The Time-Dependent Maxwellian Model of Solid Bodies

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NOTE

The Time-Dependent Maxwellian Model of Solid Bodies

By Jun-ichi OKABE

1. It is well-known that the behavior of the Maxwellian model of solid bodies (see FIGURE 1) is defined by the equation

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}, \qquad (1.1)$$

where $\varepsilon =$ strain, $\dot{\varepsilon} =$ time rate of strain,
$\sigma =$ stress, $\dot{\sigma} =$ time rate of stress,
$E =$ modulus of elasticity, and $\eta =$ coefficient of viscosity.

If we consider only the case of the constant time rate of strain, namely

$$\dot{\varepsilon} = \text{const.} \equiv \alpha, \qquad (1.2)$$

in other words, denoting the time by t,

$$\varepsilon = \alpha t, \qquad (1.3)$$

then (1.1) may be written

$$\dot{\sigma} + \frac{E}{\eta}\,\sigma = E\alpha, \qquad (1.4)$$

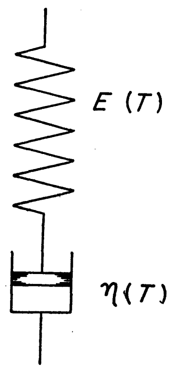

FIG. 1.

where both of E and η are the functions of the temperature of the solid body, T say, which arises in the course of the deformation owing to the internal friction of the body. The relationship between T and t will be formulated in a very complicated integro-differential equation, and we shall have to solve it together with (1.4). But to find the solution of these simultaneous equations in a general case being unfortunately impossible, the calculation which follows will be extremely simplified in order to avoid this difficulty.

Because the functional forms of $E(T)$ and $\eta(T)$ appearing in the equation (1.4) can be determined experimentally, we may regard E and η as dependent only upon t, at least conceptually, if the material, the mode of its deformation, and the condition surrounding the body have been specified definitely. Although both of these coefficients are known to decrease with T increasing, since that property is more conspicuous for η than for E, assuming for simplicity

$$E = \text{const.} \qquad (1.5)$$

in our subsequent calculations, let us consider the *material* if the surrounding condition is to be prescribed, or on the contrary the *surrounding condition* if the material is specified, such that in the course of our constant strain rate deformation η of the material varies with t in the following manner:

$$\eta = \eta_1 + \frac{\eta_0 - \eta_1}{kt+1}, \tag{1.6}$$

$$(\eta_0 > \eta_1 > 0, \quad k > 0.)$$

As readily found from (1.6),

$$\begin{aligned} \eta &= \eta_0 \quad \text{at} \quad t = 0, \\ \eta &= \eta_1 \quad \text{at} \quad t = \infty, \end{aligned} \tag{1.7}$$

and
$$\frac{d\eta}{dt} = -k(\eta_0 - \eta_1) < 0 \quad \text{at} \quad t = 0.$$

In FIGURE 2 and TABLE 1 the behavior of the function $1/(kt+1)$ is shown for various values of k and t. Again (1.6) may be rewritten in the form

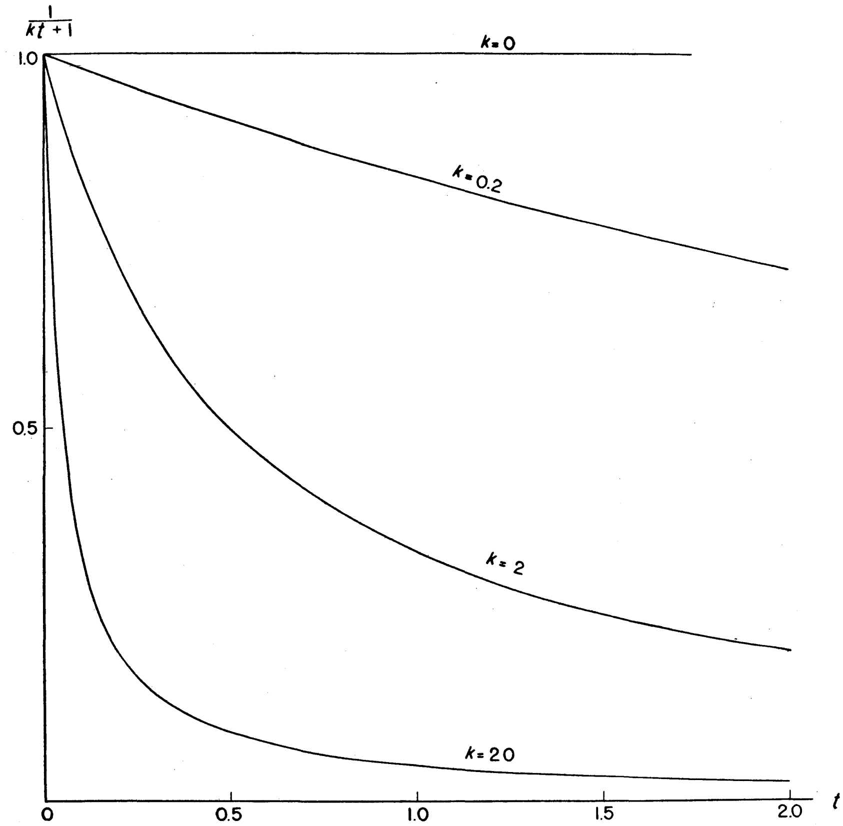

FIG. 2.

$$\eta = \eta_1 \frac{kt + \beta}{kt + 1}, \tag{1.8}$$

where
$$\beta \equiv \eta_0/\eta_1 > 1. \tag{1.9}$$

Under the initial condition that

$$\sigma = 0 \quad \text{at} \quad t = 0, \tag{1.10}$$

the solution of the equation (1.4) is obtained in the form

$$\sigma(t) = \exp\left(-\int_0^t \frac{E}{\eta} dt'\right) \int_0^t E\alpha \exp\left(\int_0^{t'} \frac{E}{\eta} dt''\right) dt', \quad (1.11)$$

and making use of the expressions (1.5) and (1.8), we are led to the result

$$\sigma(t) = e^{-\frac{E}{\eta_1}t} \left(\frac{kt+\beta}{\beta}\right)^{\frac{(\beta-1)E}{k\eta_1}} \int_0^t E\alpha e^{\frac{E}{\eta_1}t'} \left(\frac{kt'+\beta}{\beta}\right)^{-\frac{(\beta-1)E}{k\eta_1}} dt'. \quad (1.12)$$

Or putting

$$E/\eta_1 \equiv \gamma, \quad (1.13)$$

(1.12) is transformed into

$$\sigma(t) = E\alpha e^{-\gamma t}\left(\frac{k}{\beta}t+1\right)^{\frac{(\beta-1)\gamma}{k}} \int_0^t e^{\gamma t'}\left(\frac{k}{\beta}t'+1\right)^{-\frac{(\beta-1)\gamma}{k}} dt'. \quad (1.14)$$

However, the integral on the right-hand side being approximated by

$$\left(\frac{k}{\beta}t+1\right)^{-\frac{(\beta-1)\gamma}{k}} \int_0^t e^{\gamma t'} dt' \sim \left(\frac{k}{\beta}t+1\right)^{-\frac{(\beta-1)\gamma}{k}} \frac{e^{\gamma t}}{\gamma}$$

for a very large value of t, we have

$$\lim_{t\to\infty} \sigma = E\alpha/\gamma = \eta_1\alpha, \quad (1.15)$$

which is in accordance with the conclusion obtained directly from (1.4) when we put

$$\eta = \eta_1, \ \dot{\sigma} = 0 \quad \text{at} \quad t = \infty.$$

On the other hand, for the special case when

$$\eta = \text{const.} \equiv \eta_c, \text{ say}, \quad (1.17)$$

the solution of the equation (1.4) is readily found to be

$$\sigma(t) = \eta_c\alpha\left(1 - e^{-\frac{E}{\eta_c}t}\right), \quad (1.18)$$

and if we put

$$\eta_c = \eta_1 \quad (1.19)$$

specifically, we may write also

$$\sigma(t) = \eta_1\alpha\,(1-e^{-\gamma t}), \quad (1.20)$$

and

$$\lim_{t\to\infty} \sigma = \eta_1\alpha \quad (1.21)$$

as before.

Next, in order to transform (1.14) into more convenient form, let us introduce the new variable τ such that

$$\frac{k}{\beta}t + 1 \equiv \tau, \quad (1.22)$$

then it can be shown without difficulty that

$$\sigma(\tau) = E\frac{\alpha\beta}{k} e^{-\frac{\beta\gamma}{k}\tau} \tau^{\frac{(\beta-1)\gamma}{k}} \int_1^\tau e^{\frac{\beta\gamma}{k}\tau'} \tau'^{-\frac{(\beta-1)\gamma}{k}} d\tau', \tag{1.23}$$

from which we find

$$\begin{aligned} \sigma &> E\frac{\alpha\beta}{k} e^{-\frac{\beta\gamma}{k}\tau} \tau^{\frac{(\beta-1)\gamma}{k}} \tau^{-\frac{(\beta-1)\gamma}{k}} \int_1^\tau e^{\frac{\beta\gamma}{k}\tau'} d\tau' \\ &= E\frac{\alpha}{\gamma}\left\{1 - e^{-\frac{\beta\gamma}{k}(\tau-1)}\right\} \\ &= \eta_1\alpha\,(1 - e^{-\gamma t}). \end{aligned} \tag{1.24}$$

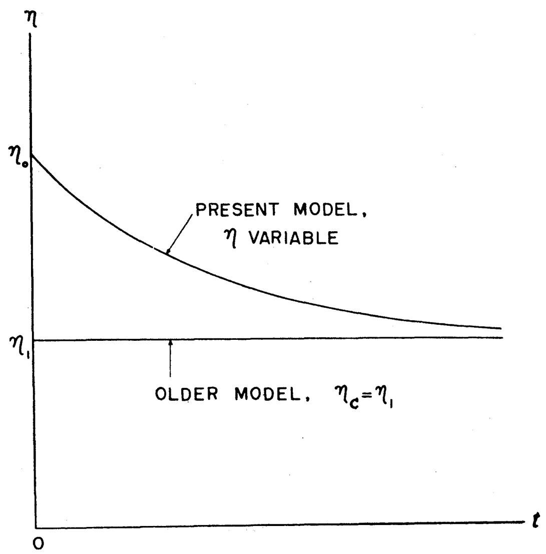

FIG. 3.

Namely, it has been proved that σ predicted from the present model is always larger than σ calculated by means of the older model.[1] (As the older model we mean the procedure in which both E and η are regarded as invariable. The behaviors of η's assumed in these two models are shown schematically in FIGURE 3.)

From (1.23) we can derive

$$\frac{d\sigma}{d\tau} = E\frac{\alpha\beta}{k}\left\{\frac{\sigma}{\eta_1\alpha}\left(\frac{\beta-1}{\beta\tau}-1\right)+1\right\}, \tag{1.25}$$

and

$$\frac{d^2\sigma}{d\tau^2} = E\frac{\alpha\beta}{k}\left\{\frac{d\sigma}{d\tau}\frac{1}{\eta_1\alpha}\left(\frac{\beta-1}{\beta\tau}-1\right) - \frac{\sigma}{\eta_1\alpha}\frac{\beta-1}{\beta\tau^2}\right\}. \tag{1.26}$$

If we denote the extreme value of σ (if any) by σ_m, the corresponding value of τ by τ_m, then from (1.25) σ_m satisfies the equation

[1] According to Professor Higuchi this result agrees with the experiment.

$$\frac{\sigma_m}{\eta_1\alpha} = \left(1-\frac{\beta-1}{\beta\tau_m}\right)^{-1}, \tag{1.27}$$

so by virtue of the restriction

$$\infty > \tau_m \geqq 1, \tag{1.28}$$

it can be verified easily that

$$\beta \geqq \sigma_m/\eta_1\alpha > 1. \tag{1.29}$$

And further in the neighborhood of $\tau = \tau_m$, σ may be proved the ***maximum***, not the ***minimum***, since we have

$$\left(\frac{d^2\sigma}{d\tau^2}\right)_{\tau=\tau_m} = -E\frac{\alpha\beta}{k}\frac{\sigma_m}{\eta_1\alpha}\frac{\beta-1}{\beta\tau_m{}^2} < 0. \tag{1.30}$$

In conclusion, there are only two possibilities for the relationship between σ and τ, both of which are shown in FIGURE 4. Curves ① and ② represent respectively the results of the present, and the older, models.

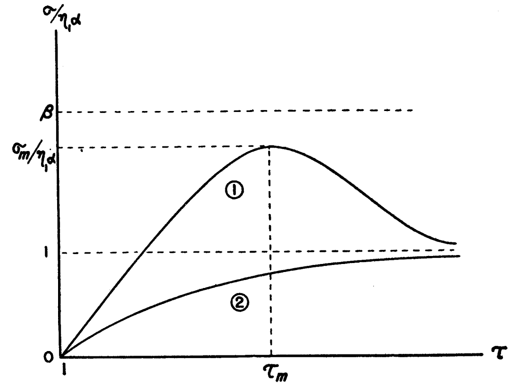

FIG. 4.

2. As the first example, let us consider the case

$$\begin{aligned} \beta &= 2, \quad \text{cf. (1.9)}, \\ r &= 2, \quad \text{cf. (1.13)}, \\ k &= 2, \quad \text{cf. (1.6)}, \end{aligned} \tag{2.1}$$

where β, r and k have the dimensions of (time^0), (time^{-1}), and (time^{-1}), respectively. Since

$$E = 2\eta_1 \tag{2.2}$$

numerically at present, it follows from (1.23) that

$$\sigma(\tau) = 2\eta_1\alpha e^{-2\tau}\tau\int_1^{\tau} e^{2\tau'}\frac{d\tau'}{\tau'}, \tag{2.3}$$

and by writing

$$2\tau \equiv \tilde{\tau}, \tag{2.4}$$

the above expression reduces to

$$\frac{\sigma(\tilde{\tau})}{\eta_1\alpha} = e^{-\tilde{\tau}}\,\tilde{\tau}\int_2^{\tilde{\tau}} e^{\tilde{\tau}'}\frac{d\tilde{\tau}'}{\tilde{\tau}'}, \tag{2.5}$$

where $\tilde{\tau}$ is connected with t by the equation

$$2t = \tilde{\tau} - 2. \tag{2.6}$$

Next, if the numerical values

$$\beta = 1.5, \quad r = 2, \quad k = 2 \tag{2.7}$$

are assumed as the second example, we have as before from (1.23)

$$\frac{\sigma(\tilde{\tau})}{\eta_1\alpha} = e^{-\tilde{\tau}}\sqrt{\tilde{\tau}}\int_{1.5}^{\tilde{\tau}} e^{\tilde{\tau}'}\frac{d\tilde{\tau}'}{\sqrt{\tilde{\tau}'}} \tag{2.8}$$

where $$1.5\,\tau = \tilde{\tau}, \tag{2.9}$$
so that $$2t = \tilde{\tau} - 1.5. \tag{2.10}$$
On the other hand, when
$$\eta \equiv \eta_c = \eta_1, \tag{2.11}$$
(1.20) reduces to
$$\frac{\sigma(t)}{\eta_1\alpha} = 1 - e^{-2t}, \tag{2.12}$$

for these two cases above-mentioned, since γ has the same value in both of our examples.

In FIGURE 5 and TABLE 2, two kinds of $\sigma/\eta_1\alpha$ corresponding to the parameters (2.1) and (2.7) are shown for various values of t; the former case was worked out using the formulas (2.5), (2.6), and the latter was calculated with the aid of (2.8), (2.10). The result of (2.12) is referred to as "the older model" in the figure and the table for ready comparison. Since there exists a relationship given by (1.3) between the strain and the time, the abscissa of FIGURE 5 is proportional to the strain itself.

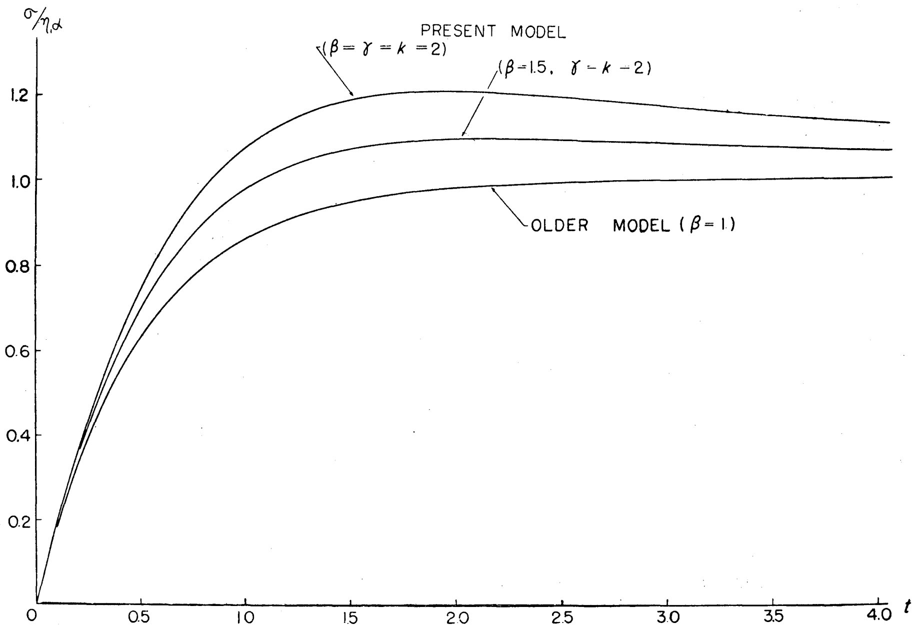

FIG. 5.

Finally, with a view to knowing the mode of variation of η with T, the temperature of the solid body, we denote by q the heat generated per unit time in the body per unit volume owing to the internal friction. Then, since q is proportional to the product of σ and $\dot{\varepsilon}$, the latter being constant and equal to α in our constant strain rate deformation, T at time t is found to be proportional to $\int_0^t \sigma\, dt$, provided that the specific heat of the body overall is independent of the temperature and that all the heat generated is stored in the body to elevate its temperature instead of escaping from it. Thus from (1.6), η and T are related to each other through t as the parameter. In FIGURE 6 and TABLE 3 for the values

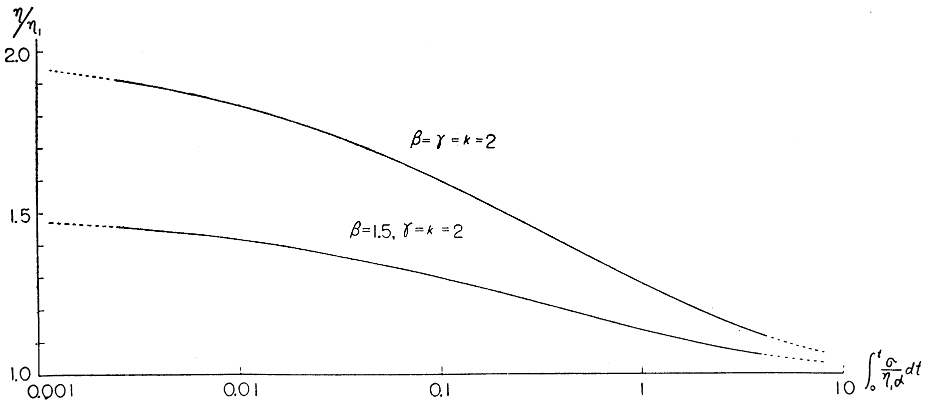

FIG. 6.

of the constants illustrated in the preceding two examples, the behavior of η/η_1 is shown as the function of $\int_0^t \sigma dt/\eta_1\alpha$ in the logarithmic scale.

The writer is very grateful to Professor M. Higuchi of the Institute for his advices and encouragements.

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TABLE 1.

t	$1/(kt+1)$ $k=0.2$	$1/(kt+1)$ $k=2$	$1/(kt+1)$ $k=20$
0	1.0000	1.0000	1.0000
0.005	0.9990	0.9901	0.9091
0.01	0.9980	0.9804	0.8333
0.02	0.9960	0.9615	0.7143
0.03	0.9940	0.9434	0.6250
0.04	0.9921	0.9259	0.5556
0.05	0.9901	0.9091	0.5000
0.1	0.9804	0.8333	0.3333
0.2	0.9615	0.7143	0.2000
0.3	0.9434	0.6250	0.1429
0.4	0.9259	0.5556	0.1111
0.5	0.9091	0.5000	0.0909
0.6	0.8929	0.4545	0.0769
0.7	0.8772	0.4167	0.0667
0.8	0.8621	0.3846	0.0588
0.9	0.8475	0.3571	0.0526
1.0	0.8333	0.3333	0.0476
1.1	0.8197	0.3125	0.0435
1.2	0.8065	0.2941	0.0400
1.3	0.7937	0.2778	0.0370
1.4	0.7813	0.2632	0.0345
1.5	0.7692	0.2500	0.0323
1.6	0.7576	0.2381	0.0303
1.7	0.7463	0.2273	0.0286
1.8	0.7353	0.2174	0.0270
1.9	0.7246	0.2083	0.0256
2.0	0.7143	0.2000	0.0244

TABLE 2.

t	$\sigma/\eta_1\alpha$ $\begin{pmatrix}\text{present}\\ \text{model}\end{pmatrix}$ $(\beta=\gamma=k=2)$	$\sigma/\eta_1\alpha$ $\begin{pmatrix}\text{present}\\ \text{model}\end{pmatrix}$ $(\beta=1.5, \gamma=k=2)$	$\sigma/\eta_1\alpha$ $\begin{pmatrix}\text{older}\\ \text{model}\end{pmatrix}$
0	0	0	0
0.025	0.0494	0.0492	0.0488
0.050	0.0975	0.0967	0.0952
0.075	0.1443	0.1425	0.1393
0.100	0.1897	0.1867	0.1813
0.125	0.2339	0.2294	0.2212
0.150	0.2767	0.2704	0.2592
0.175	0.3183	0.3098	0.2953
0.200	0.3585	0.3478	0.3297
0.225	0.3974	0.3842	0.3624
0.250	0.4350	0.4192	0.3935
0.275	0.4713	0.4528	0.4231
0.300	0.5063	0.4850	0.4512
0.325	0.5401	0.5159	0.4780
0.350	0.5727	0.5454	0.5034
0.375	0.6041	0.5737	0.5276
0.400	0.6343	0.6008	0.5507
0.425	0.6633	0.6267	0.5726
0.450	0.6912	0.6515	0.5934
0.475	0.7180	0.6752	0.6133
0.500	0.7438	0.6978	0.6321
0.525	0.7684	0.7193	0.6501
0.550	0.7921	0.7399	0.6671
0.575	0.8148	0.7596	0.6834
0.600	0.8365	0.7783	0.6988
0.625	0.8573	0.7962	0.7135
0.650	0.8772	0.8132	0.7275
0.675	0.8962	0.8294	0.7408
0.700	0.9143	0.8448	0.7534
0.725	0.9316	0.8595	0.7654
0.750	0.9482	0.8734	0.7769
0.775	0.9639	0.8867	0.7878
0.800	0.9789	0.8993	0.7981
0.825	0.9932	0.9113	0.8080
0.850	1.0068	0.9227	0.8173
0.875	1.0198	0.9335	0.8262
0.900	1.0321	0.9437	0.8347
0.925	1.0437	0.9534	0.8428
0.950	1.0548	0.9626	0.8504
0.975	1.0653	0.9713	0.8577
1.000	1.0752	0.9795	0.8647
1.050	1.0936	0.9947	0.8775
1.100	1.1099	1.0083	0.8892
1.150	1.1245	1.0204	0.8997
1.200	1.1374	1.0311	0.9093
1.250	1.1488	1.0406	0.9179
1.300	1.1587	1.0490	0.9257
1.350	1.1674	1.0564	0.9328
1.400	1.1750	1.0629	0.9392
1.450	1.1814	1.0686	0.9450

TABLE 2. *continued*

t	$\sigma/\eta_1\alpha$ (present model) ($\beta=\gamma=k=2$)	$\sigma/\eta_1\alpha$ (present model) ($\beta=1.5$, $\gamma=k=2$)	$\sigma/\eta_1\alpha$ (older model)
1.500	1.1869	1.0735	0.9502
1.550	1.1915	1 0778	0.9550
1.600	1.1954	1.0814	0.9592
1 650	1.1985	1.0845	0.9631
1.700	1.2009	1.0871	0.9666
1.750	1.2028	1.0893	0.9698
1.800	1.2041	1.0911	0.9727
1.850	1.2050	1.0925	0.9753
1.900	1.2054	1.0936	0.9776
1.950	1.2055	1.0944	0.9798
2.000	1.2052	1.0950	0.9817
2.100	1.2038	1.0955	0.9850
2.200	1.2014	1.0953	0.9877
2.300	1.1984	1.0945	0.9899
2.400	1.1948	1.0934	0.9918
2.500	1.1908	1.0919	0.9933
2.600	1.1865	1.0902	0.9945
2.700	1.1821	1.0884	0.9955
2.800	1.1776	1.0864	0.9963
2.900	1.1731	1.0844	0.9970
3.000	1.1686	1.0824	0.9975
3.100	1.1641	1.0803	0.9980
3.200	1.1597	1.0783	0.9983
3.300	1.1554	1.0763	0.9986
3.400	1.1513	1.0744	0.9989
3.500	1.1473	1.0725	0.9991
3.600	1.1434	1.0706	0.9993
3.700	1.1396	1.0688	0.9994
3.800	1.1360	1.0671	0.9995
3.900	1.1325	1.0654	0.9996
4.000	1.1292	1.0638	0.9997

TABLE 3.

t	$\beta=\gamma=k=2$		$\beta=1.5,\ \gamma=k=2$	
	$\int_0^t \sigma\, dt/\eta_1\alpha$	η/η_1	$\int_0^t \sigma\, dt/\eta_1\alpha$	η/η_1
0	0	2.000	0	1.500
0.05	0.00246	1.909	0.00244	1.455
0.10	0.00966	1.833	0.00956	1.417
0.15	0.0213	1.769	0.0210	1.385
0.20	0.0372	1.714	0.0365	1.357
0.25	0.0571	1.667	0.0557	1.333
0.30	0.0807	1.625	0.0783	1.313
0.35	0.1077	1.588	0.1041	1.294
0.40	0.1378	1.556	0.1328	1.278
0.45	0.1710	1.526	0.1641	1.263
0.50	0.207	1.500	0.1979	1.250
0.55	0.245	1.476	0.234	1.238
0.60	0.286	1.455	0.272	1.227
0.65	0.329	1.435	0.312	1.217
0.70	0.374	1.417	0.353	1.208
0.75	0.420	1.400	0.396	1.200
0.80	0.468	1.385	0.440	1.192
0.85	0.518	1.370	0.486	1.185
0.90	0.569	1.357	0 533	1.179
0.95	0.621	1.345	0.580	1.172
1.0	0.675	1.333	0.629	1.167
1.1	0.784	1.313	0.728	1.156
1.2	0.896	1.294	0.830	1.147
1.3	1.011	1.278	0.934	1.139
1.4	1.128	1.263	1.040	1.132
1.5	1.246	1.250	1.147	1.125
1.6	1.365	1.238	1.255	1.119
1.7	1.485	1.227	1.363	1.114
1.8	1.605	1.217	1.472	1.109
1.9	1.726	1.208	1.581	1.104
2.0	1.846	1.200	1.691	1.100
2.2	2.09	1.185	1.910	1.093
2.4	2.33	1.172	2.13	1.086
2.6	2.56	1.161	2.35	1.081
2.8	2.80	1.152	2.56	1.076
3.0	3.04	1.143	2.78	1.071
3.2	3.27	1.135	3.00	1.068
3.4	3.50	1.128	3.21	1.064
3.6	3.73	1.122	3.43	1.061
3.8	3.96	1.116	3.64	1.058
4.0	4.18	1.111	3.85	1.056

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Page 76, line 2, from the bottom, for "(1 8)" read "(1.8)."
Page 85, line 7, from the bottom, for "J_1'' ($j_{0\ n}$)" read "$J_1''(j_{0,n})$."
Page 87, line 12, for "approximata" read "approximate."
Page 99, TABLE 1, $x = 1.0$, $t = 1.5$, for "0 777" read "0.777."
Page 100, TABLE 3, ($\tau = 0$; $\mu = 50$), $n = 22$, for "$-$0. 80 5543" read "$-$0.0080 5543."
Page 102, TABLE 5, ($\tau = 1$; $\mu = 30$), $n = 6$, for "$-$0 0336 7637" read "$-$0.0336 7637."
Page 110, $\sigma = 0.725$, for "0 372 803" read "0.372 803," $\sigma = 0.800$, for "0 267/ 960" read "0.267 960," and σ=0.975, for "0 076 278" read "0.076 278."
Page 111, footnote, for "$j_{0\ 8} = 24{,}35\cdots$" read "$j_{0,8} = 24.35\cdots$."
Page 113, footnote, for "$j_{0\ 16}$" and "$j_{0\ 17}$" read "$j_{0,16}$" and "$j_{0,17}$" respectively.
Page 114, footnote, for "$j_{0\ 21}$" and "$j_{0\ 23}$" read "$j_{0,21}$" and "$j_{0,23}$" respectively.
Page 115, $\sigma = 0.225$, for "0 024 413" read "0.024 413," σ=0.575, for "$-$0,105 24" read "$-$0.105 24," and footnote, for "$j_{0\ 26}$" read "$j_{0,26}$."
Page 117, footnote, for "$j_{0\ 36}$" and "$j_{0\ 39}$" read "$j_{0,36}$" and "$j_{0,39}$" respectively.
Page 121, equation (39), for "ζ_2" read "ζ^2."
Page 129, TABLE 17, $\vartheta = -0.5$, $\sigma = 0.5$ ($R = 1$) for "0 0416" read "0.0416." (*J. O.*)