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ON THE THEORY OF HOMOGENEOUS AXISYMMETRIC TURBULENCE (III)*

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In the major part of this paper, the large-scale structure and the final period of decaying homogeneous axisymmetric turbulence are considered according to Batchelor-Proudman's general theory [2] the outline of which is also reproduced. The essential result is that four independent parameters appear in the relevant energy spectrum tensor, while the customary theory requires only two parameters. Correspondingly, a little modification is found to be necessary for Chandrasekhar's picture of axisymmetric turbulence in the final period [3], and some illustrative correlation curves in that period are newly computed.

In the remaining part, on the other hand, we consider a certain special case of axisymmetric turbulence in connexion with an interesting model proposed by Chandrasekhar [3] which may be regarded as a non-interacting superposition of isotropic and axisymmetric fields of turbulence.

1. Introduction. At the present time, the dynamical theory of (homogeneous) non-isotropic turbulence seems to be much less satisfactory than that of isotropic one. For instance, what is the decay law? What is the tendency to isotropy? What are the shapes of correlation or spectrum functions ?... We cannot yet answer these questions beyond some qualitative arguments resting on a few experimental evidences.

But it has been considered that 'the permanence of big eddies' and 'the final period of decay' are only exceptions for which we can establish a rigorous theory without assuming any particular symmetry of the field (see [1] §§ 5.3 and 5.4). Recently, however, the drawback of these previous results has been revealed first in the work of I. Proudman & W. H. Reid (*Philos. Trans.* A, 247 (1954) 163), and then presented by G. K. Batchelor & Proudman [2] in greater details together with a reconstruction of the whole theory. The gist is that the pressure action which depends on the entire velocity field of turbulence may be of long-range character such as to ensure statistical connexions between quantities at widely separated points. We can thus no longer admit the basic assumption of the customary theory that all integral moments of all velocity cumulants in homogeneous turbulence are *always* convergent.

Taking these circumstances into account correctly, Batchelor & Proudman have succeeded to obtain a new type of large-scale structure in homogeneous turbu-

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lence after a good deal of highly ingenious mathematics. Their results must be of considerable generality, but to be remembered, they are based on 'the hypothesis of convergent integral moments at an initial instant' which more or less idealizes the early state of grid turbulence (see next section), and how far such an idealization is actually permissible still remains an open question. In short, the dynamics of big eddies is much more complicated than has hitherto been supposed, and especially, since the large-scale structure of grid turbulence is known to be distinctly non-isotropic, the problem must bear a close relation to the theory of non-isotropic turbulence.

In this paper, we shall be concerned with the large-scale structure of homogeneous axisymmetric turbulence on the basis of Batchelor-Proudman's theory in order to see what we can now predict for this simplest case of non-isotropy. As to this, the contributions of S. Chandrasekhar [3] should be mentioned too. Considering that his picture of axisymmetric turbulence in the final period of decay depends on the customary assumption stated above, certain remarks will be necessary from the present point of view.

Supplementing his paper, Chandrasekhar has also shown the possibility of an interesting special state of axisymmetric turbulence in terms of a purely mathematical argument. In the last section below, we shall intend to throw some light on the physical aspects of this model.

Before entering into these particulars, however, let us dwell on the general subject for a moment, and sketch the outline of Batchelor-Proudman's theory in view of its importance. The notations and terminologies in author's preceding papers [5] and [6] (referred as I and II respectively) will be followed as far as possible.

2. Resume of Batchelor-Proudman's theory. As is well-known, the basic equation of homogeneous incompressible turbulence is

$$\frac{\partial R_{ij}}{\partial t}=\frac{\partial}{\partial r_k}\left(\overline{u_iu_ku_j'}-\overline{u_iu_k'u_j'}\right)+\frac{1}{\rho}\left(\frac{\partial\overline{pu_j'}}{\partial r_i}-\frac{\partial\overline{u_ip'}}{\partial r_j}\right)+2\nu\nabla^2R_{ij}, \quad (2.1)$$

where $R_{ij}(\boldsymbol{r},t)\equiv\overline{u_iu_j'}$, and by taking the divergence we have

$$\nabla^2\overline{pu_j'}=-\rho\frac{\partial^2\overline{u_iu_ku_j'}}{\partial r_i\partial r_k}\,, \quad (2.2)$$

whose appropriate solution may be written as

$$\overline{pu_j'}=\frac{\rho}{4\pi}\int\frac{\partial^2\overline{u_i''u_k''u_j'}}{\partial x_i''\partial x_k''}\cdot\frac{d\boldsymbol{x}''}{|\boldsymbol{x}''-\boldsymbol{x}|}\,. \quad (2.3)^{1)}$$

Now, with Batchelor & Proudman (B & P hereafter), let us assume that at an initial instant $t=t_0$ all integral moments of all velocity cumulants are convergent.[2] Then, at that instant, the integral on the right-hand side of (2.3) can be

1) This is the particular solution of (2.2), the complementary solution being identically zero in an infinite fluid.

2) An equivalent assumption is that at $t=t_0$ all spectrum tensors (i.e. Fourier transforms) of velocity cumulants are analytic everywhere in wave-number space. Thus, for instance, the special model worked out by T. Tatsumi (*Proc. 9th Int. Congr. Appl. Mech.* **3** (1957) 396, and [7]) for which the initial single-line energy spectrum is assumed, is excluded from the present considerations.

expanded into an infinite series of solid harmonics (i.e. contributions from various multiple poles) in such a manner as

$$[\overline{pu_j'}]_{t_0}=\frac{1}{r}\int A_j(\tilde{\boldsymbol{r}})\,\mathrm{d}\tilde{\boldsymbol{r}}-\frac{\partial}{\partial r_i}\left(\frac{1}{r}\right)\int A_j(\tilde{\boldsymbol{r}})\tilde{r}_i\mathrm{d}\tilde{\boldsymbol{r}}+\frac{1}{2!}\frac{\partial^2}{\partial r_i\partial r_k}\left(\frac{1}{r}\right)\int A_j(\tilde{\boldsymbol{r}})\tilde{r}_i\tilde{r}_k\mathrm{d}\tilde{\boldsymbol{r}}$$

$$-\frac{1}{3!}\frac{\partial^3}{\partial r_i\,\partial r_k\,\partial r_l}\left(\frac{1}{r}\right)\int A_j\,(\tilde{\boldsymbol{r}})\,\tilde{r}_i\,\tilde{r}_k\,\tilde{r}_l\,\mathrm{d}\tilde{\boldsymbol{r}}+O(r^{-5}), \qquad (2.4)$$

where $\tilde{\boldsymbol{r}}=\boldsymbol{x}''-\boldsymbol{x}'$ and $A_j(\tilde{\boldsymbol{r}})=\dfrac{\rho}{4\pi}\left[\dfrac{\partial^2\overline{u_i''u_k''u_j'}}{\partial x_i''\partial x_k''}\right]_{t_0}$.

However, since the first two terms of the series vanish by Green's theorem, and also the third by solenoidality of $A_j(\tilde{\boldsymbol{r}})$, we find that in general

$$[\overline{pu_j'}]_{t_0}\sim O(r^{-4}),\quad \text{or in view of (2.1),}\quad [\partial R_{ij}/\partial t]_{t_0}\sim O(r^{-5})$$

for large values of the separation $r=|\boldsymbol{r}|$.

Further, making $\partial^2R_{ij}/\partial t^2$, $\partial^3R_{ij}/\partial t^3,\ldots$ successively from (2.1) and examining the pressure terms at $t=t_0$ as above, it is possible to prove that no asymptotic forms of larger order than r^{-5} appear in $[\partial^nR_{ij}/\partial t^n]_{t_0}$ for any n. We may thus obtain the asymptotic law $R_{ij}(\boldsymbol{r})\sim O(r^{-5})$ at all times subsequent to the instant t_0.

This is a typical argument of B & P showing the generation of power-law forms of velocity cumulants at large r as an effect of the pressure forces which are expressed as volume integrals like (2.3). They found, in this and similar ways,

$$\overline{u_iu_j'}\sim O(r^{-5}),\qquad \overline{\omega_i\omega_j'}\sim O(r^{-8}), \qquad (2.5)$$

and

$$\overline{u_iu_ku_j'}\sim O(r^{-4}),\qquad \overline{u_iu_k\omega_j'}\sim O(r^{-6}), \qquad (2.6)$$

respectively for large values of r, where ω_i stands for the component of vorticity. Here it is of special consequence that $\overline{\omega_i\omega_j'}$, viz. the double curl of $\overline{u_iu_j'}$, is of order $r^{-8}=r^{-5}\times r^{-3}$, while $\overline{u_iu_j'}$ itself is of order r^{-5}. In other words, the largest-scale component of $\overline{u_iu_j'}$ does not contribute to $\overline{\omega_i\omega_j'}$, and the complicated tensor analysis of B & P is essentially a procedure to determine the asymptotic functional form of order r^{-5} having this semi-irrotational property. Their final result is that

$$R_{ij}(\boldsymbol{r},t)=\frac{1}{4}C_{lmnp}\left(\delta_{il}\nabla^2-\frac{\partial^2}{\partial r_i\partial r_l}\right)\left(\delta_{jm}\nabla^2-\frac{\partial^2}{\partial r_j\partial r_m}\right)\frac{\partial^2 r}{\partial r_n\partial r_p}+O(r^{-6}), \quad (2.7)$$

when r is large; the tensor coefficient C_{lmnp} may be a function of decay time t in general.

The Fourier transform version of (2.7) is

$$\Phi_{ij}(\boldsymbol{\kappa},t)=\frac{1}{4\pi^2}C_{lmnp}\left(\delta_{il}-\frac{\kappa_i\kappa_l}{\kappa^2}\right)\left(\delta_{jm}-\frac{\kappa_j\kappa_m}{\kappa^2}\right)\kappa_n\kappa_p+O(\kappa^3\ln\kappa), \qquad (2.8)$$

when $\kappa=|\boldsymbol{\kappa}|$ is small. Note that the leading term of this expression is of order κ^2 in agreement with the older result (cf. [1] p. 37), but is singular at $\boldsymbol{\kappa}=0$ because as $\boldsymbol{\kappa}\to0$ the second derivative of $\Phi_{ij}(\boldsymbol{\kappa})$ tends to the limit which depends on the orientation of the diminishing vector $\boldsymbol{\kappa}$. Just corresponding to this fact, the Loitsianskian integral

$$\int R_{ij}(\boldsymbol{r}, t)\, r_m\, r_n\, \mathrm{d}\boldsymbol{r} = 4\pi L^{mn}_{ij}(t), \text{ say,} \tag{2.9}$$

does exist as will be seen by repeating partial integrations of (2.7), although

$$\int |R_{ij}(\boldsymbol{r}, t) r_m r_n|\, \mathrm{d}\boldsymbol{r}$$

must diverge on account of the property that $R_{ij} r_m r_n \sim O(r^{-3})$ at large r. In addition, B & P pointed out the relation

$$\begin{aligned} 20C_{lmnp}(t) = &-\frac{92}{3} L^{np}_{lm} - \frac{8}{3} L^{lm}_{np} - \frac{20}{3}\left(L^{mp}_{ln} + L^{mn}_{lp} + L^{lp}_{mn} + L^{ln}_{mp}\right) \\ &+ \frac{23}{21}\delta_{lm}\left(L^{np}_{kk} - L^{kk}_{np}\right) + \frac{2}{21}\delta_{np}\left(L^{lm}_{kk} - L^{kk}_{lm}\right) \\ &+ \frac{1}{21}\delta_{mp}\left(23\,L^{ln}_{kk} - 65\,L^{kk}_{ln}\right) + \frac{1}{21}\delta_{mn}\left(23\,L^{lp}_{kk} - 65\,L^{kk}_{lp}\right) \\ &+ \frac{1}{21}\delta_{lp}\left(23\,L^{mn}_{kk} - 65\,L^{kk}_{mn}\right) + \frac{1}{21}\delta_{ln}\left(23\,L^{mp}_{kk} - 65\,L^{kk}_{mp}\right) \\ &+ \frac{1}{2}\left(\delta_{ln}\delta_{mp} + \delta_{lp}\delta_{mn}\right) L^{kk}_{ii}, \end{aligned} \tag{2.10}$$

C_{lmnp} as well as L^{np}_{lm} being obviously symmetric in the indices (l, m) and (n, p).

From the asymptotic laws (2.6), on the other hand, we arrive at

$$\overline{u_i u_k u_j'} = -\frac{5}{3}\, T^m_{ikl} \frac{\partial^3}{\partial r_j \partial r_l \partial r_m}\left(\frac{1}{r}\right) + O(r^{-5}) \tag{2.11}$$

at large r, and

$$\Upsilon_{(ik)j}(\kappa, t) = -\frac{\iota}{6\pi^2}\left(4T^l_{ikm} + T^m_{ikl}\right)\left(\delta_{jm} - \frac{\kappa\,\kappa_m}{\kappa^2}\right)\kappa_l + O(\kappa^2 \ln\kappa) \tag{2.12}$$ [3]

at small κ, where

$$T^l_{ikj}(t) = T^l_{kij}(t) = \frac{1}{4\pi}\int \overline{u_i u_k u_j'}\; r_l\, \mathrm{d}\boldsymbol{r}. \tag{2.13}$$

Again $\Upsilon_{(ik)j}(\boldsymbol{\kappa})$ is singular at $\boldsymbol{\kappa}=0$, and the moment (2.13) is not absolutely but conditionally convergent like L^{mn}_{ij} .

Further constraints on the forms of L^{mn}_{ij} and T^l_{ikj} will be imposed by the incompressibility of fluid, but we are not able to find them from the asymptotic forms obtained above, since they satisfy the solenoidal conditions (I; (4.3), (7.5)) by themselves. Nevertheless, as a matter of tensor algebra, it follows that the incompressibility demands

3) Since T^l_{ikj} is real, the leading term of $\Upsilon_{(ik)j}(\boldsymbol{\kappa})$ must be a pure imaginary. In I; p. 74, footnote (10), an erroneous statement has been made that (so far as the axisymmetric case is concerned) $\Upsilon_{(ik)j}(\boldsymbol{s};\, \boldsymbol{\kappa})$ and $\Upsilon_{i(kj)}(\boldsymbol{s};\, \boldsymbol{\kappa})$ are pure imaginaries. This is not the case; we have to suppose they are complex in general. The purely imaginary character of (2.12), on the contrary, is common to all kinds of homogeneous turbulence.

$$C_{lmmp}=0 \quad \text{and} \quad T^{l}_{ikl}=0, \quad \text{respectively.} \tag{2.14}$$

Next, with these asymptotic forms of elementary tensors, the equation (2.1) or its Fourier transform (I; (6.2)) yields after some manipulations

$$\left\{\frac{\mathrm{d}C_{lmnp}}{\mathrm{d}t}-\frac{2}{3}(4T^{p}_{lnm}+4T^{p}_{mnl}+T^{m}_{lnp}+T^{l}_{mnp})\right\}\left(\delta_{il}\nabla^2-\frac{\partial^2}{\partial r_i\partial r_l}\right)\left(\delta_{jm}\nabla^2-\frac{\partial^2}{\partial r_j\partial r_m}\right)\times\frac{\partial^2 r}{\partial r_n\partial r_p}=0, \tag{2.15}$$

or equivalently

$$\left\{\frac{\mathrm{d}C_{lmnp}}{\mathrm{d}t}-\frac{2}{3}\left(4T^{p}_{lnm}+4T^{p}_{mnl}+T^{m}_{lnp}+T^{l}_{mnp}\right)\right\}\left(\delta_{il}-\frac{\kappa_i\kappa_l}{\kappa^2}\right)\left(\delta_{jm}-\frac{\kappa_j\kappa_m}{\kappa^2}\right)\kappa_n\kappa_p=0, \tag{2.16}$$

for 'big eddies'. Our previous belief of the permanence of big eddies has been thus disproved, and the problem is found to be not less difficult than the general one, in so far as we cannot disregard the effects of higher order correlations.

In the final period of decay, however, the formal solution is still obtainable after the customary method (see [**1**] § 5.4). That is, neglecting the inertia and pressure terms, we have

$$\begin{aligned}\Phi_{ij}(\boldsymbol{\kappa},t)&=\Phi_{ij}(\boldsymbol{\kappa},t_f)\ \exp[-2\nu\kappa^2(t-t_f)]\\ &\sim\frac{1}{4\pi^2}C_{lmnp}\left(\delta_{il}-\frac{\kappa_i\kappa_l}{\kappa^2}\right)\left(\delta_{jm}-\frac{\kappa\ \kappa_m}{\kappa^2}\right)\kappa_n\kappa_p\cdot\exp[-2\nu\kappa^2(t-t_f)]\end{aligned} \tag{2.17}$$

as $(t-t_f)\to\infty$, where t_f is an initial instant for the final period, and $C_{lmnp}\equiv C_{lmnp}(t_f)$ is constant throughout this period. The corresponding expression of $R_{ij}(\boldsymbol{r},t)$ has been shown explicitly as well, from which the energy decay law

$$\overline{u_i u_j}\equiv R_{ij}(0,t)\ \propto\ [\nu(t-t_f)]^{-5/2} \tag{2.18}$$

is concluded in accordance with the older theory and experiments, whereas the shape of the longitudinal correlation coefficient is, contrary to the earlier prediction, not always Gaussian.

These are the principal outcomes of B & P's theory. Strictly speaking, they are relevant only to the case of turbulence which is homogeneous and moreover has convergent integral moments of velocity cumulants at some initial instant t_0. But, considering that any real field of turbulence is by no means homogeneous in the earliest stage of its history, such initial conditions would hardly be fulfilled in reality. Thus, B & P has proposed an extended interpretation that the large scale structure of fully-developed homogeneous turbulence (generated by a grid, say) is *statistically* the same as that of *virtual* field of turbulence which has been developed from those idealized initial conditions.[4] This is what B & P's ' hypothesis of convergent integral moments at an initial instant' implies. Evidently the hypothesis relates to the statistical nature of the phenomena, and on its physical background detailed discussions have been made by B & P themselves. Although we have not yet decisive experimental verifications in these respects at present, it seems quite unlikely that we could obtain some other plausible predictions than the above-

4) Accordingly, t_0 must be considered as a virtual time origin rather than a real instant.

mentioned one along the similar lines of approach, and so we shall refer to B & P's theory in the followings.

3. **The large scale structure of homogeneous axisymmetric turbulence.** For simplicity's sake, let us consider in $\boldsymbol{\kappa}$-space exclusively in this and next sections. Now, the axisymmetric form of the Loitsianskian integral L_{ij}^{mn} (2.9) is

$$L_{ij}^{mn}=L_1\delta_{ij}\delta_{mn}+L_2(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})+L_3s_is_j\delta_{mn}+L_4s_ms_n\delta_{ij}$$
$$+L_5(s_is_m\delta_{jn}+s_js_m\delta_{in}+s_is_n\delta_{jm}+s_js_n\delta_{im})+L_6s_is_js_ms_n, \tag{3.1}$$

in view of the indices-symmetry in (i, j) and (m, n), where s_i is the direction cosine of the axis of symmetry and $L_1, \dots, L_6$ are undetermined scalars. Substituting from (3.1) into (2.10), we have then

$$C_{lmnp}=C_1\delta_{lm}\delta_{np}+C_2(\delta_{ln}\delta_{mp}+\delta_{lp}\delta_{mn})+C_3s_ls_m\delta_{np}+C_4s_ns_p\delta_{lm}$$
$$+C_5(s_ls_n\delta_{mp}+s_ms_n\delta_{lp}+s_ls_p\delta_{mn}+s_ms_p\delta_{ln})+C_6s_ls_ms_ns_p, \tag{3.2}$$

in which

$$\left.\begin{aligned}
84\,C_1&=-140\,L_1-112\,L_2+5L_3-5L_4,\\
840\,C_2&=-875\,L_1-2170\,L_2+115\,L_3-197\,L_4+84\,L_5+21\,L_6,\\
84\,C_3&=-130\,L_3-10\,L_4-112\,L_5,\\
84\,C_4&=-25\,L_3-115\,L_4-112\,L_5,\\
420\,C_5&=-335\,L_3-71\,L_4-1148\,L_5-42\,L_6,\\
C_6&=-3\,L_6,
\end{aligned}\right\} \tag{3.3}$$

but the incompressibility condition (2.14) requires

$$C_1+4C_2+C_5=C_3+C_4+5C_5+C_6=0, \tag{3.4}$$

or, because of (3.3),

$$5L_1+10L_2+L_4+2L_5=5L_3+2L_4+14L_5+3L_6=0, \tag{3.5}$$

showing only four of C's or L's are independent.

On using (3.2) in (2.8), it turns out that near $\kappa=0$

$$4\pi^2\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})=-[C_1-(C_3-C_4)\mu^2-C_6\mu^4]\,\kappa_i\kappa_j+(C_1+C_4\mu^2)\kappa^2\delta_{ij}$$
$$+(C_3+C_6\mu^2)[\kappa^2s_is_j-\kappa\mu(s_i\kappa_j+\kappa_is_j)]+O(\kappa^3\ln\kappa). \tag{3.6}$$

The fact that there appear terms like $C_3\mu^2$, $C_4\mu^2$, $C_6\mu^4$, $C_6\kappa\mu^3$, which are of the form $C\kappa^m\kappa^n$ with $m<n$, is a manifestation of the singular character of $\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ at the origin mentioned in the last section (and hence $R_{ij}(\boldsymbol{s}\,;\boldsymbol{r})\sim O(r^{-3})$ at large r[5)]). On the other hand, the asymptotic expression of the vorticity spectrum tensor

$$\Omega_{ij}(\boldsymbol{\kappa})=\frac{1}{8\pi^3}\int\overline{\omega_i\omega_j'}\ \mathrm{e}^{-\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$
$$=(\kappa^2\,\delta_{ij}-\kappa_i\kappa_j)\ \Phi_{kk}(\boldsymbol{\kappa})-\kappa^2\Phi_{ji}(\boldsymbol{\kappa}) \tag{3.7}$$

5) This ensures the convergency of the integral II; (4.3) and hence of its inversion II; (2.7), justifying the premise remarked in II; p. 155, footnote (3).

becomes

$$4\pi^2\Omega_{ij}(\boldsymbol{s};\boldsymbol{\kappa})=-[(C_1+C_3)+(C_4+C_6)\mu^2]\kappa^2\kappa_i\kappa_j$$
$$+[(C_1+C_4\mu^2)+(C_3+C_6\mu^2)(1-\mu^2)]\kappa^4\delta_{ij}-(C_3+C_6\mu^2)[\kappa^4 s_i s_j-\kappa^3\mu(s_i\kappa_j+\kappa_i s_j)]$$
$$+O(\kappa^5\ln\kappa), \tag{3.8}$$

whose leading term is no longer singular in the above sense.

By the way, the energy spectrum functions (I; § 5(2)) and their analogues for the mean-square vorticity such that

$$\left.\begin{aligned}\frac{\overline{u_{//}^2}}{2}&=\int_0^\infty \mathcal{E}_{//}(\kappa)\mathrm{d}\kappa\,, \quad \frac{\overline{u_\perp^2}}{2}=\int_0^\infty \mathcal{E}_\perp(\kappa)\,\mathrm{d}\kappa,\\ \text{and}\qquad \frac{\overline{\omega_{//}^2}}{2}&=\int_0^\infty \mathcal{Q}_{//}(\kappa)\mathrm{d}\kappa\,, \quad \frac{\overline{\omega_\perp^2}}{2}=\int_0^\infty \mathcal{Q}_\perp(\kappa)\,\mathrm{d}\kappa\end{aligned}\right\} \tag{3.9}$$

are now found to be

$$\left.\begin{aligned}\mathcal{E}_{//}(\kappa)&=\frac{\kappa^4}{2\pi}\left(\frac{2}{3}C_1+\frac{8}{15}C_3+\frac{2}{15}C_4+\frac{8}{105}C_6\right)+O(\kappa^5\ln\kappa),\\ \mathcal{E}_\perp(\kappa)&=\frac{\kappa^4}{2\pi}\left(\frac{2}{3}C_1+\frac{1}{15}C_3+\frac{4}{15}C_4+\frac{3}{105}C_6\right)+O(\kappa^5\ln\kappa),\\ \mathcal{Q}_{//}(\kappa)&=\frac{\kappa^6}{2\pi}\left(\frac{2}{3}C_1+\frac{2}{15}C_4\right)+O(\kappa^7\ln\kappa),\\ \mathcal{Q}_\perp(\kappa)&=\frac{\kappa^6}{2\pi}\left(\frac{2}{3}C_1+\frac{5}{15}C_3+\frac{4}{15}C_4+\frac{7}{105}C_6\right)+O(\kappa^7\ln\kappa),\end{aligned}\right\} \tag{3.10}$$

at small κ, respectively. Note that $\kappa^2(\mathcal{E}_{//}+2\mathcal{E}_\perp)=\mathcal{Q}_{//}+2\mathcal{Q}_\perp$.

Turning to the third-order tensor, the integral moment (2.13) must be of the form

$$T^{\,l}_{ikj}=T_1\delta_{ik}\delta_{jl}+T_2(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj})+T_3 s_i s_k\delta_{jl}+T_4 s_j s_l\delta_{ik}$$
$$+T_5(s_i s_j\delta_{kl}+s_k s_j\delta_{il})+T_6(s_k s_l\delta_{ij}+s_i s_l\delta_{kj})+T_7 s_i s_k s_j s_l \tag{3.11}$$

in axisymmetric case, where $T_1,\cdots,T_7$ are time-dependent scalars satisfying

$$3T_1+2T_2+T_4=3T_3+2T_5+2T_6+T_7=0, \tag{3.12}$$

which corresponds to the second condition of (2.14). Then (2.12) yields, in comparison with I; (7.7),

$$\left.\begin{aligned}&6\pi^2\kappa^2\Upsilon_1\sim-5T_2,\quad \Upsilon_2\sim0,\quad 6\pi^2\Upsilon_3\sim5\mu T_4,\quad 6\pi^2\Upsilon_4\sim5\mu T_7,\\ &6\pi^2\kappa\Upsilon_5\sim5\mu(T_5+T_6),\quad 6\pi^2\Upsilon_6\sim-(4T_5+T_6),\end{aligned}\right\} \tag{3.13}$$

where the terms of $O(\kappa\ln\kappa)$ and of higher order are omitted. Consequently, we get from I; (7.9)

$$\left.\begin{aligned}\Gamma_1&\sim-\frac{5}{3\pi^2}\left[T_2+\mu^2(T_4+T_5+T_6)\right],\\ \Gamma_2&\sim\frac{\kappa^2}{3\pi^2}\left[5T_2+\mu^2(T_5+4T_6)\right],\\ \Gamma_3&\sim\frac{\kappa^2}{3\pi^2}\left[(4T_5+T_6)+5\mu^2T_7\right],\\ \Gamma_4&\sim\Gamma_5\sim\frac{5\kappa\mu}{6\pi^2}(T_4-\mu^2T_7),\end{aligned}\right\} \tag{3.14}$$

and from I; (7.11)

$$\Pi \sim \frac{\mu}{6\pi^2}(5T_4+8T_5+2T_6+5\mu^2 T_7). \tag{3.15}$$

It is readily seen that, so far as the largest eddies are concerned, $\Gamma_{ij}(s;\boldsymbol{\kappa})$ as well as $\Pi_{ij}(s;\boldsymbol{\kappa})$ is entirely real and is symmetric in the indices i, j by itself. Each of the inertia and the pressure forces, therefore, acts so as to preserve the symmetry condition of $\Phi_{ij}(s;\boldsymbol{\kappa})$.[6)]

Now, by inserting (3.2) and (3.11) into (2.16) we find

$$\left.\begin{aligned} &\frac{dC_1(t)}{dt}=\frac{20}{3}T_2(t), \qquad \frac{dC_3(t)}{dt}=\frac{16T_5(t)+4T_6(t)}{3}, \\ &\frac{dC_4(t)}{dt}=\frac{4T_5(t)+16T_6(t)}{3}, \quad \frac{dC_6(t)}{dt}=\frac{20}{3}T_7(t), \end{aligned}\right\} \tag{3.16}$$

but so far we have been able to do nothing more than recording mere equations. It seems that some kind of further hypothesis or principle is indispensable for the use of these equations.

Lastly, another way of representing the axisymmetric spectrum tensor $\Phi_{ij}(s;\boldsymbol{\kappa})$ introduced in I; § 8 is quoted. That is, the expression (3.6) is equivalent to take either

$$\left.\begin{aligned} &4\pi^2 Z_{\mathrm{I}}=C_1+C_4\mu^2+(1-\mu^2)(C_3+C_6\mu^2)+O(\kappa\ln\kappa), \\ &4\pi^2 Y_{\mathrm{I}}=-(C_3+C_6\mu^2)+O(\kappa\ln\kappa), \\ &\qquad (Z_{\mathrm{I}}\geqq 0, \quad Y_{\mathrm{I}}\geqq 0), \end{aligned}\right\} \tag{3.17}$$

or

$$\left.\begin{aligned} &4\pi^2 Z_{\mathrm{II}}=(C_1+C_4\mu^2)+\mu^2(1-\mu^2)(1+\mu^2)^{-1}(C_3+C_6\mu^2)+O(\kappa\ln\kappa), \\ &4\pi^2 X_{\mathrm{II}}=(1+\mu^2)^{-1}(C_3+C_6\mu^2)+O(\kappa\ln\kappa), \\ &\qquad (Z_{\mathrm{II}}\geqq 0, \quad X_{\mathrm{II}}\geqq 0). \end{aligned}\right\} \tag{3.18}$$

Thus the large-scale structure of the turbulence may be pictured as is composed of isotropic and transverse ones, if and only if $C_3=C_4<0$ and $C_6=0$.

4. Special cases and comparison with the older results. In this section we shall examine more closely the special case just mentioned above. Suppose that

$$C_3=C_4(=C, \text{ say}) \quad \text{and} \quad C_6=0, \tag{4.1}$$

the sign of C being arbitrary for a moment. Clearly, (4.1) is the only case where the singularity in the leading term of $\Phi_{ij}(s;\boldsymbol{\kappa})$ disappears (cf. (3.6)), and as is stressed by B & P, the term of order r^{-5} in the asymptotic expression of $R_{ij}(s;\boldsymbol{r})$ vanishes then identically. The simplified formulas are as follows: using (4.1) in (3.3) and (3.5), we have

$$L_1=-2L_2(=-C_1), \quad L_3=L_4=-2L_5(=-C), \quad L_6=0, \tag{4.2}$$

by means of which (3.1) and (3.6) reduce to

6) Generalizing, in view of the properties

$$\Phi_{ij}(\boldsymbol{\kappa})\sim\Phi_{ij}(-\boldsymbol{\kappa}) \text{ and } \Upsilon_{(ik)j}(\boldsymbol{\kappa})\sim-\Upsilon_{(ik)j}(-\boldsymbol{\kappa})$$

of the asymptotic forms (2.8) and (2.12), the largest eddies of homogeneous turbulence are supposed (according to B & P's theory) to have reflexional symmetry, which is not destroyed by each of the inertia and pressure forces.

$$L_{ij}^{mn}=L_1\Big[\delta_{ij}\delta_{mn}-\frac{1}{2}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})\Big]$$
$$+L_3\Big[s_is_j\delta_{mn}+s_ms_n\delta_{ij}-\frac{1}{2}(s_is_m\delta_{jn}+s_js_m\delta_{in}+s_is_n\delta_{jm}+s_js_n\delta_{im})\Big], \quad (4.3)$$

and

$$4\pi^2\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})\sim L_1\kappa_i\kappa_j-(L_1+\mu^2L_3)\kappa^2\delta_{ij}-L_3[\kappa^2s_is_j-\kappa\mu(s_i\kappa_j+\kappa_is_j)], \quad (4.4)$$

respectively. This form of $\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ would be persistent, so long as $\mathrm{d}C_3/\mathrm{d}t\equiv\mathrm{d}C_4/\mathrm{d}t$ and $\mathrm{d}C_6/\mathrm{d}t\equiv0$, namely $T_5\equiv T_6$ and $T_7\equiv0$ (see (3.16)). Then it follows from (3.11), (3.12) and (3.13) that

$$T_{ikj}^{l}=T_2\Big(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj}-\frac{2}{3}\delta_{ik}\delta_{jl}\Big)+T_4\Big(s_js_l\delta_{ik}-\frac{1}{3}\delta_{ik}\delta_{jl}\Big)$$
$$+T_5\Big(s_is_j\delta_{kl}+s_ks_j\delta_{il}+s_ks_l\delta_{ij}+s_is_l\delta_{kj}-\frac{4}{3}s_is_k\delta_{jl}\Big), \quad (4.5)$$

and

$$6\pi^2\Upsilon_{(ik)j}(\boldsymbol{s}\,;\boldsymbol{\kappa})\sim-\frac{10T_2}{\kappa^2}\kappa_i\kappa_k\kappa_j+5T_2(\kappa_i\delta_{kj}+\kappa_k\delta_{ij})-5T_4\mu^2\kappa_j\delta_{ik}$$
$$+5\kappa\mu[T_5(s_i\delta_{kj}+s_k\delta_{ij})+T_4s_j\delta_{ik}]-5T_2\Big[2\frac{\mu}{\kappa}(s_i\kappa_k\kappa_j+\kappa_is_k\kappa_j)-(s_i\kappa_ks_j+\kappa_is_ks_j)\Big], \quad (4.6)$$

together with

$$\frac{\mathrm{d}L_1(t)}{\mathrm{d}t}=-\frac{20}{3}T_2(t),\qquad \frac{\mathrm{d}L_3(t)}{\mathrm{d}t}=-\frac{20}{3}T_5(t). \quad (4.7)$$

In contrast with the case of $\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$, the singular character of the leading term of $\Upsilon_{(ik)j}(\boldsymbol{s}\,;\boldsymbol{\kappa})$, and hence the fact that $\overline{u_iu_ku_j'}\sim O(r^{-4})$ at large r, are still retained here.

It has been anticipated at the end of the last section that the present special case includes the following varieties:

(1) Isotropic turbulence ($C=0$, $C_1>0$, i.e. $4\pi^2Z_{\mathrm{I}}\sim C_1$, $Y_{\mathrm{I}}=0$)

This is the case already considered in B & P's original paper ([2] § 8). In their notation,

$$L=\overline{u^2}\int_0^\infty r^4f(r)\mathrm{d}r\quad(=-3L_1\ ;\ L_3=0), \quad (4.8)$$

and

$$T=-\frac{1}{30}(\overline{u^2})^{3/2}\lim_{r\to\infty}r^4k(r)\quad(=-2/3T_2\ ;\ T_4=T_5=0), \quad (4.9)$$

$f(r)$ and $k(r)$ being the well-known Kármán's double and triple correlation coefficients in isotropic turbulence. The quantity L has been long known as Loitsianski's invariant, which however is shown to be not an invariant but governed by the equation

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t}=-30\,T(t). \quad (4.10)$$

(2) Transverse turbulence ($C=-C_1<0$, i.e. $Z_{\mathrm{I}}=0$, $4\pi^2Y_{\mathrm{I}}\sim-C$)

In this case, no axial components of velocities ($u_{//}$) are present (see I; § 8). It is interesting that here too the energy spectrum tensor is analytic at the origin and the velocity covariance tends to zero as $r\to\infty$ at a rate faster than $O(r^{-5})$. Further discussions on this model are reserved for the later section (§ § 6, 7).

As a general matter of fact, we may suppose the big eddies are composed of (1) and (2) when $C<0$, but, when $C>0$, such a simple picture is no longer possible.

To end this section, let us recall the customary analysis a little while. The previous assumption of the analytic energy spectrum tensor has been that in place of (2.8)

$$4\pi^2 \Phi_{ij}(\boldsymbol{\kappa}) = - L_{ij}^{mn} \kappa_m \kappa_n + O(\kappa^3), \tag{4.11}$$

near $\boldsymbol{\kappa}=0$, which requires

$$L_{ij}^{mn} + L_{im}^{nj} + L_{in}^{jm} = 0 \tag{4.12}$$

in view of the solenoidality of $\Phi_{ij}(\boldsymbol{\kappa})$ (see [1] § 3.1). If L_{ij}^{mn} is of the axisymmetric form (3.1) in particular, the condition (4.12) yields the same relations as (4.2), and consequently the older asymptotic expression of $\Phi_{ij}(\boldsymbol{s};\boldsymbol{\kappa})$ is identical with (4.4). It is, therefore, a special case of our revised expression (3.6), or the latter is a generalization of the former.[7] In other words, the kinematical restrictions for the large-scale structure due to the incompressibility of fluid are found to be less severe than have been predicted before.

On the other hand, assuming that $\Upsilon_{(ik)j}(\boldsymbol{\kappa})$ is analytic, we have

$$2\pi^2 \Upsilon_{(ik)j}(\boldsymbol{\kappa}) = -\iota T_{ikj}^{l} \kappa_l + O(\kappa^2), \tag{4.13}$$

for which

$$T_{ikj}^{l} + T_{ikl}^{j} = 0 \tag{4.14}$$

is the solenoidal condition (see [1] § 5.3). Thus (3.11) reduces to

$$T_{ikj}^{l} = T_5(s_i s_j \delta_{kl} + s_k s_j \delta_{il} - s_i s_l \delta_{kj} - s_k s_l \delta_{ij}), \tag{4.15}$$

and (4.13) to

$$2\pi^2 \Upsilon_{(ik)j}(\boldsymbol{s};\boldsymbol{\kappa}) = -\iota T_5 [s_i \kappa_k s_j + \kappa_i s_k s_j - \kappa\mu(s_i \delta_{kj} + s_k \delta_{ij})]. \tag{4.16}$$

The corresponding dynamical relations are, unlike (4.7),

$$\frac{\mathrm{d}L_1}{\mathrm{d}t} = 8T_5\mu^2 \quad \text{and} \quad \frac{\mathrm{d}L_3}{\mathrm{d}t} = -4T_5, \tag{4.17}$$

but since L_1 as well as T_5 cannot depend on μ, T_5 must be identically zero, showing the permanence of big eddies (L_1=const., L_3=const.) in the previous theory. So we have to bear in mind that the coincidence between (4.4) and the older

7) Rewriting (3.5) as

$$(5L_1+10L_2)+(L_4+2L_5)=0 \quad \text{and} \quad (5L_3+10L_5)+(2L_4+4L_5)+(3L_6)=0,$$

it is seen at once that (4.2) can be obtained by equating each of the expressions within the round brackets to zero.

result is only apparent except in the final period of decay.

5. The final period of decay of axisymmetric turbulence. In the final period of decay, the defining functions of $\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ can be represented by

$$\left.\begin{aligned}&4\pi^2\Phi_1=-[C_1+C_4\mu^2-(C_3+C_6\mu^2)\mu^2]\cdot\exp[-2\nu\kappa^2(t-t_f)],\\ &\text{and}\\ &4\pi^2\Phi_3=(C_3+C_6\mu^2)\cdot\exp[-2\nu\kappa^2(t-t_f)],\end{aligned}\right\}\tag{5.1}$$

where C_1, C_3, C_4, C_6 are constants (cf. (2.17), (3.6)). From now on, we shall simply write t instead of $t-t_f$.

The expressions (5.1) show that even in the final period Φ_1 and Φ_3 depend on μ in general and the same is also the case for Z_1 and Y_1 (see (3.17)). According to II ; § 5, this means that Chandrasekhar's correlation functions q_1 and q_2 depend on $m(=\boldsymbol{r}\cdot\boldsymbol{s}/r)$, such simple relations as II ; (5.11), II ; (5.12) being not available. Some years ago Chandrasekhar [3] came to the contrary conclusion that as $t\to\infty$, q_1 and q_2 *necessarily* become independent of m.[8] What is the origin of this discrepancy ?

Chandrasekhar assumed the series expansions like

$$\left.\begin{aligned}&q_1(r,m,0)=\sum_{n=0}^{\infty}q_{2n}^{(1)}(r)\,C_{2n}^{3/2}(m),\\ &\text{and}\\ &q_2(r,m,0)=\sum_{n=0}^{\infty}q_{2n}^{(2)}(r)\,C_{2n}^{3/2}(m),\end{aligned}\right\}\tag{5.2}$$

where $C_{2n}^{3/2}(m)$ is the Gegenbauer polynomial of order $2n$. Then the final period equations for q_1 and q_2 permit the aysmptotic solution (not containing m)

$$q_1(r,m,t)\sim\frac{-\Lambda_1}{48\sqrt{2\pi}(\nu t)^{5/2}}\exp[-r^2/8\nu t],\quad q_2(r,m,t)\sim\frac{-\Lambda_2}{48\sqrt{2\pi}(\nu t)^{5/2}}\exp[-r^2/8\nu t],\tag{5.3}$$

at large t, so long as

$$\Lambda_1=-\int_0^\infty q_0^{(1)}(r)r^4dr\quad\text{and}\quad\Lambda_2=-\int_0^\infty q_0^{(2)}(r)r^4dr\tag{5.4}$$

exist. The mathematical procedure is quite similar to that of heat conduction in which the conditions of the type (5.4) are usually justified from the physical point of view. In the present problem, however, the state of affairs is somewhat different. For, noting

$$q_0^{(1)}(r)=\frac{3}{4}\int_{-1}^{1}q_1(r,m,0)(1-m^2)dm,\quad q_0^{(2)}(r)=\frac{3}{4}\int_{-1}^{1}q_2(r,m,0)\,(1-m^2)dm,\tag{5.5}$$ [9]

$q_0^{(1)}$ and $q_0^{(2)}$, together with q_1 and q_2, must be of the order r^{-5} at large r as

8) Here q_1 and q_2 stand for Q_1 and Q_2 in the original paper [3]. Chandrasekhar used the symbols q_1, q_2 in another sense. (ibid. p. 362).

9) $\int_{-1}^{1}C_m^{3/2}(x)C_n^{3/2}(x)(1-x^2)dx=\delta_{mn}\dfrac{(n+2)(n+1)}{n+(3/2)}$; $C_0^{3/2}(x)\equiv1$ (e.g. [4] pp. 174–175).

$R_{ij}(s\,;r)$ itself; the existence of Λ_1 and Λ_2 is thus not guaranteed by the *conditional* convergence of the Loitsianskian integral L_{ij}^{mn}. This shows the limitation of the heat conduction analogy in turbulence problems. The method of sources used by Chandrasekhar, i.e. the method of superposing the elementary contributions due to a continuous distribution of sources seems to be successful only when the total heat is absolutely convergent.

Under these circumstances, it is clear that the earlier result (5.3) corresponds to the special case considered in § 4. When (4.1) is assumed, we have

$$Z_{\mathrm{I}}=-(L_1+L_3)\exp[-2\nu\kappa^2t]\Big/4\pi^2 \text{ and } Y_{\mathrm{I}}=L_3\exp[-2\nu\kappa^2t]\Big/4\pi^2, \tag{5.6}$$

for the final period. Inserting these into II; (5.11) and comparing with (5.3), we find

$$\Lambda_1=-32(L_1+L_3), \qquad \Lambda_2=32L_3, \tag{5.7}$$

particularly in that case.

Let us now turn to the correlation functions. In [2] B & P have derived the general expression of $R_{ij}(r,t)$ equivalent to (2.17), but here we shall make use of II; (2.9) and II; (2.10) to evaluate the two representative longitudinal correlation coefficients $f_{//}$ and $f_{\perp}$ (see II; (6.2) and II; Fig. 2) for the sake of simplicity. From (5.1) in this way, we arrive at

$$\overline{u_{//}^2}f_{//}(\xi)=\frac{3}{64}\sqrt{\frac{2}{\pi}}(\nu t)^{-5/2}\Big[\Big(\frac{C_3-C_4-2C_6}{\xi^5}+\frac{15C_6}{2\xi^7}\Big)\sqrt{\pi}\ \mathrm{erf}\ \xi$$
$$+\Big\{\frac{2}{3}(C_1+C_4)-\frac{4}{3\xi^2}(C_3-C_4+C_6)-\frac{2}{\xi^4}(C_3-C_4+3C_6)-\frac{15}{\xi^6}C_6\Big\}\exp(-\xi^2)\Big]\ , \tag{5.8}$$

and

$$\overline{u_{\perp}^2}f_{\perp}(\xi)=\frac{3}{64}\sqrt{\frac{2}{\pi}}(\nu t)^{-5/2}\Big[-\Big(\frac{2C_3-2C_4+3C_6}{4\xi^5}-\frac{45C_6}{16\xi^7}\Big)\sqrt{\pi}\ \mathrm{erf}\ \xi$$
$$+\Big\{\frac{2}{3}\Big(C_1+\frac{C_3}{2}\Big)+\frac{2}{3\xi^2}\Big(C_3-C_4-\frac{3}{4}C_6\Big)+\frac{1}{\xi^4}\Big(C_3-C_4-\frac{9}{4}C_6\Big)-\frac{45}{8\xi^6}C_6\Big\}\exp(-\xi^2)\Big], \tag{5.9}$$

where

$$\xi=r/\sqrt{8\nu t}\,,\quad \sqrt{\pi}\ \mathrm{erf}\ \xi=2\int_0^{\xi}\exp\,(-x^2)\,dx, \tag{5.10}$$

and

$$\left.\begin{aligned}\overline{u_{//}^2}&=\frac{3}{64}\sqrt{\frac{2}{\pi}}(\nu t)^{-5/2}\Big(\frac{2}{3}C_1+\frac{8}{15}C_3+\frac{2}{15}C_4+\frac{8}{105}C_6\Big),\\ \overline{u_{\perp}^2}&=\frac{3}{64}\sqrt{\frac{2}{\pi}}(\nu t)^{-5/2}\Big(\frac{2}{3}C_1+\frac{1}{15}C_3+\frac{4}{15}C_4+\frac{3}{105}C_6\Big),\end{aligned}\right\} \tag{5.11}$$

(cf. (3.10)). The correlation coefficients are self-preserving in ξ, and are characterized by the ratios

$$C_1 : C_3 : C_4 : C_6. \tag{5.12}$$

Furthermore, it is convenient to relate them to another set of parameters of physical significance. The choice of these parameters is rather arbitrary, but the following three may be appropriate for the time being:

$$\left.\begin{array}{lll} \text{(i)} & \text{the energy partition ratio} & e=\overline{u_{//}^2}/\overline{u_{\perp}^2}, \\ \text{(ii)} & \text{the vorticity partition ratio} & d=\overline{\omega_{//}^2}/\overline{\omega_{\perp}^2}, \\ \text{(iii)} & \text{the length-scale ratio} & l=L_{//}/L_{\perp}. \end{array}\right\} \tag{5.13}$$

Here let $L_{//}$ and $L_{\perp}$ be the longitudinal integral scales represented by

$$\left.\begin{aligned} L_{//}&=\sqrt{8\nu t}\int_0^\infty f_{//}(\xi)\,d\xi=\frac{\sqrt{8\pi\nu t}}{3\overline{u_{//}^2}}(C_1+C_3), \\ L_{\perp}&=\sqrt{8\nu t}\int_0^\infty f_{\perp}(\xi)\,d\xi=\frac{\sqrt{8\pi\nu t}}{3\overline{u_{\perp}^2}}(C_1+\frac{1}{2}C_4). \end{aligned}\right\} \tag{5.14}$$

Then in our case it turns out that

$$\left.\begin{aligned} e&=\frac{70C_1+56C_3+14C_4+8C_6}{70C_1+7C_3+28C_4+3C_6}, \\ d&=\frac{10C_1+2C_4}{10C_1+5C_3+4C_4+C_6}, \\ l&=\frac{1}{e}\times\frac{2C_1+2C_3}{2C_1+C_4}, \end{aligned}\right\} \tag{5.15}$$

respectively. The structure of homogeneous axisymmetric turbulence in the final period is thus specified in terms of the values of the three constants e, d and l.

6. Additional remarks and examples. Prior to the illustrative examples of $f_{//}$ and $f_{\perp}$, the special case of § 4 is to be reexamined in the present connexion. That is on assuming $C_3=C_4$; $C_6=0$ (4.1), we have

$$f_{//}(\xi) = f_{\perp}(\xi) = \exp(-\xi^2) \tag{6.1}$$

together with

$$e=\frac{1+(C/C_1)}{1+(C/2C_1)}, \quad d=\frac{1+(C/5C_1)}{1+(9C/10C_1)}, \quad l=1, \tag{6.2}$$

from which

$$d=\frac{8-3e}{1+4e} \quad \text{or} \quad e=\frac{8-d}{3+4d} \tag{6.3}$$

is obtained. The essential features of this case can be summarized as follows:

(1) The longitudinal correlation coefficient is of Gaussian form in agreement with the older prediction.

(2) Fig. 1 shows the relation (6.3), e and d being not independent each

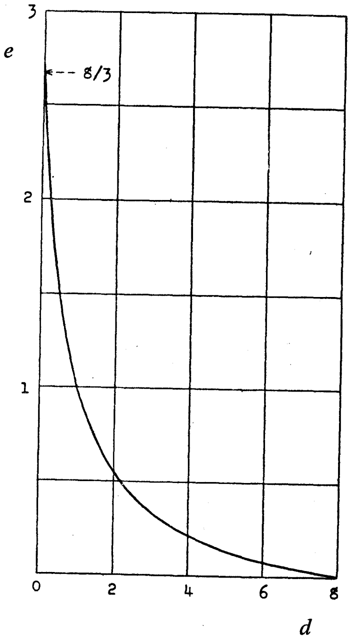


Fig. 1.

other. Moreover, since they cannot be negative by definition, their possible values must be confined within the ranges

$$0 \leq e < 8/3 \quad \text{and} \quad 0 \leq d < 8. \tag{6.4}$$

(3) $d \gtreqless 1$ according as $e \gtreqless 1$, and *vice versa*.

It has been pointed out by B & P [2] that in isotropic turbulence the final shape of the longitudinal correlation coefficient is Gaussian notwithstanding the considerable modifications of the customary theory. As is seen above, however, the similar possibility is not excluded in axisymmetric turbulence as well. At the same time, (5.8) and (5.9) indicate that no other case than (4.1) has such a property. If the observed f-profile of axisymmetric turbulence is Gaussian in the final period, therefore, the field in question may be supposed as a superposition of isotropic and transverse ones (provided that $e<1$). On the contrary, we can readily prove that whether $C_6=0$ or not the relation (6.3) between e and d is equally valid so long as $C_3=C_4$, namely (6.3) is not a sufficient condition of (4.1). Nevertheless it will be also interesting to plot the measured values of e and d on Fig. 1.

Figs. 2 to 5, on the other hand, visualize some examples of theoretical f-

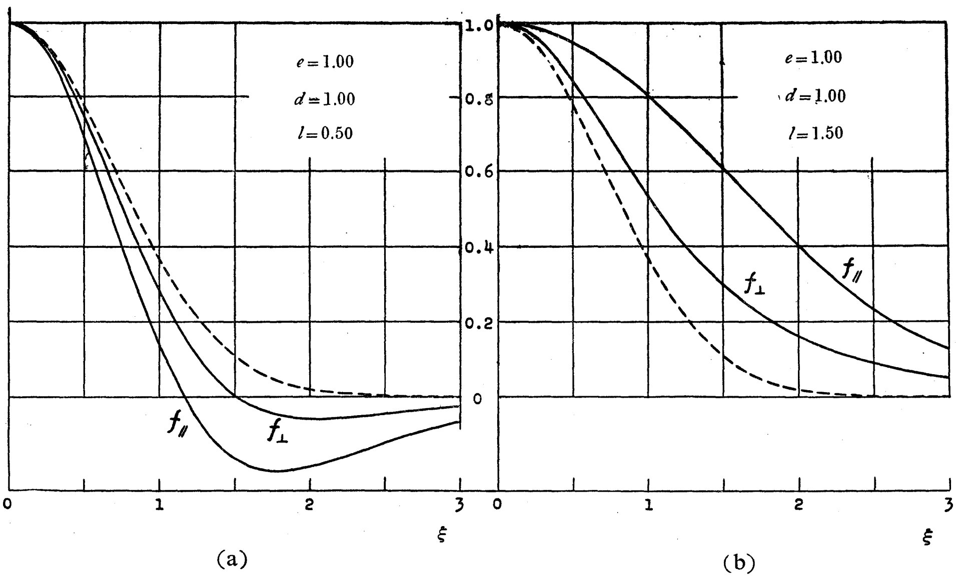


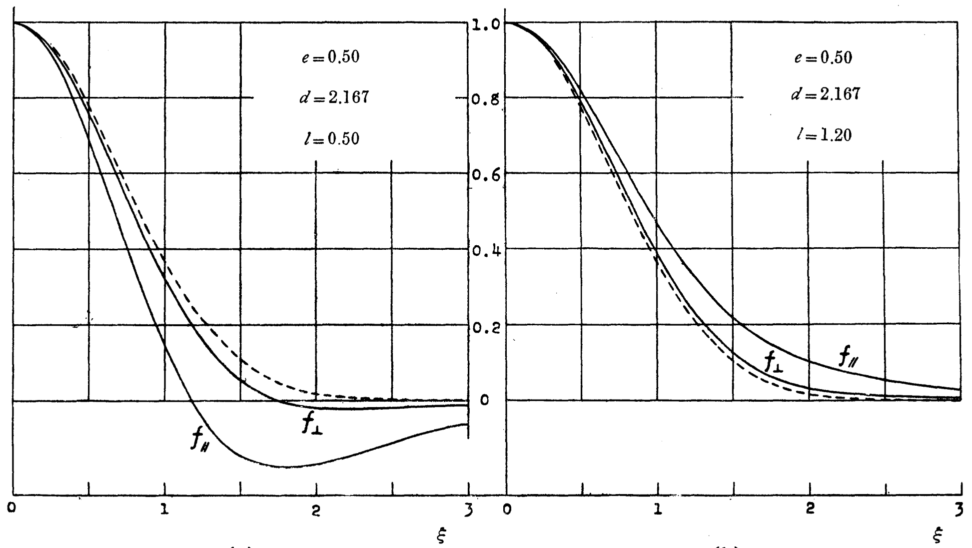
Fig. 2.

profile computed through (5.8) and (5.9) for more general values of structural parameters. Of course, the primary purpose of those figures is not the systematic representation of all cases but mere illustrations.

Fig. 2 corresponds to somewhat artificial cases where $e=d=1$ as in isotropic turbulence but $l\neq1$.

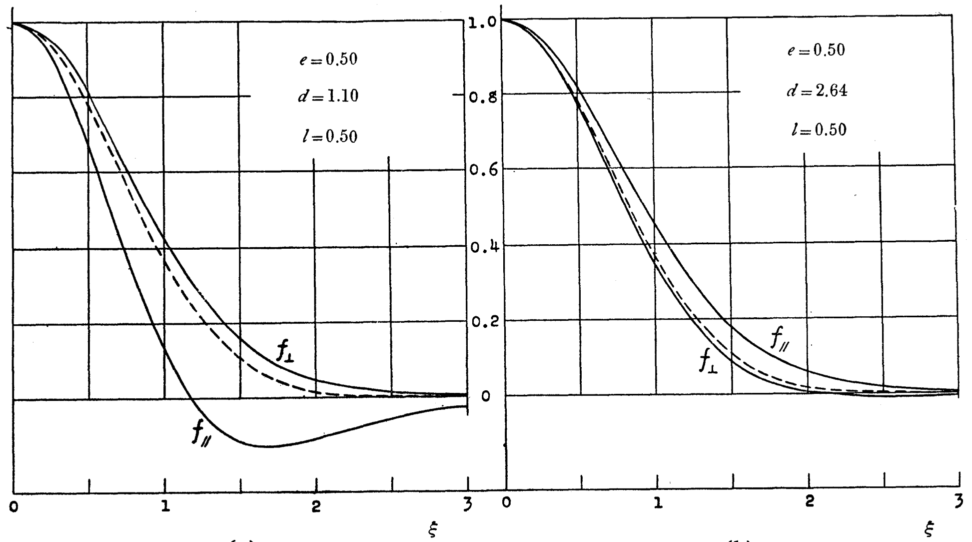
Whereas in Fig. 3 another set of e- and d-values is chosen.

In Fig. 4 the effect of changing the d-value is illustrated, the e- and l-values



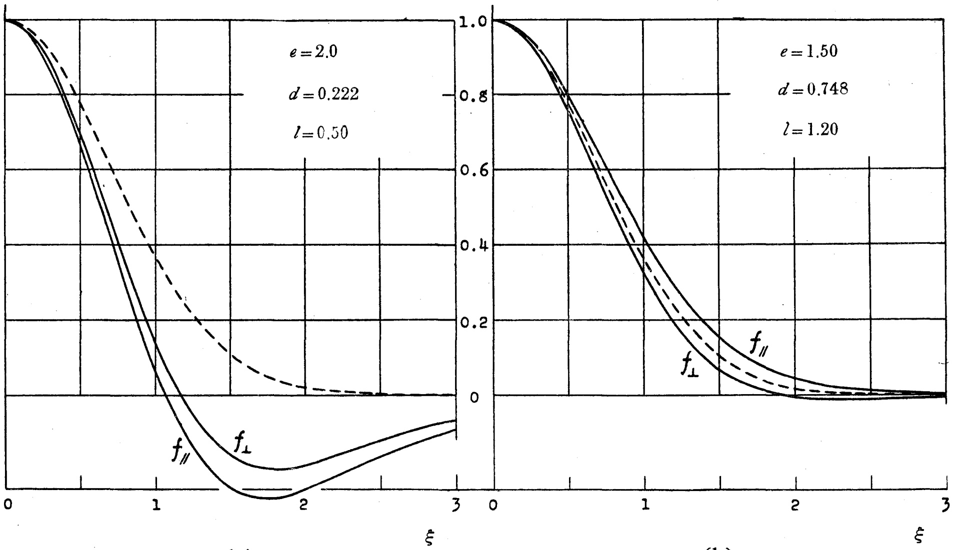
(a) (b)

Fig. 3.



(a) (b)

Fig. 4.



(a) (b)

Fig. 5.

being the same as in Fig. 3.

Fig. 5 shows the cases of $e>1$.

In contrast with the Gaussian profile (drawn in a broken line), these newly computed f-functions sometimes become negative for a certain range of separations. Although no observation has supported this up to the present, the final check will be a matter of future investigations.

7. Chandrasekhar's special model of axisymmetric turbulence and related discussions.

Chandrasekhar [3] has pointed out the formal possibility of a certain special state of axisymmetric turbulence which may be pictured as a superposition of two non-interacting fields represented by his correlation functions q_1 and q_2 respectively, and he wrote, 'it would be of interest to know whether this example of axisymmetric turbulence is merely a mathematical curiosity or has a deeper physical meaning.' Being suggested from this, we shall make a few further considerations here.

In Chandrasekhar's special model, q_1 is required to be independent of m, and q_2 to be free from the effects of higher order correlations. This implies, after our preceding analysis (I; §§ 8, 9 and II; § 5), that the turbulence is composed of an isotropic field (q_1 in $\boldsymbol{r}$-space, or Z_{I} in $\boldsymbol{\kappa}$-space) and a transverse one (q_2 or Y_{I}),[10] where the latter is not affected by neither inertial nor pressure forces just as in the final period of decay. The dynamical equations in $\boldsymbol{\kappa}$-space (I; (8.5)) are then

$$\frac{\partial Z_{\mathrm{I}}}{\partial t}+2\nu\kappa^2 Z_{\mathrm{I}}=\frac{\Gamma_2}{\kappa^2}\,,\qquad \frac{\partial Y_{\mathrm{I}}}{\partial t}+2\nu\kappa^2 Y_{\mathrm{I}}=0. \tag{7.1}$$

10) In II; § 5(2), the author proved the correspondence $q_1 \leftrightarrows Z_{\mathrm{I}}$, $q_2 \leftrightarrows Y_{\mathrm{I}}$ on assuming that Z_{I} and Y_{I} are independent of μ. But as a matter of fact the μ-independence of Y_{I} is not necessary in that proof at all.

To be somewhat generalized, Z_{I} may depend on μ in the present formulation—— both fields of Z_{I} and Y_{I} are non-interacting if only $\Gamma_3=0$. In addition, the equations I; (8.5) will assume another simplified form

$$\frac{\partial Z_{\mathrm{I}}}{\partial t}+2\nu\kappa^2 Z_{\mathrm{I}}=0, \qquad \frac{\partial Y_{\mathrm{I}}}{\partial t}+2\nu\kappa^2 Y_{\mathrm{I}}=-\frac{\Gamma_3}{\kappa^2}, \tag{7.2}$$

when

$$\Gamma_2+(1-\mu^2)\Gamma_3=0 \tag{7.3}$$

is satisfied. Thus the problem is to discuss the physical aspects of these special equations.

First of all, it is not likely that the transverse trubulence is persistent by itself in the presence of pressure forces which would transfer energy between different velocity components. For instance, let us suppose a field of transverse turbulence were set up in some way (at least conceptually) at an instant $t=0$, say. Then by definition

$$[u_{//}]_0=0 \tag{7.4}$$

everywhere (the suffix 0 indicates the initial value), and it follows

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\overline{(u_{//}^2)}\right]_0=2\left[\overline{u_{//}\frac{\partial u_{//}}{\partial t}}\right]_0=0, \tag{7.5}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}\overline{(u_{//}^2)}\right]_0=2\left[\overline{\left(\frac{\partial u_{//}}{\partial t}\right)^2}+\overline{\left(u_{//}\frac{\partial^2 u_{//}}{\partial t^2}\right)}\right]_0=2\left[\overline{\frac{\partial u_{//}}{\partial t}}\right]_0^2, \tag{7.6}$$

and so on. On the other hand, differentiating (2.1) with respect to t and making use of (7.4), we readily find

$$\left[\frac{\partial^2\overline{u_{//}u'_{//}}}{\partial t^2}\right]_0=\left[\frac{\partial^2 R_{ij}}{\partial t^2}\right]_0 s_i s_j=\frac{1}{\rho}\frac{\partial}{\partial s}\left[\overline{p\frac{\partial u_{//}'}{\partial t}}-\overline{\frac{\partial u_{//}}{\partial t}p'}\right]_0, \tag{7.7}$$

($\partial/\partial s=s_i\partial/\partial r_i$) which does not vanish in general even at $\boldsymbol{r}=0$. In view of (7.6), therefore, $[\partial u_{//}/\partial t]_0$ is not zero, showing that axial velocity component will develop at times subsequent to $t=0$ as is expected. Accordingly, unless the motion of transverse turbulence is so slow as the pressure term or, in effect, the triple correlations can be neglected (the case of (7.1)), the non-interacting model would be unnatural.

However, it may be worth-while to note that because of (7.4) and (7.5), a Taylor's series expansion of $\overline{u_{//}^2}=2E_{//}$ at $t=0$ begins from the order of t^2, i.e.,

$$\overline{u_{//}^2}=\alpha t^2+O(t^3)\,; \qquad \alpha=\overline{[\partial u_{//}/\partial t]_0^2}\,. \tag{7.8}$$

The value of α will be determined by specifying the detailed statistical properties of the initial motion which may depend on the mechanism of turbulence-producing device. Without knowledge of these particulars, we cannot proceed beyond (7.8), so that we shall make a reference to the relevant work of G. I. Taylor & A. E. Green [**8**] for the time being. Starting from the initial velocity field of purely periodic and monochromatic pattern, these authors calculated the decay and modulation of energy in terms of an iteration process. After them let the initial velocity components be of the form

$$\left.\begin{aligned} [u_1]_0 &= 0, \\ [u_2]_0 &= V\cos kr_1 \sin kr_2 \sin kr_3, \\ [u_3]_0 &= -\ V\sin kr_1 \cos kr_2 \sin kr_3, \end{aligned}\right\} \tag{7.9}$$

where V and k are constants, and $u_1 \equiv u_{//}$. Introducing now the dimensionless quantities

$$R = V/k\nu, \quad T = Vkt, \quad \tau = 3k^2\nu t,$$

and putting $u_0^2 \equiv V^2/8 (= \overline{[u_2]_0^2} = \overline{[u_3]_0^2})$, in which the spatial mean values are to be taken, their essential result can be expressed as

$$\frac{\overline{u_1^2}}{u_0^2} = O(R^2\tau^2)\ ; \quad \frac{\overline{u_2^2}}{u_0^2} = \frac{\overline{u_3^2}}{u_0^2} = e^{-2\tau} + O(R^2\tau^2), \tag{7.10 a}$$

or alternatively

$$\frac{\overline{u_1^2}}{u_0^2} = \frac{T^2}{16} - \frac{7T^3}{8R} + O(T^4)\ ; \quad \frac{\overline{u_2^2}}{u_0^2} = \frac{\overline{u_3^2}}{u_0^2} = 1 - \frac{T^2}{32} + \frac{11T^3}{48R} + O(T^4). \tag{7.10 b}$$

In spite of the non-rondom, non-statistical character of this problem, it has been demonstrated by T. Tatsumi [7] that his elaborated calculations based on the quasi-normal probability distribution of the velocity fluctuations (for the isotropic case) lead to a expression of the total energy decay completely similar to Taylor-Green's one apart from the numerical coefficients included there. Considering this, we may interprete the expressions (7.10) in the following way:

(a) When the Reynolds number R is sufficiently small, we can neglect the terms of $O\ (R^2\tau^2)$ in (7.10 a), and only the viscous effect would predominate as has been stated above.

(b) When R is not so small and further $T \ll 1$, (7.10 b) suggests that $\overline{u_{//}^2}$ is still negligible in comparison with $\overline{u_\perp^2}$. The condition $T \ll 1$ corresponds to $t \ll (Vk)^{-1}$ where V and k^{-1} stand for the characteristic velocity and scale, respectively, of initial turbulent motion; in other words, during a period which is short compared with its characteristic time, the energy of transverse turbulence would be approximately 'frozen' within the transverse plane, and we can speak of isolated transverse turbulence in such a limited sense.[11)]

Consider now the state of axisymmetric turbulence where $u_{//}$ together with its Fourier components remain small quantities of the first-order represented simply by ε. Then we have

$$\overline{u_{//}u_k u_j'} \sim \overline{u_i u_{//} u_j'} \sim \overline{u_i u_k u'_{//}} \sim \varepsilon,$$

or

$$\Upsilon_{(ik)j} s_i \sim \Upsilon_{(ik)j} s_k \sim \Upsilon_{(ik)j} s_j \sim \varepsilon, \tag{7.11}$$

11) Under the present condition, the flow of energy between different wave-numbers (the energy cascade as it is called) too must be comparatively small, but it will be of the order $O(T)$ rather than $O(T^2)$ for the wave-number range over which the initial energy distributes.

from which the third-order spectrum functions are found to be written as

$$\left.\begin{array}{lll} \Upsilon_1 \sim (1-\mu^2)\Upsilon, & \Upsilon_2 \sim -2\mu\Upsilon, & \Upsilon_3 \sim \varepsilon, \\ \Upsilon_4 \sim -2\kappa^2\mu\Upsilon, & \Upsilon_5 \sim 2\kappa\mu\Upsilon, & \Upsilon_6 \sim -\kappa^2(1+\mu^2)\Upsilon, \end{array}\right\} \tag{7.12}$$

in virtue of I; (7.7) $\Upsilon(\kappa, \mu)$ being a single defining scalar. The corresponding transfer functions are

$$\left.\begin{array}{ll} \Gamma_1 \sim 2\kappa^2(1-\mu^2)\Upsilon^e, & \Gamma_2 \sim -\kappa^2(1-\mu^3)\Gamma_1, \\ \Gamma_3 \sim \kappa^2\Gamma_1, & \Gamma_4 \sim \Gamma_5 \sim -\kappa\mu\Gamma_1, \end{array}\right\} \tag{7.13}$$

so that

$$\Gamma_2 + (1-\mu^2)\Gamma_3 \sim \varepsilon,$$

and

$$\Pi \sim \varepsilon \tag{7.14}$$

is also obtained (cf. I; (7.9) and I; (7.11)). The above-mentioned fact (7.8) that the pressure forces produce the second order effect only is thus explicitly proved. Hence, for instance, the generation of r^{-5}-law forms of velocity covariance, and the change of Loitsianskian integral (see §§ 2, 4) must be of the order ε^2 in these circumstances.

Lastly, can this (at least *nearly*) transverse field of turbulence be non-interacting with an isotropic field superposed upon it? In general, we are not able to expect such a possibility, so long as there exist certain non-linear coupling terms between them. However, if the scales of isotropic and transverse turbulence are effectively separated, the two respective motions can be considered as statistically almost independent each other, and the neglect of couplings may be a plausible approximation in a sense. Namely, for the isotropic range of wave-numbers the equation (7.1), while for the transverse range the equation (7.2) should be applied. Whether this is the case or not in reality will need further investigations.

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