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STABILITY OF LAMINAR SHEAR LAYER BETWEEN PARALLEL UNIFORM STREAMS (I)

By Hikoji YAMADA*

Résumé First paper on the stability of a shear layer between two parallel uniform streams, whose velocity distribution is erf-function. In the former half inviscid fluid is discussed by means of polynomial approximation, dividing the half domain of existence $0 \leq y < \infty$ into three sub-domains. Instability is exhibited only by antisymmetrical oscillations, the wave number of neutral stability being a little larger than 1.

In the latter half we concern with cases of finite Reynolds number and employ the method of expansion by means of Hermite functions. To see the pertinence of our method inviscid flow is treated again by this procedure, results according well with the former ones. Stability curve given by our method seems satisfactory except when the Reynolds number is small and shall be the subject of next paper.

Stability of a laminar shear layer between two parallel uniform streams is a well-known problem and several efforts have been devoted for solution of this difficult problem. Among these efforts those of Helmholtz, Rayleigh, and Rosenfeld are well-known and they dealt with a separation surface between two inviscid liquid flows. Recent investigations of viscous flow on the basis of the Orr-Sommerfeld's stability equation are those of Lessen(1), Carrier(2), Esch(3), and of Tatsumi-Gotoh(4). By these works the region of large Reynolds number and that of extremely small one have become clear, but the region between them which has an intermediate Reynolds number remains obscure. In Esch's paper whole region has really been dealt with, but haply by reason of the singular velocity distribution he adopted his stability curve presents a few aspects which are alluded by Tatsumi-Gotoh and, we think, require further discussions. We then have intended to study this problem and alike in some details, this report being the first one. In this and a following

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(1) M. Lessen : On stability of free laminar boundary layer between parallel streams, NACA Report no. 979 (1950).

(2) G. Carrier : Interface stability of the Helmholtz type, Los Alamos Internal Report (1954).

(3) R. E. Esch : The instability of a shear layer between two parallel streams, J. Fluid Mech., vol. 3, pp. 289-303 (1957).

(4) T. Tatsumi and K. Gotoh : The stability of free boundary layers between two uniform streams, J. Fluid Mech., vol. 7, pp. 433–441 (1960).

we discuss stability of a more natural shear layer, assuming the erf-function distribution of mean flow velocity :

$$w(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-\xi^2} d\xi \qquad (1)$$

which is the one used by Carrier in his inviscid case (a fact we read in the Esch's paper). Not only we may assume sufficient approximation to a real shear layer, but also (1) is free of any singular point from the mathematical point of view.

§ 1. **Stability of inviscid flow (1)**——As our mean flow $w(y)$ is parallel we adopt the coordinate-system, in which x-axis is along and y-axis is across the flow. The stream function ψ of disturbance, which is superimposed on mean flow, is assumed of the form:

$$\psi = \varphi(y) e^{i\alpha(x-ct)} \qquad (2)$$

and the determination of amplitude $\varphi(y)$ is to be done by means of the Orr-Sommerfeld's equation:

$$L(\varphi) \equiv \frac{1}{i\alpha R}(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi) - (w - c)(\varphi'' - \alpha^2\varphi) - w''\varphi = 0. \quad (3)$$

All the quantities are reduced in non-dimensional forms by the characteristic length L and characteristic velocity U; R is the Reynolds number UL/ν, ν the kinematic viscosity of the liquid; dash denotes differentiation by y. At infinity $(y = \pm\infty)$ φ and φ' have to vanish.

As is well known, this problem has to be unterstood as a characteristic value problem for c and φ, being α and R two given real positive constants. Usually c has a complex value $c = c_r + ic_i$ and according as c_i is positive, zero, or negative the disturbance increases, remains constant, or decreases. The middle case i.e. the case of neutral stability divides stability from instability and the condition $c_i = 0$ imposes a functional relation between α and R, which is called stability curve.

Now turning to our problem we determine, at first, the wave number α of neutral stability $(c_i = 0)$ of inviscid flow $(R = \infty)$. This number has been calculated by G. Carrier and value a little larger than 1 reported (so we read in Esch's paper). His paper being, however, beyond our reach we calculate this case again as follows.

In this case stability equation (3) reduces to

$$(w - c)(\varphi'' - \alpha^2\varphi) - w''\varphi = 0, \qquad (4)$$

and by means of the well-known theorem of Tollmien characteristic value c has to be equal to the velocity of flow at a point y_c, where w'' vanishes. In our case as w'' vanishes at the origin $(y = 0)$ we have $c = 0$, and then (4) rewrites itself into

$$\varphi'' - \{\alpha^2 + F(y)\}\varphi = 0, \qquad (5)$$

where

$$F(y) = \frac{w''}{w} = -\frac{4}{\sqrt{\pi}} y e^{-y^2} \Big/ \frac{2}{\sqrt{\pi}} \int_0^y e^{-\xi^2} d\xi \qquad (6)$$

by (1).

In solving the characteristic problem (4) we note that the equation admits even and odd functions. We are then to manipulate only with the half infinite region $y \geq 0$ with the boundary condition at $y = 0$:

for even $\varphi_s(y)$ $\qquad \varphi_s'(0) = 0,$ $\qquad$ (7)

for odd $\varphi_a(y)$ $\qquad \varphi_a(0) = 0.$ $\qquad$ (7')

We divide then the region $0 \leq y < \infty$ into three subregions I($0 \leq y < 1.6$), II ($1.6 \leq y < 3.2$), and III ($3.2 \leq y < \infty$) ; in I and II we approximate $F(y)$ by polynomials of 6 degrees, and in III by identical zero, polynomials being determined by the values of F at five points of equal intervals and the values of $F'(y)$ at both end-points ($y = 0$, 1.6 and $y = 1.6$, 3.2). In Table 1 compared are the approximate values of $F(y)$ with the exact ones, in which we see sufficient accordance.

Table 1.

	y	$w(y)$	$\left(\frac{w''}{w}\right)_{\text{exact}}$	$\left(\frac{w''}{w}\right)_{\text{approx.}}$	$\left(\frac{w''}{w}\right)'_{\text{approx.}}$
I	0.0	0.0000	−2.0000	−2.0000	0.0000
	0.2	0.2227	−1.9473	−1.9475	
	0.4	0 4284	−1.7956	−1.7956	
	0.6	0.6039	−1.5644	−1.5642	
	0.8	0.7421	−1.2829	−1.2829	
	1.0	0.8427	−0.9852	−0.9855	
	1.2	0.9103	−0.7049	−0.7049	
	1.4	0.9523	−0.4673	−0.4670	
II	1.6	0.9764	−0.2859	−0.2859	0.7617
	1.8	0.9891	−0.1608	−0.1610	
	2.0	0.9953	−0.0831	−0.0831	
	2.2	0.9981	−0.0393	−0.0393	
	2.4	0.9993	−0.0171	−0.0171	
	2.6	0.9998	−0.0068	−0.0068	
	2 8	0.9999	−0.0025	−0.0023	
	3.0	1.0000	−0.0008	−0.0002	
III	3.2	〃	−0.0003	0.0000	0.0000
	3.4	〃	−0.0001	〃	
	3.6	〃	−0.0000	〃	
	∞	〃	〃	〃	

Using these expressions of $F(y)$ into (5), and changing y-variable into η:

$$\eta = \frac{y}{1.6} \text{ in I }, \qquad \eta = \frac{y-1.6}{1.6} \text{ in II},$$

we have

$$\text{I}: \frac{d^2\varphi_1}{d\eta^2} - f_1(\eta)\varphi_1 = 0, \qquad f_1(\eta) = \lambda_1 + \sum_{n=1}^{6} \beta_n^{(1)} \eta^n, \qquad (8)$$

and

$$\text{II}: \frac{d^2\varphi_2}{d\eta^2} - f_2(\eta)\,\varphi_2 = 0, \qquad f_2(\eta) = \lambda_2 + \sum_{n=1}^{6} \beta_n^{(2)}\eta^n, \tag{8'}$$

where φ_1, φ_2 are amplitude functions in the regions I and II, and the constants λ's and β's are as follows:

$\lambda_1 \doteqdot 2.56\,(\alpha^2 - 2.0000)$; $\beta_1^{(1)} = 0.0000$, $\beta_2^{(1)} = 8.6088$,
$\beta_3^{(1)} = 1.1346$, $\beta_4^{(1)} = -9.3330$, $\beta_5^{(1)} = 4.0378$, $\beta_6^{(1)} = -0.0601$.
$\lambda_2 = 2.56(\alpha^2 - 0.2859)$; $\beta_1^{(2)} = 3.1193$, $\beta_2^{(2)} = -4.7618$,
$\beta_3^{(2)} = 1.8528$, $\beta_4^{(2)} = 2.7749$, $\beta_5^{(2)} = -3.2666$, $\beta_6^{(2)} = 1.0132$.

As we know from (8) and (8′) that φ_1 and φ_2 are intergral functions of η, we assume power series expansions:

$$\varphi_j = \sum_{n=0} a_n^{(j)}\eta^n \qquad (j=1, \text{ or } 2), \tag{9}$$

and insert them in each equation, obtaining

$$a_{n+2}^{(j)} = \frac{1}{(n+1)(n+2)} \sum_{m=0}^{n} a_{n-m}^{(j)}\, \beta_m^{(j)} \qquad (j = 1, \text{ or } 2), \tag{10}$$

which determine the expansion coefficients a's. The first two of them, i.e. ($a_0^{(j)}$, $a_1^{(j)}$), which remain arbitrary, are fixed once to (1, 0) and the other time to (0, 1), the resulting functions being denoted by $\varphi_j^{(s)}$ and $\varphi_j^{(a)}$ respectively. $a_n^{(j)}$'s are determined up to $n = 20$, and each of them, being polynomials of λ_j, up to λ_j^5 (see below).

In the region III y is replaced by $\eta = (y - 3.2)/1.6$ and then (5) reduces to

$$\frac{d^2\varphi_3}{d\eta^2} - 2.56\,\alpha^2\varphi_3 = 0; \tag{11}$$

solution of this, which fits to our condition at infinity, is evidently

$$\varphi_3 = e^{-1.6\alpha\eta}. \tag{12}$$

Now we have to join together the solutions of three regions. At first we take up the symmetrical solution $\varphi_s(y)$, which is equal to $A\varphi_1^{(s)}$ in I. In II and III it can be expressed as $B\varphi_2^{(s)} + C\varphi_2^{(a)}$ and $D\varphi_3$ respectively, and analytical continuation of them requires, as is well known, that the functions and their first derivatives should have the same values at the joining points, i.e. $y = 1.6$ and 3.2, and thus we have

$$\left.\begin{aligned} A\varphi_1^{(s)}(1) - \{B\varphi_2^{(s)}(0) + C\varphi_2^{(a)}(0)\} &= 0,\\ A\varphi_1^{(s)\prime}(1) - \{B\varphi_2^{(s)\prime}(0) + C\varphi_2^{(a)\prime}(0)\} &= 0,\\ -\{B\varphi_2^{(s)}(1) + C\varphi_2^{(a)}(1)\} + D\varphi_3(0) &= 0,\\ -\{B\varphi_2^{(s)\prime}(1) + C\varphi_2^{(a)\prime}(1)\} + D\varphi_3'(0) &= 0, \end{aligned}\right\} \tag{13}$$

where dash denotes differentiation with regard to η.

The condition of solvability of this system of linear homogeneous equations regarding to A, B, C, D is

$$D^{(s)}(\alpha)\equiv\begin{vmatrix}\varphi_1^{(s)}(1) & 1 & 0 & 0\\ \varphi_1^{(s)\prime}(1) & 0 & 1 & 0\\ 0 & \varphi_2^{(s)}(1) & \varphi_2^{(a)}(1) & 1\\ 0 & \varphi_2^{(s)\prime}(1) & \varphi_2^{(a)\prime}(1) & -1.6\alpha\end{vmatrix}=0, \qquad (14)$$

which develops easily into

$$D^{(s)}(\alpha)=\varphi_1^{(s)}(1)\{1.6\alpha\varphi_2^{(s)}(1)+\varphi_2^{(s)\prime}(1)\}$$
$$+\varphi_1^{(s)\prime}(1)\{1.6\alpha\varphi_2^{(a)}(1)+\varphi_2^{(a)\prime}(1)\}=0, \qquad (14')$$

and is nothing but the characteristic equation for the determination of α, through λ_1 and λ_2. The antisymmetric solution $\varphi_a(y)$ is dealt with in the same way and the characteristic equation reads:

$$D^{(a)}(\alpha)=\varphi_1^{(a)}(1)\{1.6\alpha\varphi_2^{(s)}(1)+\varphi_2^{(s)\prime}(1)\}$$
$$+\varphi_1^{(a)\prime}(1)\{1.6\alpha\varphi_2^{(a)}(1)+\varphi_2^{(a)\prime}(1)\}=0. \qquad (15)$$

To solve (14′) and (15) we notice at first that the roots α of these equations are confined within $(0, \sqrt{2})$, for if α^2 is not less than 2 α^2+F is everywhere non-negative and original equation (5) indicates that φ does not vanish at infinity, whose vanishing being one of our boundary conditions. When $\alpha^2<2$ the magnitudes of λ_1 and λ_2 do not exceed 5.12 by definition and power series defining a_n's can be cut off at a proper exponent. If we permit inaccuracies in the 4th decimal places of each numerical value of $\varphi_i(1)$, and accordingly in the 3rd decimal places of $\varphi_i'(1)$, a_n's have to be taken into up to $n=20$, and every a_n up to λ^5, thus each φ in $D(\alpha)$ being polynomials of λ_1 or λ_2 of fifth degree.

Using these expressions $D(\alpha)$ can easily be calculated for each value of α, arbitrarily given, and we know that $D^{(s)}(\alpha)=0$ has only one root α_0 in the neighborhood of $\alpha=1.00$, and $D^{(a)}(\alpha)=0$ has no root except $\alpha=0$. By trial and

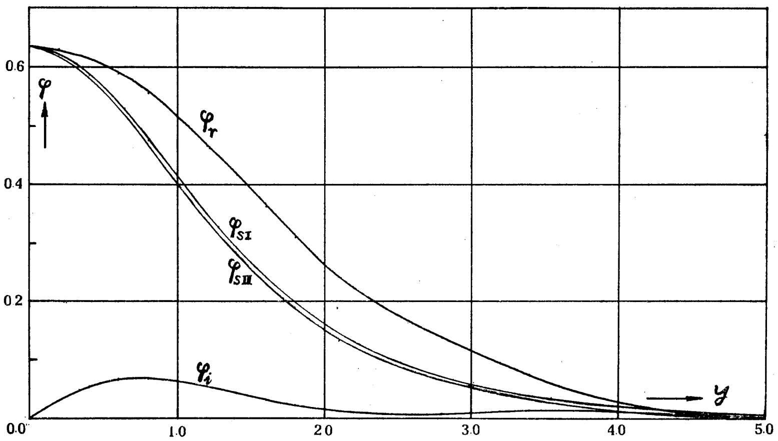

Fig. 1. Amplitude functions of neutral stability; φ_{sI} and φ_{sII} are inviscid, $\varphi_r+i\varphi_i$ is for $\alpha R=11.50$, $\alpha=0.800$.

error α_0 is fixed to 1.035. We know then that antisymmetrical disturbance (i.e. φ_s) is instable when $0 \leq \alpha < \alpha_0$ and stable when $\alpha_0 < \alpha$, and symmetrical disturbance (i.e. φ_a) is always stable.

The amplitude of neutral stability oscillation $\varphi_s(y, \alpha_0)$ is shown in Fig. 1, in an arbitrary scale, denoting it by $\varphi_{s\text{I}}$. To obtain this function we determine B, C and D (assuming $A=1$ say) by (13), the coefficients φ_i's being fixed by use of the values $\alpha = 1.035$, $\lambda_1 = -2.378$ and $\lambda_2 = 2.010$. With B, C and D thus determined $\varphi_s(y, \alpha_0)$ is expressed by

$\varphi_1^{(s)}(\eta)$ in I,

$B\varphi_2^{(s)}(\eta) + C\varphi_2^{(a)}(\eta) \equiv \varphi_2(\eta)$ in II,

$2.018e^{-1.6\alpha\eta}$ in III.

Of course φ_i's in these expressions are functions defined in (9) and now polynomials of η of degree 20. Their coefficients are fixed with values of λ_1 and λ_2 above given, and shown in Table 2.

Table 2. $\varphi_{s\text{I}}$

	$\varphi_1^{(s)}$	φ_2
a_0	1.0000	0.3788
a_1	0.0000	−0.5932
a_2	−1.1883	0.3810
a_3	0.0000	−0.0019
a_4	0.9527	−0.2407
a_5	0.0567	0.2356
a_6	−0.7275	−0.0784
a_7	0.0608	−0.0582
a_8	0.3743	0.0911
a_9	−0.0469	−0.0525
a_{10}	−0.1768	0.0060
a_{11}	0.0284	0.0159
a_{12}	0.0809	−0.0146
a_{13}	−0.0228	0.0051
a_{14}	−0.0273	0.0014
a_{15}	0.0097	−0.0029
a_{16}	0.0093	0.0017
a_{17}	−0.0041	−0.0003
a_{18}	−0.0030	−0.0003
a_{19}	0.0017	0.0003
a_{20}	0.0007	−0.0001

§ 2. Method of solution in general case— Solution φ of (3) i.e. $L(\varphi)=0$ is in general a linear combination of four fundamental solutions, the two of them being so-called inviscid integrals, the other two being viscous integrals. Now the existing region of oscillation being an entire space $(-\infty < y < \infty)$, the latter two, however, drop by reason of the boundary condition at infinity $(y = \pm\infty)$; the remaining two are slowly varying functions and vanish exponentially at infinity ($\varphi \sim \exp(\pm \alpha y)$). Then φ is quadratic integrable and can be approximated arbitrarily accurately in the mean by a linear combination of functions which belong to a complete set of functions.

We take as such a set of functions the Hermite functions[5] :

$$\psi_n(y) = K_n H_n(y) e^{-y^2/2}, \quad K_n = (2^n n! \sqrt{\pi})^{-1/2}, \qquad n = 0, 1, 2, \cdots, \tag{16}$$

which are not only complete but orthogonal:

$$\int_{-\infty}^{\infty} \psi_m(y) \psi_n(y)\, dy = \delta_{mn}, \tag{17}$$

and approximate $\varphi(y)$ by a linear combination of them:

$$\varphi(y) = \sum_{n=0} a_n \psi_n(y), \tag{18}$$

(5) Cf. Courant-Hilbert : Methoden der mathematischen Physik, vol. 1 (2nd ed.), chap. 2 § 9, and Jahnke-Emde-Lösch: Tafeln höherer Funktionen (Teubner 1960).

the number of terms being appropriately chosen. As φ vanishes more slowly at large y according as α becomes smaller, more terms of the series should have been taken in for an approximation required. Here we use (18) with the number of terms fixed, and cases of small α will be discussed separately in another way in a subsequent paper.

Several formulas concerning to $\psi_n(y)$ are, by the difinition (16), derived from those of so-called Hermite polynomials $H_n(y)$:

$$H_n(y) = (-)^n e^{y^2}\left(\frac{d}{dy}\right)^n e^{-y^2} \quad (n\geq 0), \tag{19}$$

$$\frac{d}{dy} H_n(y) = 2n H_{n-1}(y) \qquad (n\geq 0), \tag{19'}$$

$$H_{n+1}(y) - 2yH_n(y) + 2n H_{n-1}(y) = 0 \quad (n\geq 0), \tag{19''}$$

where H_{-1} is identically zero. From these we have following two relations:

$$\sqrt{2(n+1)}\ \psi_{n+1} - 2y\psi_n + \sqrt{2n}\psi_{n-1} = 0 \quad (n\geq 0), \tag{20}$$

$$\frac{d}{dy}\psi_n = \sqrt{\frac{n}{2}}\ \psi_{n-1} - \sqrt{\frac{n+1}{2}}\ \psi_{n+1} \qquad (n\geq 2), \tag{20'}$$

which are fundamental, ψ_{-1} being identically zero. By the former we calculate numerical values of Hermite functions successively and by the latter we express the derivatives of a Hermite function by the functions themselves. Especially by repeated use of it we have

$$\psi_n'' = \frac{\sqrt{(n-1)n}}{2}\psi_{n-1} - \frac{2n+1}{2}\psi_n + \frac{\sqrt{(n+1)(n+2)}}{2}\psi_{n+2}, \tag{21}$$

$$\begin{aligned}\psi_n'''' = & \frac{\sqrt{(n-3)(n-2)(n-1)n}}{4}\psi_{n-4} - \frac{\sqrt{(n-1)n}(4n-2)}{4}\psi_{n-2} \\ & + \frac{6n^2+6n+3}{4}\psi_n - \frac{\sqrt{n(n+1)}\,(4n+6)}{4}\psi_{n+2} \\ & + \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{4}\psi_{n+4},\end{aligned} \tag{21'}$$

which we will require immediately, functions with negative suffixes being identically zero. Evidently ψ_n is even or odd function of y according as n is even or odd integer.

Each term of (3) i.e. $L(\varphi)$ vanishes exponentially when y tends to infinity and then $L(\varphi)$ is expressible by means of ψ_n, i.e.

$$L(\varphi) = \sum_{m=0}^{\infty} \beta_m \psi_m(y), \tag{22}$$

and (3) is equivalent to

$$\beta_m = 0, \quad m = 0, 1, 2, \cdots, \tag{22'}$$

where β_m's are:

$$\beta_m = \int_{-\infty}^{\infty} \psi_m \left(\begin{array}{l} \frac{1}{i\alpha R}(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi) \\ -\{(w-c)(\varphi''-\alpha^2\varphi) - w''\varphi\} \end{array} \right) dy.$$

When we integrate every term of this expression by parts several times the derivatives of φ in the integrand are reduced to φ itself, and afterwards we have to use the expansion (18); in this way we know that the degree of approximation of our calculation depends on the accuracy of the expansion (18), and not on those of its derivatives. The result is the same as direct insertion of (18) and termwise integration, such that

$$\beta_m = \sum_{n=0} a_n \left(\frac{1}{i\rho} A_{mn} - B_{mn} + \gamma C_{mn} \right) , \tag{23}$$

where

$$\left. \begin{array}{l} A_{mn} = A_{nm} = \int_{-\infty}^{\infty} \psi_m(\psi_n'''' - 2\alpha^2\psi_n'' + \alpha^4\psi_n)dy, \\ C_{mn} = C_{nm} = \int_{-\infty}^{\infty} \psi_m(\psi_n'' - \alpha^2\psi_n)dy, \end{array} \right\} \tag{24}$$

and

$$B_{mn} = \int_{-\infty}^{\infty} \psi_m\{w_1(\psi_n'' - \alpha^2\psi_n) - w_1''\psi_n\}\, dy\, ; \tag{25}$$

the notations:

$$\left. \begin{array}{ll} \rho = \frac{2}{\sqrt{\pi}}\alpha R\, , & \gamma = \frac{\sqrt{\pi}}{2}\, c\ ; \\ w_1 = \frac{\sqrt{\pi}}{2}\, w\ = \int_0^y e^{-\xi^2}\, d\xi & \end{array} \right\} \tag{26}$$

being used.

Using (23) into (22′) we have a set of linear homogeneous equations for the expansion coefficients a_n's, and the condition for non-zero solution is the determinant equation:

$$\left| \frac{1}{i\rho} A_{mn} - B_{mn} + \gamma C_{mn} \right| = 0. \tag{27}$$

This equation determines numerical values of γ i.e. $c = c_r + ic_i$, for a given pair of values (α, ρ) i.e. (α, R). Especially for the case of neutral stability $(c_i=0)$ γ is real (unknown), and the equation separates into two real equations for ρ and γ, α being a given wave number. Stability curve $\alpha(R)$ and neutral wave velocity $c(\alpha)$ follow from them.

The constants A_{mn} and C_{mn} are easily calculated by the formulas (21) and

(21′), and all vanish except the followings:

$$\left.\begin{aligned} A_{n-4,n} &= A_{n,n-4} = \frac{\sqrt{(n-3)(n-2)(n-1)n}}{4}, \\ A_{n-2,n} &= A_{n,n-2} = -\sqrt{(n-1)n}\left(\frac{2n-1}{2} + \alpha^2\right), \\ A_{n,n} &= \frac{3}{4}(2n^2+2n+1) + (2n+1)\alpha^2+\alpha^4; \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} C_{n-2,n} &= C_{n,n-2} = \frac{\sqrt{(n-1)n}}{2}, \\ C_{n,n} &= -\frac{2n+1}{2} - \alpha^2; \end{aligned}\right\} \tag{28'}$$

the ones which have negative suffixes being zero.

Determination of the constants B_{mn} is a little lengthy. We replace ψ_n'' in (25) by (21) and use the notations:

$$I_{mn} = \int_{-\infty}^{\infty} \psi_m\, w_1\, \psi_n\, dy, \tag{29}$$

$$J_{mn} = \int_{-\infty}^{\infty} \psi_m\, w_1''\, \psi_n\, dy, \tag{29'}$$

then

$$B_{mn} = \frac{\sqrt{(n-1)n}}{2} I_{m,n-2} - \left(\frac{2n+1}{2} + \alpha^2\right) I_{m,n} + \frac{\sqrt{(n+1)(n+2)}}{2} I_{m,n+2} - J_{m,n}, \tag{30}$$

and we have to calculate I_{mn} and J_{mn}. For I_{mn} we rewrite the relations (19), (19′) for Hermite polynomials into those of Hermite functions:

$$\frac{d}{dy}\left((\psi_{m-1}e^{-y^2/2}\right) = -\sqrt{2m}\ \psi_m\, e^{-y^2/2} \qquad (m \geq 1), \tag{31}$$

$$\frac{d}{dy}\left(\psi_n\, e^{y^2/2}\right) = \sqrt{2n}\ \psi_{n-1}\, e^{y^2/2} \qquad (n \geq 0), \tag{31'}$$

and use in (29). Then we see easily that

$$\begin{aligned} I_{mn} &= -\frac{1}{\sqrt{2m}} \int_{-\infty}^{\infty} \frac{d}{dy}(\psi_{m-1}e^{-y^2/2})\, w_1\, \psi_n\, e^{y^2/2}\, dy \\ &= \frac{1}{\sqrt{2m}}[m-1, n] + \sqrt{\frac{n}{m}}\, I_{m-1,n-1} \qquad (m, n \geq 1), \end{aligned} \tag{32}$$

where another notation

$$[m, n] = \int_{-\infty}^{\infty} \psi_m e^{-y^2} \psi_n \, dy \tag{33}$$

being used. With this recurrence formula and relations:

$$I_{m,0} = \frac{1}{\sqrt{2m}} [m-1, 0], \qquad I_{0,0}=0, \tag{32'}$$

which can easily be proved, all I_{mn} are reduced to the integrals of the type $[m, n]$. The same is true for the integrals J_{mn}. We use the explicit expression of w'' i.e. $-2ye^{-y^2}$ in (29′) and rewrite $2y\varphi_m$ by means of (20), resulting into

$$J_{mn} = -\sqrt{2(m+1)}\ [m+1, n] - \sqrt{2m}\ [m-1, n]\ , \tag{34}$$

and especially

$$J_{0,0} = 0. \tag{34'}$$

We have thus been led to integrals of the type $[m, n]$, and these, in turn, are reducible to simpler ones, for by means of (19)

$$[m, n] = (-)^{m+n} K_m K_n \int_{-\infty}^{\infty} \left(\frac{d}{dy}\right)^m e^{-y^2} \left(\frac{d}{dy}\right)^n e^{-y^2} dy$$

$$= (-)^m K_m K_n \int_{-\infty}^{\infty} e^{-y^2} \left(\frac{d}{dy}\right)^{m+n} e^{-y^2} dy = (-)^n K_m K_n [m+n], \tag{35}$$

where $[m+n]$ is an integral of the type:

$$[m] = \int_{-\infty}^{\infty} e^{-2y^2} H_m(y)\ dy\ , \tag{36}$$

and the latter, having a simple recurrence formula:

$$[m] = -(m-1)\ [m-2]\ , \tag{36'}$$

which can easily be proved by use of (19″) and then (19′), we can evaluate starting from the special cases:

$$[1] = 0, \qquad [0] = \sqrt{\frac{\pi}{2}}\ . \tag{36''}$$

We have then completed the integrations. First the table of $[m]$, and then table of $[m, n]$ are constructed. With the latter I_{mn} and J_{mn} are easily calculated, and by use of these into (30) the required numerical values of B_{mn} are found, as we see in Table 3.

§ 3. **Stability of inviscid flow (2)**—— To see the pertinence of our method of integration and to obtain some indications for management of genaral cases of finite Reynolds number, we have engaged ourselves again to the inviscid flow, which had been discussed in section 1. In this case equation has been taken up in the form (4), i.e.

Table 3. B_{mn}

m \ n	0	1	2	3	4
0	—	$-0.37500-0.50000\alpha^2$	—	$0.73995+0.10206\alpha^2$	—
1	$0.62500-0.50000\alpha^2$	—	$-1.01646-0.53033\alpha^2$	—	$1.39699+0.12756\alpha^2$
2	—	$0.04420-0.53033\alpha^2$	—	$-1.45241-0.54127\alpha^2$	—
3	$0.12758+0.10206\alpha^2$	—	$-0.36988-0.54127\alpha^2$	—	$-1.85548-0.54688\alpha^2$
4	—	$0.63152+0.12758\alpha^2$	—	$-0.76174-0.54688\alpha^2$	—
5	$0.05419-0.03423\alpha^2$	—	$1.14875+0.14120\alpha^2$	—	$-1.14642-0.55028\alpha^2$
6	—	$-0.08208-0.04891\alpha^2$	—	$1.66988+0.14977\alpha^2$	—
7	$-0.04423+0.01321\alpha^2$	—	$-0.26264-0.05883\alpha^2$	—	$2.19280+0.15567\alpha^2$
8	—	$-0.00198+0.02101\alpha^2$	—	$-0.46457-0.06605\alpha^2$	—
9	$0.02703-0.00545\alpha^2$	—	$0.07455+0.02724\alpha^2$	—	$-0.67876-0.07155\alpha^2$
10	—	$0.01598-0.00947\alpha^2$	—	$0.17259+0.03232\alpha^2$	—

m \ n	5	6	7	8	9	10
0	$-0.28814-0.03423\alpha^2$	—	$0.14068+0.01321\alpha^2$	—	$-0.08330-0.00545\alpha^2$	—
1	—	$-0.57120-0.04891\alpha^2$	—	$0.29212+0.02101\alpha^2$	—	$-0.15840-0.00947\alpha^2$
2	$1.99599+0.14120\alpha^2$	—	$-0.85093-0.05883\alpha^2$	—	$0.45589+0.02724\alpha^2$	—
3	—	$2.56856+0.14977\alpha^2$	—	$-1.12504-0.06605\alpha^2$	—	$0.62510+0.03232\alpha^2$
4	$-2.24695-0.55028\alpha^2$	—	$3.12684+0.15567\alpha^2$	—	$-1.39426-0.07155\alpha^2$	—
5	—	$-2.63305-0.55256\alpha^2$	—	$3.67664+0.15999\alpha^2$	—	$-1.65962-0.07589\alpha^2$
6	$-1.52795-0.55256\alpha^2$	—	$-3.01615-0.55420\alpha^2$	—	$4.22092+0.16329\alpha^2$	—
7	—	$-1.90772-0.55420\alpha^2$	—	$-3.39747-0.55544\alpha^2$	—	$4.76148+0.16589\alpha^2$
8	$2.71675+0.15999\alpha^2$	—	$-2.28660-0.55544\alpha^2$	—	$-3.77754-0.55640\alpha^2$	—
9	—	$3.24121+0.16329\alpha^2$	—	$-2.66476-0.55640\alpha^2$	—	$-4.15681-0.55717\alpha^2$
10	$-0.90072-0.07589\alpha^2$	—	$3.76612+0.16589\alpha^2$	—	$-3.04248-0.55717\alpha^2$	—

$$L(\varphi) = w(\varphi'' - \alpha^2\varphi) - w''\varphi = 0 \qquad (37)$$

which continues directly to the case of finite Reynolds number (3), and not to (5). As the waves of neutral stability have zero wave velocity ($c=0$) in general, which we will show in next section, we have taken it in (37) beforehand.

As in section 1 φ is symmetrical or antisymmetrical with regard to x-axis, and then we have

$$\varphi_s(y) = a_0\psi_0 + a_2\psi_2 + \cdots + a_8\psi_8 + \cdots \qquad (38)$$

$$\varphi_a(y) = a_1\psi_1 + a_3\psi_3 + \cdots + a_9\psi_9 + \cdots \qquad (38')$$

in place of (18). Inserting these in (37) in turn, $L(\varphi)$ is antisymmetrical or symmetrical, and (37) is replaced by the orthogonality of $L(\varphi)$ to the set of functions $(\psi_1, \psi_3, \psi_5, \cdots)$ or to $(\psi_0, \psi_2, \psi_4, \cdots)$, i.e.

$$\sum_{n=0} B_{2m+1,2n}\, a_{2n} = 0 \quad (m=0, 1, 2, \cdots), \qquad (39)$$

or

$$\sum_{n=0} B_{2m,2n+1}\, a_{2n+1} = 0 \quad (m=0, 1, 2, \cdots), \qquad (39')$$

each being nothing but the condition (22′) and (23) ; evidently A_{mn} and C_{mn} do not appear in present case.

Eliminating a_{2n}'s from (39) and a_{2n+1}'s from (39′) we have two characteristic equations:

$$D^{(s)}(\alpha) \equiv \left| B_{2m+1,2n} \right| = 0, \qquad (40)$$

$$D^{(a)}(\alpha) \equiv \left| B_{2m,2n+1} \right| = 0, \qquad (40')$$

which by virtue of Table 4 can be developed into power series of α^2. We cut short the expansions (38) and (38′) to first five terms, and consequently D's in (40) and (40′) are determinants of fifth rank, which when developed give

$$D^{(s)}(\alpha) = -1.1733 - 0.6779\alpha^2 + 0.5903\alpha^4 + 0.6434\alpha^6 + 0.12991\alpha^8 + 0.007788\alpha^{10}, \qquad (41)$$

$$D^{(a)}(\alpha) = 1.3173 + 9.6941\alpha^2 + 10.0900\alpha^4 + 3.2640\alpha^6 + 0.3861\alpha^8 + 0.01446\alpha^{10}. \qquad (41')$$

(41) gives one and only one root $\alpha_0 = 1.067$ and (41′) no root; this character is the same as in section 1 and the wave number α_0 of neutral disturbance is identical within about 3 % error.

Making use of the value α_0 in (39) we obtain the coefficients a_{2n}/a_0 ($n = 1, 2, 3, 4$), and then by (38) the neutral oscillation amplitude $\varphi_s(y)$:

$$\varphi_s(y) = 1.3719\psi_0 + 0.1382\psi_2 + 0.1033\psi_4 + 0.0301\psi_6 + 0.0205\psi_8, \qquad (42)$$

a_0 being fixed so as $\varphi_s(0) = 1.0000$. This function, calculated by means of a table of Hermite functions, is drawn in Fig. 1 as φ_{sII}, scale being changed arbitrarily. We see good accordance with φ_{sI} of section 1 all along y-axis, which proves pertinence of our method of approximation.

Thus we have recognized that the expansion of φ into a series of Hermite functions is a convenient method of approximate calculation of our problem, and series with a few terms would be sufficient in the general case of not very small Reynolds number. We proceed then to this case.

§ 4. **Calculation of general case** —— As we see in (27), (28), and in Table 3, elements of the determinant (27) have such a special character that $(i\rho)^{-1}A_{mn} + \gamma C_{mn}$ and B_{mn} have non-vanishing elements alternately in rows, and also in columns, so that (27) is written in

$$\begin{vmatrix} \frac{1}{i\rho}A_{00}+\gamma C_{00}, & -B_{01}, & \frac{1}{i\rho}A_{02}+\gamma C_{02}, & -B_{03}, & \cdots\cdots \\ -B_{10}, & \frac{1}{i\rho}A_{11}+\gamma C_{11}, & -B_{12}, & \frac{1}{i\rho}A_{13}+\gamma C_{13}, & \cdots\cdots \\ \frac{1}{i\rho}A_{20}+\gamma C_{20}, & -B_{21}, & \frac{1}{i\rho}A_{22}+\gamma C_{22}, & -B_{23}, & \cdots\cdots \\ -B_{30}, & \frac{1}{i\rho}A_{31}+\gamma C_{31}, & -B_{32}, & \frac{1}{i\rho}A_{33}+\gamma C_{33}, & \cdots\cdots \\ \cdots\cdots & & & & \\ \cdots\cdots & & & & \end{vmatrix} = 0,$$

and then, if we multiply rows of even number by i, the imaginary unit, and also columns of odd number by i, the latter multiplication being equivalent to the introduction of new coefficients $\bar{a}_{2n+1} = -ia_{2n+1}$ in places of the coefficients of expansion of odd number a_{2n+1}, i.e. equivalent to the adoption of expansion:

$$\varphi = a_0\psi_0 + i\bar{a}_1\psi_1 + a_2\psi_2 + i\bar{a}_3\psi_3 + \cdots \tag{43}$$

and elimination of $a_0, \bar{a}_1, a_2, \bar{a}_3, \cdots$ from simultaneous linear equations, we have

$$\begin{vmatrix} \frac{1}{\rho}A_{00}+i\gamma C_{00}, & B_{01}, & \frac{1}{\rho}A_{02}+i\gamma C_{02}, & B_{03}, & \cdots\cdots \\ -B_{10}, & \frac{1}{\rho}A_{11}+i\gamma C_{11}, & -B_{12}, & \frac{1}{\rho}A_{13}+i\gamma C_{13}, & \cdots\cdots \\ \frac{1}{\rho}A_{20}+i\gamma C_{20}, & B_{21}, & \frac{1}{\rho}A_{22}+i\gamma C_{22}, & B_{23}, & \cdots\cdots \\ -B_{30}, & \frac{1}{\rho}A_{31}+i\gamma C_{31}, & -B_{32}, & \frac{1}{\rho}A_{33}+i\gamma C_{33}, & \cdots\cdots \\ \cdots\cdots & & & & \\ \cdots\cdots & & & & \end{vmatrix} = 0. \tag{44}$$

For a real value of α A_{mn}, B_{mn}, C_{mn} are all real, as will be seen from definition equations. For a neutral disturbance c, and accordingly γ, is real also, and

(44) has to reduce into two independent real equations for two real unknowns γ and ρ^{-1}. As one of these equations manifests the vanishing of the imaginary part of (44), and as imaginary unit i appears in (44) always combined with γ, we see the imaginary part giving a root $\gamma = 0$ certainly, irrespective of value α. Non-exisence of another real root is very probable in view of the single root $\gamma = 0$ when $R = \infty$, but not being proved here; for proof it seems necessary to discuss (44) numerically, which would be a difficult task. We here assume[6] simply the sole root $\gamma = 0$.

Substituting $\gamma = 0$ in (44), characteristic equation for ρ reduces to

$$\begin{vmatrix} \frac{1}{\rho}A_{00}, & B_{01}, & \frac{1}{\rho}A_{02}, & B_{03}, & \cdots\cdots \\ -B_{10}, & \frac{1}{\rho}A_{11}, & -B_{12}, & \frac{1}{\rho}A_{13}, & \cdots\cdots \\ \frac{1}{\rho}A_{20}, & B_{21}, & \frac{1}{\rho}A_{22}, & B_{23}, & \cdots\cdots \\ -B_{30}, & \frac{1}{\rho}A_{31}, & -B_{32}, & \frac{1}{\rho}A_{33}, & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0, \qquad (44')$$

and solution ρ i. e. R for an arbitrarily given α determines the required stability curve.

When $R = \infty$ i.e. $\rho^{-1} = 0$, (44′) becomes to

$$\begin{vmatrix} 0, & B_{01}, & 0, & B_{03}, & \cdots\cdots \\ B_{10}, & 0, & B_{12}, & 0, & \cdots\cdots \\ 0, & B_{21}, & 0, & B_{23}, & \cdots\cdots \\ B_{30}, & 0, & B_{32}, & 0, & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0, \qquad (45)$$

and in this case the system of linear equations for the coefficients a_{2m}, $\bar{a}_{2m+1}$ divides into two independent sets, the one the set of $\bar{a}_{2m+1}$, the other that of a_{2m}, the former giving an antisymmetrical function $\varphi_a(y)$, the latter a symmetrical one $\varphi_s(y)$. This character had been used in section 3 at the outset and two determinants $D^{(a)}(\alpha)$, $D^{(s)}(\alpha)$ there appeared are now the constituents of (45), such that (45) is equivalent to

$$D^{(a)}(\alpha) \cdot D^{(s)}(\alpha) = 0. \qquad (45')$$

Turning to the case of finite R we have to abridge the infinite determinant (44′) to one of such a few rank, that we are able to advance numerical calcula-

(6) This was also the case in the Esch's paper, cited in footnote (3).

tions. As we have seen in section 3, the cut off there adopted is sufficiently good, and it is then natural in present case to accept it also, i.e. to adopt determinant composed of the first ten or eleven rows and columns. When we choose eleven, attention has to be paid on the selection of rows, if we want to have the proper decomposition (45′) in the limiting case $\rho \to \infty$. At least when ρ is not large, however, such caution is meaningless, and larger rank of determinant will promise more accurate result.

We have adopted first ten, and then first eleven, and developed them into algebraic equations of 5th degree in ρ^2, both being the same degree. Their coefficients, which are polynomials of α^2, are too complex to be obtained in their general form, so that we assigned several numerical values for α at first, reducing thus A_{mn}, B_{mn} to numerical values, and then developed the determinants. This development has been the most tedious task in our calculations. Result is expressed in the form:

$$G_n(\rho) \equiv 1 + c_1\rho^2 + c_2\rho^4 + c_3\rho^6 + c_4\rho^8 + c_5\rho^{10} = 0, \quad (n = 9, 10) \qquad (50)$$

and the coefficients c_i are shown in Table 4. The roots ρ of these equations, then

Table 4.

n	α	c_1	$c_2\times 10$	$c_3\times 10^3$	$c_4\times 10^5$	$c_5\times 10^8$	ρ	αR	R
9	0.3	−0.2753	−1.047	−4.162	−2.750	−3.291	1.413	1.252	4.173
	0.4	−0.03935	−0.6209	−2.730	−1.865	−2.225	1.862	1.651	4.126
	0.5	+0.1574	−0.2574	−1.481	−1.078	−1.303	2.765	2.450	4.901
	0.6	+0.2773	−0.02144	−0.6371	−0.5350	−0.6792	4.428	3.924	6.541
	0.7	+0.3245	+0.09886	−0.1683	−0.2220	−0.3169	7.370	6.532	9.331
	0.8	+0.3255	+0.1422	+0.04896	−0.06399	−0.1320	12.975	11.50	14.37
	0.9	+0.2930	+0.1364	+0.1168	+0.002300	−0.04640	24.03	21.30	23.67
	0.95	+0.2861	+0.1375	+0.1358	+0.01887	−0.02409	35.57	31.52	33.18
	0.98	+0.2759	+0.1320	+0.1364	+0.02420	−0.01497	45.56	40.38	41.20
	1.00	+0.2713	+0.1299	+0.1377	+0.02710	−0.01040	55.24	48.96	48.96
	1.05	+0.2424	+0.1174	+0.1284	+0.02951	−0.00181	1.30×10^2	1.15×10^2	1.1×10^2
	1.065	+0.2461	+0.1141	+0.1256	+0.02975	−0.00016	4×10^2	4×10^2	4×10^2
	1.067					0.00000	∞	∞	∞
10	0.0	−0.859	−2.80	−12.75	−11.8	−22.9	0.945	0.838	
	0.1	−0.802	−2.66	−12.20	−11.3	−22.0	0.970	0.860	
	0.2	−0.629	−2.28	−10.69	−10.0	−19.6	1.056	0.936	
	0.3	−0.331	−1.629	−8.03	−7.69	−15.17	1.274	1.13	
	0.4	−0.00527	−0.8894	−4.92	−4.899	−9.85	1.75	1.55	
	0.5	+0.2302	−0.3237	−2.474	−2.653	−5.53	2.68	2.37	
	0.6	+0.3605	+0.01265	−0.961	−1.229	−2.75	4.36	3.87	
	0.7	+0.3972	+0.1671	−0.1711	−0.4489	−1.177	7.365	6.53	
	0.8	+0.3851	+0.2126	+0.1586	−0.08981	−0.4180	13.32	11.80	
	0.9	+0.3508	+0.2046	+0.2546	+0.04888	−0.09093	29.73	26.35	
	0.95	+0.3279	+0.1914	+0.2586	+0.07485	−0.01476	73.49	65.13	
	0.964					0.00000	∞	∞	
	0.98	+0.3148	+0.1822	+0.2534	+0.08259	+0.01445	—	—	
	1.00	+0.3089	+0.1788	+0.2530	+0.08718	+0.02863	—	—	

αR, and then R, are shown also in that table. For the case $n=9$, to each α which is less than a certain number α_0 one and only one ρ was found, and none above

α_0. α_0 is here nothing but the wave number corresponding to the inviscid liquid, and is of course equal to $\alpha_0 = 1.067$ found in section 3.

Some remarks are necessary for the case $n = 10$. As is said above, values of ρ which are not too large are almost equal to the corresponding values of $n = 9$, and are expected to be more accurate ones than the latters. There will then be not any solid reason to reject $G_{10}(\rho) = 0$ as irrational in cases of large ρ, and especially of $\rho = \infty$. In fact the determinant of first eleven rows and columms does vanish and not determine the corresponding wave number when $\rho = \infty$. This means zero absolute term in the expansion of determinant into a polynomial of ρ^{-1}, and results in an algebraic equation of 10th degree in ρ, i.e. $G_{10}(\rho) = 0$, one degree smaller than the rank of determinant. In this case, as in the other case, α_0 is to be determined as a vanishing point of the coefficient c_5 of $G_{10}(\rho)$, and interpolating values given in Table 4, we find $\alpha_0' = 0.964$, which can be accepted as a rough approximation to $\alpha_0 = 1.035$ of section 1.

If we write $\bar{a}'_{2n+1}/\rho$ instead of $\bar{a}_{2n+1}$ in (43) we have the characteristic equation

$$\begin{vmatrix} A_{00}, & B_{01}, & A_{02}, & B_{03}, & \cdots\cdots \\ -B_{10}, & \rho^{-2}A_{11}, & -B_{12}, & \rho^{-2}A_{13}, & \cdots\cdots \\ A_{20}, & B_{21}, & A_{22}, & B_{23}, & \cdots\cdots \\ -B_{30}, & \rho^{-2}A_{31}, & -B_{32}, & \rho^{-2}A_{33}, & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0, \qquad (51)$$

which replaces (44′), and the development of this equation, cut off to first eleven rows and columns is, as is easily understood, nothing but the equation $G_{10}(\rho) = 0$. Above all, c_5 is proportional to the left hand side of (51), abridged of course to eleven rank, and made ρ^{-1} vanish; this determinant really vanishes at $\alpha_0' = 0.964$. The characteristic function corresponding to this α_0' is, as seen from the substitution above introduced, in a form:

$$\varphi = a_0\psi_0 + a_2\psi_2 + a_4\psi_4 + \cdots , \qquad (51')$$

a fact which accords well with the result of the other case in section 3.

The value α_0' depends, however, on the elements $A_{2m,2n}$ $(m, n = 0, 1, 2, 3, 4, 5)$, this character being irrational from theoretical point of view. Presumably this value approaches gradually to $\alpha_0 = 1.035$ or so, we think, when the rank of determinant is increased, and becomes free of the elements. If we cut off (51) to first ten rows and columns, and make ρ^{-1} vanish, it must be identical with the corresponding one of (45). Apparent difference between them is the appearance of $A_{2m,2n}$ in the former, which has no effect on the symmetrical characteristic function, and excludes unnecessary discussion of antisymmetrical one.

The stability curve α as a function of αR, above obtained, is shown in Fig. 2 and Fig. 3, in the former the general feature and in the latter the feature when α is small being indicated. In these figures we see no hump in the intermediate region of αR which has been a remarkable point of Esch's results, and see rather

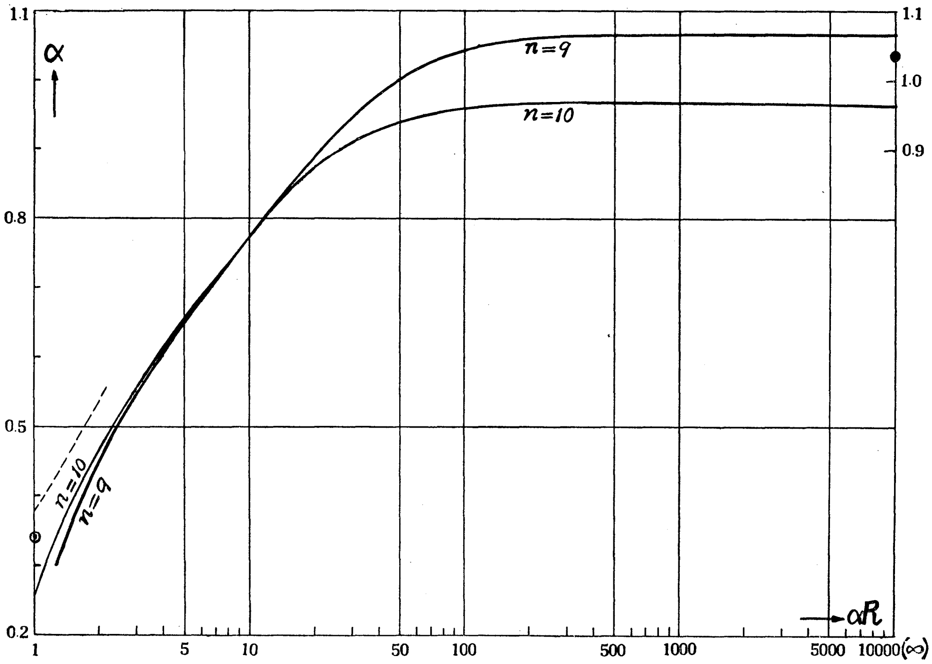

Fig. 2. Stability curves; solid curves are those of § 4, broken line is Esch's approximation, ⊙ that of Esch by computer, ● inviscid point of § 1.

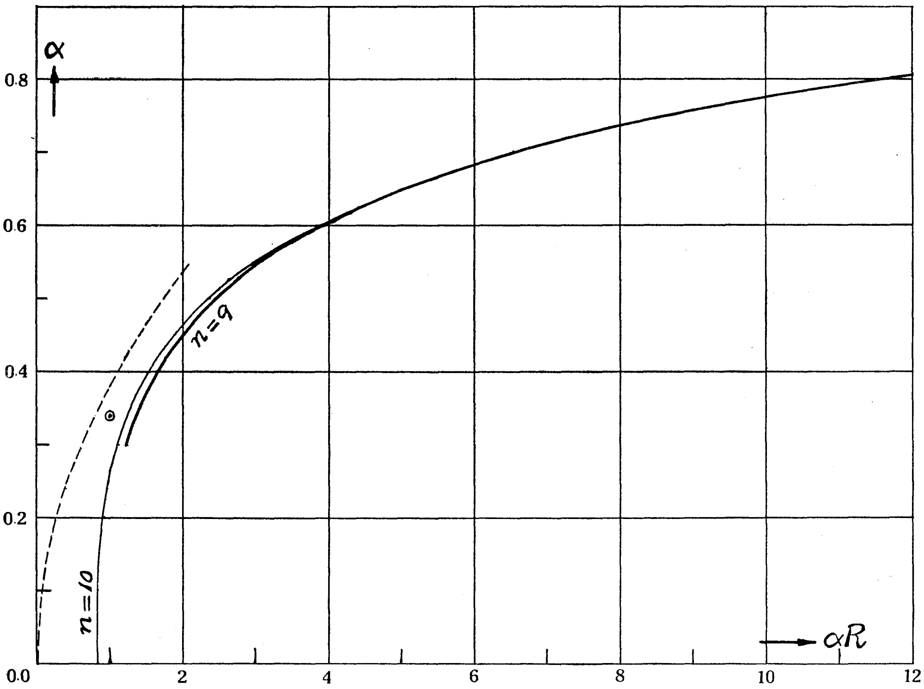

Fig. 3. Stability curves for small αR; marks being the same as in Fig. 2.

an accordance with that of Lessen in its general feature.

When α is small our approximation is expected to be rough, as is stated at the outset. Compared with Esch's first approximation for very small α, which has acquired more general meanings through the work of Tatsumi and Gotoh, and a value α at one point $\alpha R=1$, which has been calculated by Esch by means of an electronic digital computer, both being shown in the figures, our stability curve seems to be rough when αR is 3 or smaller (α is 0.55 or smaller). These cases of small α will be the principal theme of the second paper.

The velocity distribution of neutral waves, which is a stationary flow pattern by reason of $c=0$, is given by (2), and $\varphi(y)$ there by (43). Now the characteristic value ρ being inserted, linear equations for $a_0, \bar{a}_1, a_2, \bar{a}_3, \cdots$ have all the coefficients real (set of these coefficients forms the determinant (44′)), and therefore a_{2m}, $\bar{a}_{2m+1}$ are also all real ($a_0=1$ assumed). Then we have

$$\left.\begin{aligned} \varphi(y) &= \varphi_r(y) + i\varphi_i(y)\,; \\ \varphi_r(y) &= a_0\psi_0 + a_2\psi_2 + \cdots\,, \\ \varphi_i(y) &= \bar{a}_1\psi_1 + \bar{a}_3\psi_3 + \cdots\,; \end{aligned}\right\} \tag{52}$$

and the disturbance stream function (2) rewritten into

$$\psi = \varphi_r(y)\cos(\alpha x) - \varphi_i(y)\sin(\alpha x). \tag{53}$$

As an example we have taken the case ($\alpha=0.800$, $\alpha R=11.50$), in which a's are as follows:

$$\begin{aligned} &a_0 = 1.0000, \quad \bar{a}_1 = 0.0916, \quad a_2 = 0.2821, \quad \bar{a}_3 = -0.0172, \\ &a_4 = 0.1119, \quad \bar{a}_5 = 0.0199, \quad a_6 = 0.0624, \quad \bar{a}_7 = 0.0085, \\ &a_8 = 0.0228, \quad \bar{a}_9 = 0.0121. \end{aligned}$$

Using a table of Hermite functions we calculate numerical values of φ_r and φ_i, which we see indicated in Fig. 1, in comparison with φ_{sI} and φ_{sII}. Stream lines of these disturbance flows will be given in the second paper. Remarkable is the small imaginary part, and antisymmetricity of flow about x-axis (the centre line of our shear layer) seems nearly conserved down to a pretty small Reynolds number, and therefore flow pattern remaining almost the same (*To be continued*)[7].

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(7) Almost all numerical computations of the paper are undertaken by Miss S. Hoshino, to whom our thanks are due.