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NOTE

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1. For the separating laminar boundary layers in two dimensions, by making use of the approximate procedure due to VON KÁRMÁN and POHLHAUSEN, a similar solution may be obtained in the forms

$$\frac{\delta}{\delta_0}=\left(1+120\frac{\nu}{U_0\delta_0}\frac{x}{\delta_0}\right)^{0.55}, \tag{1.1}$$

and

$$\frac{U}{U_0}=\left(1+120\frac{\nu}{U_0\delta_0}\frac{x}{\delta_0}\right)^{-0.1}, \tag{1.2}$$

where δ is the thickness of the laminar boundary layer and U denotes the velocity of the fluid just outside the boundary layer. x is the distance measured along the surface of the body; δ_0 and U_0 stand for the respective values of δ and U at the station $x=0$. Finally ν denotes the kinematic viscosity of the fluid as usual.[1)]

2. In order to construct a corresponding similar solution for the separating turbulent boundary layers in two dimensions, let us start from the momentum equation of the flow, just as we did in the laminar case, which may be written down in the form

$$\frac{\tau_0}{\rho U^2}=\frac{d\theta}{dx}+(2+H)\frac{\theta}{U}\frac{dU}{dx}, \tag{2.1}$$

where τ_0 is the surface traction, θ denotes the momentum thickness of the boundary layer, H defined by the ratio of the displacement thickness to the momentum thickness is the non-dimensional parameter characterizing the velocity profiles within the boundary layer,[2)] and ρ means the density of the fluid.

Now, since we are interested solely in the *separating* boundary layers, τ_0 on the left-hand side of the equation may be put equal to zero, and moreover from the condition of similarity H should be assumed to remain in the constant value, H_s say, corresponding to the separating velocity profile.

Accordingly we have

$$\frac{d\theta}{\theta}+(2+H_s)\frac{dU}{U}=0, \tag{2.2}$$

1) DURAND, W. F. (ed.), *Aerodynamic Theory*, Vol. III (1935), Division G, *The Mechanics of Viscous Fluids* (by PRANDTL, L.) Section 18, p. 112 *et seq.*, especially equations (18.1) and (18.3). The numeral within the parentheses was slightly modified from the original text, assuming $\lambda=-12$ (instead of -10, λ being the POHLHAUSEN's parameter defined by $\delta^2 dU/\nu dx$) throughout the boundary layer; cf. p. 115, *op. cit.*

2) VON DOENHOFF, A. E. and TETERVIN, N., Determination of General Relations for the Behavior of Turbulent Boundary Layers, *NACA Technical Report* No. 772 (1943), especially p. 11.

and by integration we can arrive at the result

$$\frac{\theta}{\theta_0}=\left(\frac{U_0}{U}\right)^{2+Hs}. \tag{2.3}$$

where use is made of the initial conditions

$$U=U_0,\ \ \theta=\theta_0 \quad \text{at } x=0. \tag{2.4}$$

On the other hand, from various experimental data concerning the turbulent boundary layers, GARNER induced the following relationship among θ, H, dU/dx, and dH/dx,[1] namely

$$\theta\frac{dH}{dx}=e^{5(H-1.4)}\left\{-\frac{\theta}{U}\frac{dU}{dx}-\frac{0.0135(H-1.4)}{R_\theta^{1/6}}\right\}, \tag{2.5}$$

where

$$R_\theta=U\theta/\nu \tag{2.6}$$

is the so-called Reynolds number of the boundary layer. Since in our case, however,

$$H=H_s=\text{constant}, \quad \text{and} \quad dH/dx=0, \tag{2.7}$$

we should have simply

$$\frac{\theta}{U}\frac{dU}{dx}=-0.0135(H_s-1.4)\left(\frac{U\theta}{\nu}\right)^{-1/6}. \tag{2.8}$$

If (2.3) is substituted into the above expression, the latter can be readily integrated from ($x=0$, $U=U_0$) to (x, U), yielding

$$U/U_0=(1+\xi)^{-6/(13+7Hs)}, \tag{2.9}$$

where for shortness we write

$$\xi=\frac{0.0135}{6}(13+7H_s)(H_s-1.4)\left(\frac{U_0\theta_0}{\nu}\right)^{-1/6}\frac{x}{\theta_0}. \tag{2.10}$$

Further from (2.3)

$$\theta/\theta_0=(1+\xi)^{6(2+Hs)/(13+7Hs)}. \tag{2.11}$$

(2.9) and (2.11) correspond to (1.2) and (1.1) respectively in the laminar flow.

Finally there still remains a question regarding H_s, which is the critical value of H for the separation. According to VON DOENHOFF and TETERVIN, "Separation has not been observed for a value of H less than 1.8 and appears definitely to have occurred for a value of H of 2.6," (*loc. cit.* p. 11) and they adopted the criterion

$$H_s=2.6 \tag{2.12}$$

for their calculations. GARNER, on the other hand, states: "At a certain stage H rises very sharply; this is taken as an indication of the onset of turbulent separation, and occurs when H is about 2." (*loc. cit.* p. 118.)*

The values of $6/(13+7H_s)$, $6(2+H_s)/(13+7H_s)$, and $0.0135(13+7H_s)(H_s-1.4)/6$ were calculated for various values of H_s ranging from 1.8 to 2.6 and are reproduced in TABLE 1. In TABLE 2, assuming tentatively $H_s=2.6$, the behaviors of

$$U/U_0=(1+\xi)^{-0.1923}, \tag{2.13}$$

and

$$\theta/\theta_0=(1+\xi)^{0.885} \tag{2.14}$$

are shown for various values of ξ.

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1) GARNER, H. C., The Development of Turbulent Boundary Layers, *Reports and Remoranda*, No. 2133 (1944), equation (7).

TABLE 1.

H_s	$\frac{6}{13+7H_s}$	$\frac{6(2+H_s)}{13+7H_s}$	$h(H_s)$
1.8	0.234	0.891	0.0230
1.9	228	890	0296
2.0	222	889	0365
2.1	217	888	0436
2.2	0.211	0.887	0.0511
2.3	206	887	0589
2.4	201	886	0671
2.5	1967	885	0755
2.6	**0.1923**	**0.885**	**0.0842**

$$h(H_s)=\frac{0.0135}{6}(13+7H_s)(H_s-1.4)$$

$$\xi=h(H_s)\left(\frac{\theta_0 U_0}{\nu}\right)^{-1/6}\frac{x}{\theta_0}$$

* Note added in proof:

In the experiment of SCHUBAUER and KLEBANOFF, H was found to have the value 2.7 at the separation point; cf. SCHUBAUER, G. B. and KLEBANOFF, P. S., Investigation of Separation of the Turbulent Boundary Layer, *NACA Technical Report* No. 1030 (1951), especially p. 7.

TABLE 2.

ξ	$(1+\xi)^{-0.1923}$	$(1+\xi)^{0.885}$
0	1.000	1.000
0.1	0.982	088
0.2	966	175
0.3	951	261
0.4	937	347
0.5	0.925	1.431
0.6	914	516
0.7	903	599
0.8	893	682
0.9	884	764
1.0	0.875	1.846
1.1	867	928
1.2	859	2.01
1.3	852	09
1.4	845	17
1.5	0.838	2.25
1.6	832	33
1.7	826	41
1.8	820	49
1.9	815	56
2.0	0.810	2.64
2.2	800	80
2.4	790	95
2.6	782	3.11
2.8	774	26
3.0	0.766	3.41
3.2	759	56
3.4	752	71
3.6	746	86
3.8	740	4.01
4.0	0.734	4.15
4.2	728	30
4.4	723	45
4.6	718	59
4.8	713	74

ξ	$(1+\xi)^{-0.1923}$	$(1+\xi)^{0.885}$
5.0	0.709	4.88
5.2	704	5.02
5.4	700	17
5.6	696	31
5.8	692	45
6.0	0.688	5.59
6.5	679	94
7.0	670	6.29
7.5	663	64
8.0	0.655	6.98
8.5	649	7.33
9.0	642	7.67
9.5	636	8.01
10	0.631	8.34
11	620	9.01
12	611	9.67
13	602	10.32
14	594	10.97
15	0.587	11.62
16	580	12.26
17	574	12.90
18	568	13.53
19	562	14.16
20	0.557	14.78
25	534	17.85
30	517	20.9
35	502	23.8
40	0.490	26.7
45	479	29.6
50	469	32.4
55	461	35.2
60	0.454	38.0
65	447	40.7
70	441	43.4
75	435	46.1

ξ	$(1+\xi)^{-0.1923}$	$(1+\xi)^{0.885}$
80	0.430	48.8
85	425	51.4
90	420	54.1
95	416	56.7
100	0.412	59.3
110	404	64.5
120	398	69.6
130	392	74.6
140	386	79.7
150	0.381	84.6
160	376	89.6
170	372	94.5
180	368	99.4
190	364	104.2
200	0.361	109.0
220	354	118.5
240	348	128.0
260	343	137.3
280	338	146.6
300	0.334	155.8
350	324	178.5
400	316	201
450	309	223
500	303	245