relation (4.3). The next approximation is clearly the "zero fifth-order cumulant" (05C) which assumes $\langle u_i u_j u_k'' u_l''' u_m''''\rangle^* = 0$, or $\langle u_i u_m u_j' u_k'' u_l'''\rangle^{**} = +$ [the same terms as in (5.4)]. The "quasi-normal relation" would be more appropriate as a general term which covers various zero cumulant relations (04C), (05C), and so on.

$$\left(\frac{\partial}{\partial t}+2\nu D^{(1)}\right)\Phi_{ij}(\boldsymbol{\kappa})=-\iota\sum_{\mathrm{C}}^{(2)}\sigma_m\Delta_{in}(\boldsymbol{\sigma})\Psi_{(nm)j}(\boldsymbol{\kappa}), \tag{A2}$$

where

$$\mathfrak{D}^{(1)}=\frac{\partial^2}{\partial r_m\partial r_m}\quad;\quad D^{(1)}=\kappa^2\quad;\quad \boldsymbol{\sigma}=-\boldsymbol{\kappa}. \tag{A3}$$

(ii) Third-order [7, 8] :

$$\frac{\partial}{\partial t}\langle u_iu_j'u_k''\rangle=2\nu\mathfrak{D}^{(2)}\langle u_iu_j'u_k''\rangle+\left(\frac{\partial}{\partial r_m}+\frac{\partial}{\partial r_m'}\right)\langle u_iu_mu_j'u_k''\rangle$$
$$-\frac{\partial}{\partial r_m}\langle u_iu_j'u_m'u_k''\rangle-\frac{\partial}{\partial r_m'}\langle u_iu_j'u_k''u_m''\rangle+\left(\frac{\partial}{\partial r_i}+\frac{\partial}{\partial r_i'}\right)\langle Pu_j'u_k''\rangle$$
$$-\frac{\partial}{\partial r_j}\langle u_iP'u_k''\rangle-\frac{\partial}{\partial r_k'}\langle u_iu_j'P''\rangle, \tag{A4}$$

or

$$\left(\frac{\partial}{\partial t}+2\nu D^{(2)}\right)\Phi_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=-\iota\sum_{\mathrm{C}}^{(3)}\sigma_m'\Delta_{in}(\boldsymbol{\sigma}')\Psi_{(nm)jk}(\boldsymbol{\kappa},\boldsymbol{\kappa}'), \tag{A5}$$

where

$$\mathfrak{D}^{(2)}=\frac{\partial^2}{\partial r_m\partial r_m}+\frac{\partial^2}{\partial r_m\partial r_m'}+\frac{\partial^2}{\partial r_m'\partial r_m'}\;;\qquad D^{(2)}=\kappa^2+\kappa_m\kappa_m'+\kappa'^2\,;$$
$$\boldsymbol{\sigma}'=-(\boldsymbol{\kappa}+\boldsymbol{\kappa}'). \tag{A6}$$

In (A2) and (A3), the summations run over simultaneous cyclic changes of tensor indices (other than dummy indices) and the vector arguments. Cf. footnote 4) p. 4.

Appendix B

Derivation of Eq. (2.7) : In order to derive the four-point cumulant equation (2.7), we write

$$\frac{\partial}{\partial t}\langle u_iu_j'u_k''u_l'''\rangle^*-[\text{the right-hand side of (2.7)}]=\alpha_{ijkl} \tag{B1}$$

By substituting (2.3a)$\sim$(2.5a) into (B1), the quantity α_{ijkl} is found to be the sum of the following three parts:

(1) $\langle u_k''u_l'''\rangle[\boldsymbol{r}]_{ij}+\langle u_j'u_l'''\rangle[\boldsymbol{r}']_{ik}+\langle u_j'u_k''\rangle[\boldsymbol{r}'']_{il}$,

(2) $\langle u_iu_j'\rangle[\boldsymbol{r}',\boldsymbol{r}'']_{kl}+\langle u_iu_k''\rangle[\boldsymbol{r},\boldsymbol{r}'']_{jl}+\langle u_iu_l'''\rangle[\boldsymbol{r},\boldsymbol{r}']_{jk}$,

(3) $$\frac{\partial}{\partial r_m}\langle u_iu_j'\rangle\left(\frac{\partial}{\partial r_m'}+\frac{\partial}{\partial r_m''}\right)\langle u_k''u_l'''\rangle+\frac{\partial}{\partial r_m'}\langle u_iu_k''\rangle\left(\frac{\partial}{\partial r_m}+\frac{\partial}{\partial r_m''}\right)\langle u_j'u_l'''\rangle$$
$$+\frac{\partial}{\partial r_m''}\langle u_iu_l'''\rangle\left(\frac{\partial}{\partial r_m}+\frac{\partial}{\partial r_m'}\right)\langle u_j'u_k''\rangle,$$

where [ ] represents a certain combination of the spatial and the temporal derivatives; for instance,

$$[\boldsymbol{r}]_{ij}=\frac{\partial}{\partial r_m}\langle u_i u_m u_j'\rangle-\frac{\partial}{\partial r_m}\langle u_i u_j' u_m'\rangle+\frac{\partial}{\partial r_i}\langle P u_j'\rangle-\frac{\partial}{\partial r_j}\langle u_i P'\rangle$$

$$+2\nu\frac{\partial^2}{\partial r_m \partial r_m}\langle u_i u_j'\rangle-\frac{\partial}{\partial t}\langle u_i u_j'\rangle=0, \qquad \text{from (A1)}.$$

Similarly, we have

$$[\boldsymbol{r}']_{ik}=[\boldsymbol{r}'']_{il}=0.$$

Another example is

$$[\boldsymbol{r}',\boldsymbol{r}'']_{kl}=-\frac{\partial}{\partial r_m'}\langle u_k'' u_m'' u_l'''\rangle-\frac{\partial}{\partial r_m''}\langle u_k'' u_l''' u_m'''\rangle-\frac{\partial}{\partial r_k'}\langle P'' u_l'''\rangle$$

$$-\frac{\partial}{\partial r_l''}\langle u_k'' P'''\rangle+2\nu\left(\frac{\partial^2}{\partial r_m' \partial r_m'}+\frac{\partial^2}{\partial r_m' \partial r_m''}+\frac{\partial^2}{\partial r_m'' \partial r_m''}\right)\langle u_k'' u_l'''\rangle$$

$$-\frac{\partial}{\partial t}\langle u_k'' u_l'''\rangle. \tag{B2}$$

Now, putting $\boldsymbol{r}''-\boldsymbol{r}'=\boldsymbol{s}$, we have

$$\frac{\partial}{\partial r_i'}=-\frac{\partial}{\partial s_i}, \qquad \frac{\partial}{\partial r_i''}=\frac{\partial}{\partial s_i} \tag{B3}$$

for operations on functions of $\boldsymbol{s}$. Thus (B2) reduces to

$$[\boldsymbol{r}',\boldsymbol{r}'']_{kl}=\frac{\partial}{\partial t}\langle u_k'' u_l'''\rangle-\frac{\partial}{\partial t}\langle u_k'' u_l'''\rangle=0,$$

with the aid of (A1), and similarly we have

$$[\boldsymbol{r},\boldsymbol{r}'']_{jl}=[\boldsymbol{r},\boldsymbol{r}']_{jk}=0.$$

Finally, by virtue of (B3) and the like, it is readily found that

$$\left(\frac{\partial}{\partial r_m'}+\frac{\partial}{\partial r_m''}\right)\langle u_k'' u_l'''\rangle=\left(\frac{\partial}{\partial r_m}+\frac{\partial}{\partial r_m''}\right)\langle u_j' u_l'''\rangle=\left(\frac{\partial}{\partial r_m}+\frac{\partial}{\partial r_m'}\right)\langle u_j' u_k''\rangle=0,$$

from which we obtain $\alpha_{ijkl}=0$, and hence Eq. (2.7).

References

[1] Batchelor, G. K., *The theory of homogeneous turbulence.* Cambridge University Press (1953).

[2] Batchelor, G. K. and Proudman, I., The large-scale structure of homogeneous turbulence. *Phil. Trans.* **A 248** (1956) 369.

[3] Deissler, R. G., On the decay of homogeneous turbulence before the final period. *Phys. Fluids* **1** (1958) 111.

[4] Deissler, R. G., A theory of decaying homogeneous turbulence. *Phys. Fluids* **3** (1960) 176.

[5] Ohji, M., A note on the hypothesis of zero fourth-order cumulant in homogeneous turbulence. *Rep. Res. Inst. Appl. Mech.* **9** (1961) 163.

[6] Ohji, M., On four-point dynamical equations in homogeneous turbulence. *J. Phys. Soc.*

Japan **17** (1962) 1321.

[7] Proudman, I. and Reid, W. H., On the decay of a normally distributed and homogeneous turbulent velocity field. *Phil. Trans.* **A 247**(1954) 163.

[8] Tatsumi, T., The theory of decay process of incompressible isotropic turbulence. *Proc. Roy. Soc.* **A 239** (1957) 16.

[9] Tatsumi, T., The energy spectrum of incompressible isotropic turbulence. *Proc. IX. Int. Congr. Appl. Mech.* **3** (1957) 396.

(Received Oct. 12, 1962)