ON THE DAMPING FORCE AND ADDED MASS OF SHIPS HEAVING AND PITCHING

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ON THE DAMPING FORCE AND ADDED MASS OF SHIPS HEAVING AND PITCHING

By Fukuzō TASAI

ABSTRACT. When cylinders of Lewis form sections heave and pitch on the free surface, their damping force and the added mass were exactly calculated by Ursell's method. And many figures in this paper show the results of the calculation. Then the author calculated, applying the results of the above-mentioned calculation and by Strip Method, the damping force and the added mass of the two ships which were respectively put to test by Golovato and Gerritsuma. The added mass and the added moment of inertia which the reporter gained by Strip Method showed good coincidence with the results of Golovato's and Gerritsuma's experiments. To get better estimation on the damping force by Strip Method, it is necessary to calculate more closely the three dimensional effect, non-linear effect, etc. As to the added mass the three dimensional correction could be negligible except for small ξ^*_1.

I. Introduction

On the evaluation of the damping force and the added mass of a ship heaving or pitching, the three-dimensional method is extremely complicated, involves many assumptions, and also it is impracticable. Therefore the two-dimensional approximate calculation by means of the Strip Method has been generally used. To do this, the values for cylinders having section contours very similar to ship forms are necessary. T. Havelock [1] calculated the damping force by means of the Source Method. In the Golovato's experiment [2], It is clear that the values by the Source Method differ considerably from the experimental values. F. Ursell [3] exactly calculated the damping force and the added mass for the semi-circular cylinder, and O. Grim [4] also calculated for cylinders having Lewis form sections by different method. Then O. Grim derived an approximate method. Values of the added mass obtained by O. Grim are given in several figures [4]. But it seems that the O. Grim's values are doubtful. K. Kroukovsky [5] used coefficient K_4 as the free surface correction and suggested that K_4 was 0.75 for heaving and 1.20 for pitching, then in [6] he used the Ursell's K_4 of the semi-circular cylinder in calcualting the sectional added mass of a ship. Ass. Prof. Nakamura [7] employed the same method. In this paper, using Ursell's exact method, the author calculated the progressive wave height and added mass of cylinders of Lewis form sections heaving on the free surface. Then using the results of these exact calculations and Strip Method, the author calculated the damping force and the added mass of the two ships which had been respectively put to test by Golovato [2] and Gerritsuma [8]. Now the author is carrying the measurement of two-dimensional progressive wave hights by forced heaving of miscellanous cylinders. These results will be reported in the near future.

2. Calculation of progressive wave height and added mass produced by oscillatory heaving of cylinders.

2.1 Boundary conditions and fundamental conditions

An infinitely long cylinder having such a section contour in the z-plane of Fig. 1 is immersed in a fluid of infinite depth with its axis in the free surface. When the cylinder is given a forced heaving oscillation of small amplitude about its initial position, it produces waves of two Kinds. One is standing wave rapidly diminishing in amplitude as the distance from the cylinder increases, and the other is regular progressive wave. Therefore hydro-dynamic force newly works to the cylinder. This force separates into two forces, namely inertial force and damping force. The former causes to the added mass of the cylinder. As the cylinder is in finitely long, the motion becomes two-dimensional. And also it is symmetrical about the y-axis which is positive downward. Viscosity and surface tension will be neglected, then a velocity potential ϕ and a stream function ψ exist which respectively satisfies the Laplace equation. Neglecting the second order terms, the free surface condition is expressed as follows:

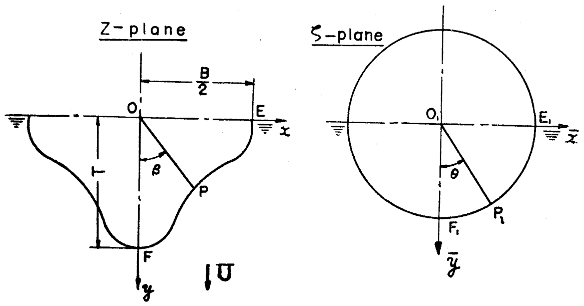


Fig. 1

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y=0,\ x> \frac{B}{2} \tag{1}$$

where $K = \frac{\omega^2}{g}$, ω is circular frequency of forced heaving. Suppose now that the cylinder oscillate vertically with a small displacement

$$y_h = h \cos(\omega t + \delta)$$

Write

$$\frac{dy_h}{dt} = -h\omega \sin(\omega t + \delta) \equiv U \tag{2}$$

Then, to the first order,the boundary condition on the surface of the cylinder is

$$\frac{\partial \phi}{\partial \nu} = U\left(\frac{\partial y}{\partial \nu}\right) \tag{3}$$

where ν is outward normal of the cylinder surface.

2.2 Mathematical representation of the section contours

Take the image of the section in Fig. 1 about the x-axis. It will be supposed that the conformal transformation of the exterior of this double figure in the z-plane into the exterior of a unit circle in the ζ-plane is known and given in the from

$$\frac{Z}{M} = \zeta + \sum_{n=1}^{\infty} a_{2n-1} \cdot \zeta^{-(2n-1)} \tag{4}$$

where Z and ζ are the complex variables.

$$Z = x+iy, \quad \zeta = \bar{x}+i\bar{y} = i\,e^{\alpha}e^{-i\theta}$$

and the coefficients a^{2n-1} are real. M is a scale factor and real.

When $n=2$, we have

$$\frac{Z}{M} = \zeta + \frac{a_1}{\zeta} + \frac{a_3}{\zeta^3} \tag{5}$$

The Lewis froms [9] are derived from the Equation (5). In this paper this Lewis froms were studied, but we can apply the same method to the general forms which are derived from the Equation (4). From the Equation (5), its equation may be expressed in the parametric form

$$\left.\begin{aligned} \frac{x}{M} &= e^{\alpha} \sin\theta + a_1 e^{-\alpha} \sin\theta - a_3 e^{-3\alpha} \sin 3\theta \\ \frac{y}{M} &= e^{\alpha} \cos\theta - a_1 e^{-\alpha} \cos\theta + a_3 e^{-3\alpha} \cos 3\theta \end{aligned}\right\} \tag{6}$$

on the circumference, put $\alpha=0$, then

$$\left.\begin{aligned} \frac{x_0}{M} &= (1+a_1)\sin\theta - a_3 \sin 3\theta \\ \frac{y_0}{M} &= (1-a_1)\cos\theta + a_3 \cos 3\theta \end{aligned}\right\} \tag{7}$$

Let B denote the beam of the section at the waterline and T the draft, we obtain

$$M = \frac{\frac{B}{2}}{(1+a_1+a_3)} \tag{8}$$

and

$$\frac{\frac{B}{2}}{T} = H_0 = \frac{1+a_1+a_3}{1-a_1+a_3} \tag{9}$$

also then sectional area S and area coefficient $\sigma = \frac{S}{B.T}$ are given in the form

$$S = \frac{\pi}{2} \cdot \left(\frac{B}{2}\right)^2 \cdot \frac{1-a_1^2-3a_3^2}{(1+a_1+a_3)^2}$$

$$\sigma = \frac{S}{B \cdot T} = \frac{\pi}{4} \cdot H_0 \cdot \frac{1-a_1^2-3a_3^2}{(1+a_1+a_3)^2} \tag{10}$$

When H_0 and σ are given, a_1 and a_3 can be obtained readily from Equations (9) and (10).

2.3 Results of the calculations

The method of the calculation are shown briefly in the Appendix.

The calculated values of the amplitude ratio $\bar{A}$ and of the free surface coefficient of added mass K_4 for the elliptic cylinder of which $H_0 = 1.5$, were compared with the Grim's results [4].

where
$$\bar{A} = \frac{\text{amplitude of progressive wave}}{\text{amplitude of forced heaving}}$$

$$K_4 = \frac{\text{added mas of cylinder}}{\frac{1}{2}\rho\pi\left(\frac{B}{2}\right)^2 \cdot C_0}$$

$$C_0 = \frac{\text{added mass of cylinder in case of } \omega \to \infty}{\frac{1}{2}\rho\pi\left(\frac{B}{2}\right)^2}$$

therefore K_4 is written also in the form

$$K_4 = \frac{\text{added mass of cylinder}}{\text{added mass of cylinder in case of } \omega \to \infty}$$

This is shown in Fig. 2. Grim's and author's exact values of $\bar{A}$ agree considerably well. In Fig. 2 the dotted line is Grim's approximate value shown in Bild. 2 of [4]. Grim gave an approximate formula for the semi-circular section, but not other sections. For the general sections which are derived from the Equation (4), if we applied Grim's approximate method, we obtain

$$\bar{A} = 2\xi_0 \int_1^\infty \frac{\left\{\frac{1}{\beta^2} + \sum_{n=1}^{N} \frac{a_{2n-1}\cdot(2n-1)}{\beta^{2n}}\right\}}{1 + \sum_{n=1}^{N} a_{2n-1}} \cos\left[\xi_0 \left\{\frac{\beta + \sum_{n=1}^{N} \frac{a^{2n-1}}{\beta^{2n-1}}}{1 + \sum_{n=1}^{N} a^{2n-1}} - 1\right\}\right] d\beta$$

where

$$\xi_0 = K\frac{B}{2} = \frac{\omega^2}{g}\cdot\frac{B}{2}$$

Then for Lewis forms it becomes

$$\bar{A} = 2\xi_0 \int_1^\infty \frac{\left(\frac{1+a_1}{\beta^2} + \frac{3a_3}{\beta^4}\right)}{1+a_1+a_3} \cos\left[\xi_0 \left\{\frac{\beta^4 + a_1\beta^2 + a_3}{(1+a_1+a_3)\beta^3} - 1\right\}\right] d\beta \qquad (11)$$

In the semi-circular section, put $a_1 = a_3 = 0$, then we have

$$\bar{A} = 2\xi_0 \int_1^\infty \frac{\cos \xi_0(\beta - 1)}{\beta^2} d\beta$$

This is the formula given by O. Grim [4].

Calculated values by the Equation (11) are shown in chain line in Fig. 2. When $\xi_0 < 2.5$, this gives an approximate value whose error is samller than 10 percent.

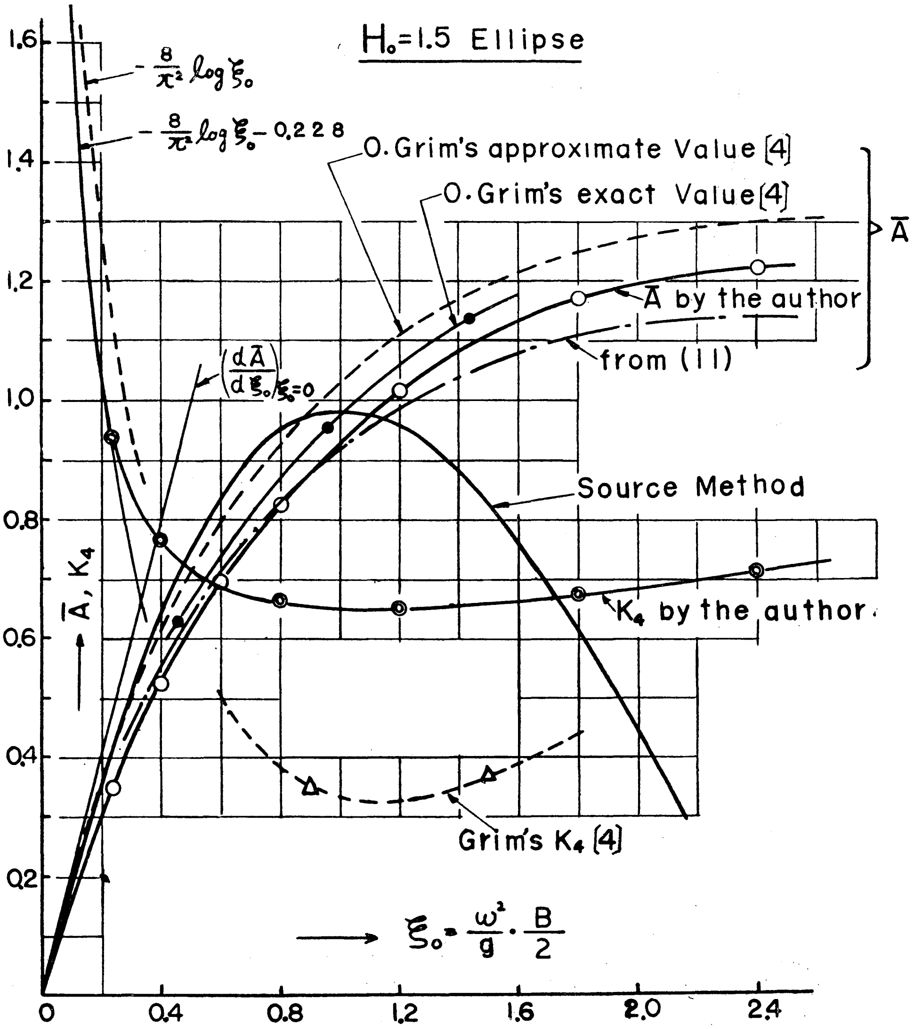


Fig. 2.

$\bar{A}$ calculated by the Source Method is also shown in Fig. 2, but it gives considerable large error. Exact values by the author are shown in double circles. Grim's K_4 values [4] are too small.

When ξ_0 is small, Uresell [10] gave a formula for the ellipse.

$$K_4 = -\frac{8}{\pi^2}\left[\log\xi_0 + \log\left(1+\frac{1}{H_0}\right) - 0.23\right] \tag{12}$$

Then, when $H_0=1.5$, we obtain $K_4 = -\frac{8}{\pi^2}\log\xi_0 - 0.228$

For $\xi_0=0.24$, the value K_4calculated from the above formula and author's exact va-

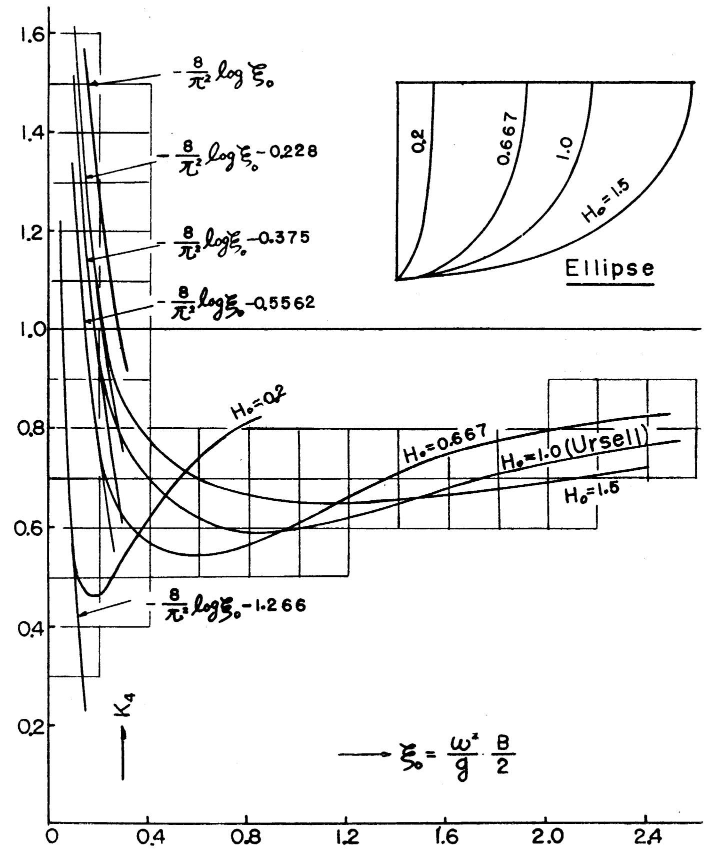


Fig. 3.

lue showed good coincidence. $\bar{A}$ and K_4 for the various Lewis forms are shown in ten figures. Fig. 3 gives K_4 for the various elliptic cylinders. In Fig. 5, approximate values calculated by the Equation (11) are also indicated. When $a_3 \neq 0$ and ξ_0 is small, it was assumed that K_4 can be approximated by the Equation (12) for the elliptic section which has the same H_0. The dotted lines of Fig. 8–12 indicated the values by this assumption.

3. Comparision of the calculated values with the experimental results

Using the results of these exact calculation and the Strip Method, the author calculated the damping force, the added mass and the added mass moment of inertia of the two ships' models which had been put to test by Golovato [2] and Gerritsuma [8]. The calculated values were compared with the experimental results. The x, y, z co-ordinate system is intorduced, which moves in space to x-direction with mean velocity of ships. The x-y-plane lies in the undisturbed water surface and z-axis points vertically downward. We assume that the centre of gravity of the ship coincides with the origine.

The following nomenclature is used in this chapter.

L = length of ship

B^{*} = breadth of ship, Δ = displacement of ship

B = beam on the L. W. L. at the distance x from the orgin

S = immersed sectional area at the distance x from the origin

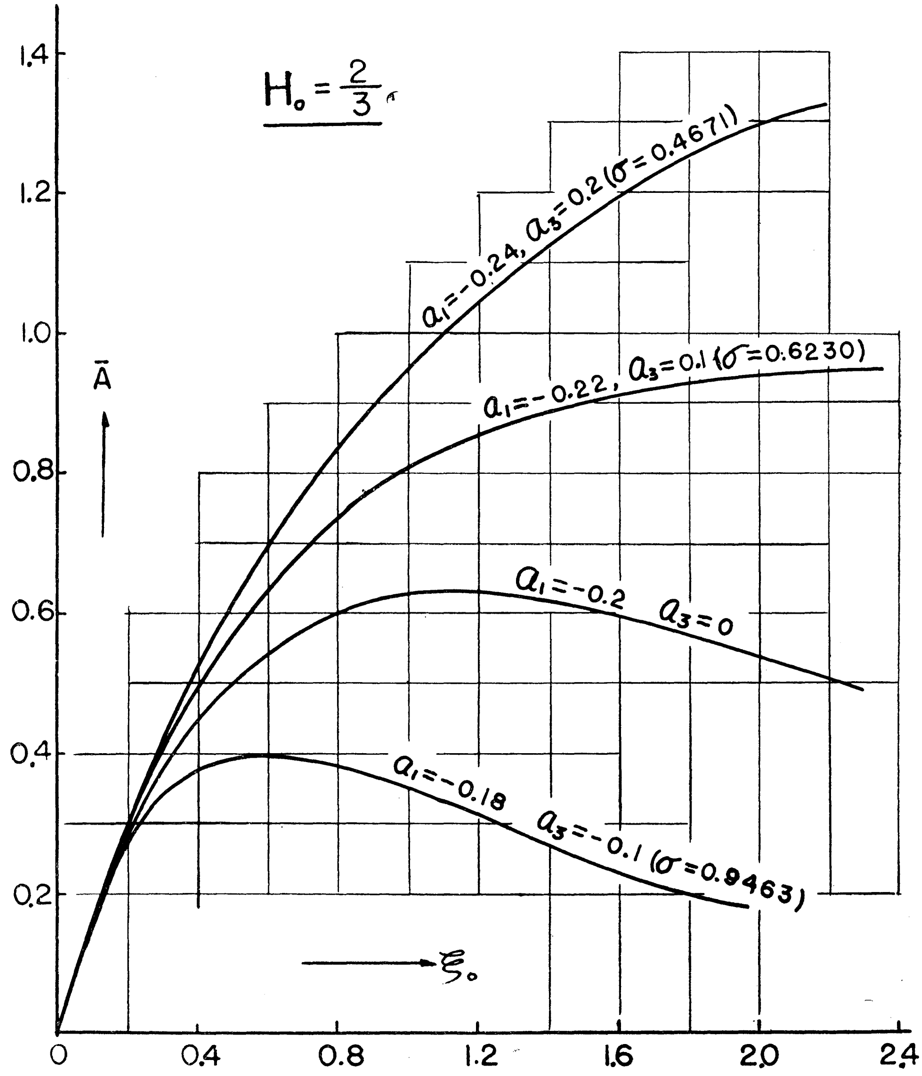


Fig. 4.

$m = \Delta/g$ = mass of ship, g = acceralation of gravity
I = longitudinal mass moment of inertia of ship
N = sectional damping force due to unit vertical velocity

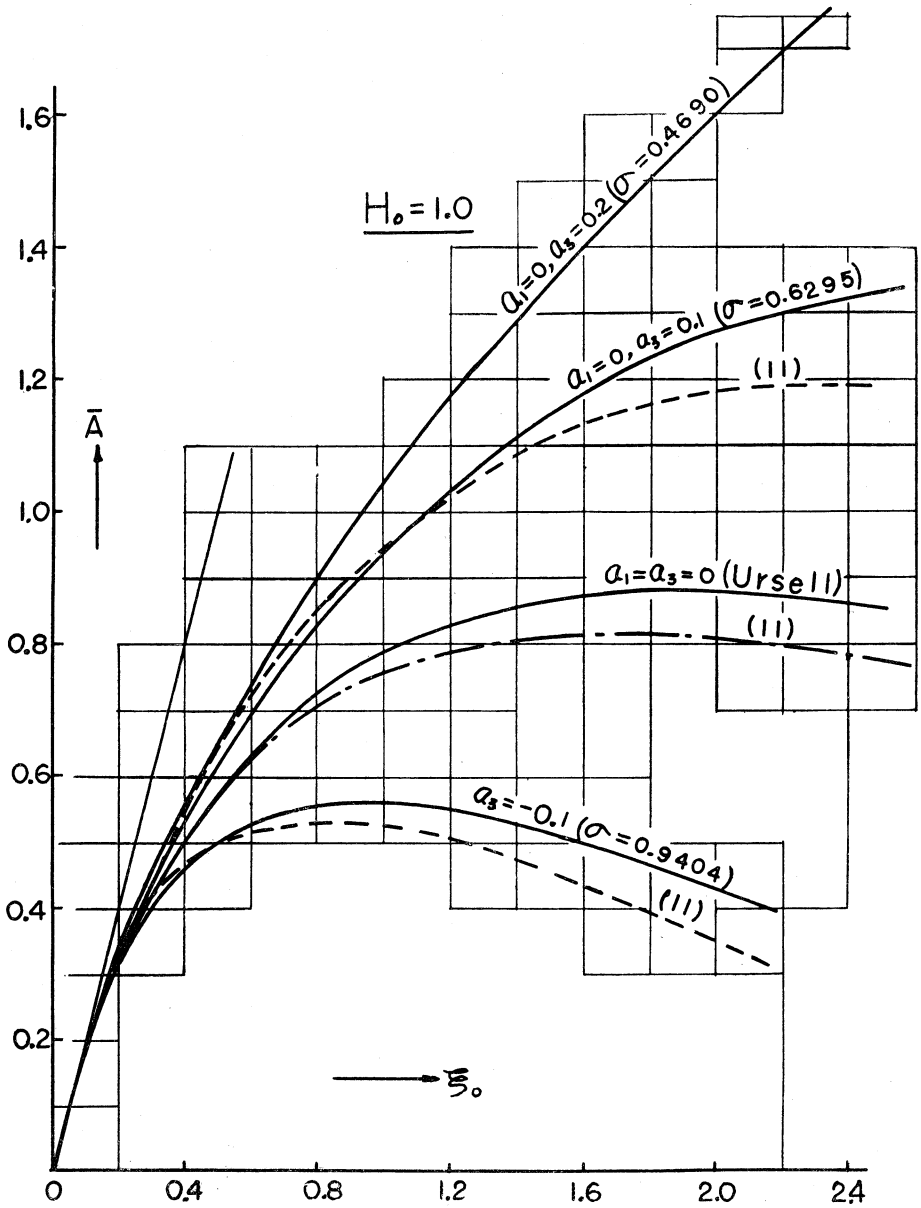


Fig. 5.

N_h= damping force of ship due to unit vertical velocity for pure heaving

N'_h= non-dimensional coefficient $\frac{N_h \cdot \sqrt{gL}}{\Delta}$

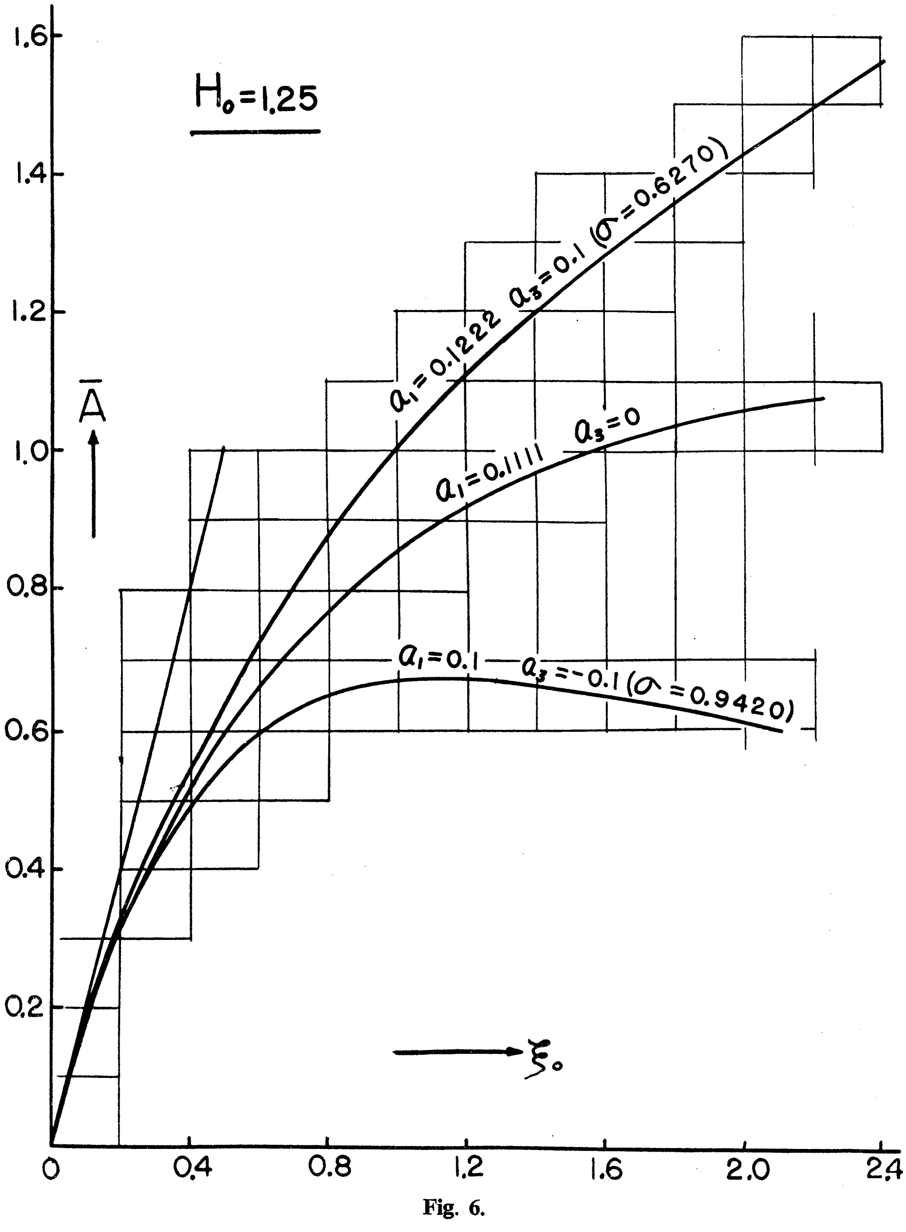
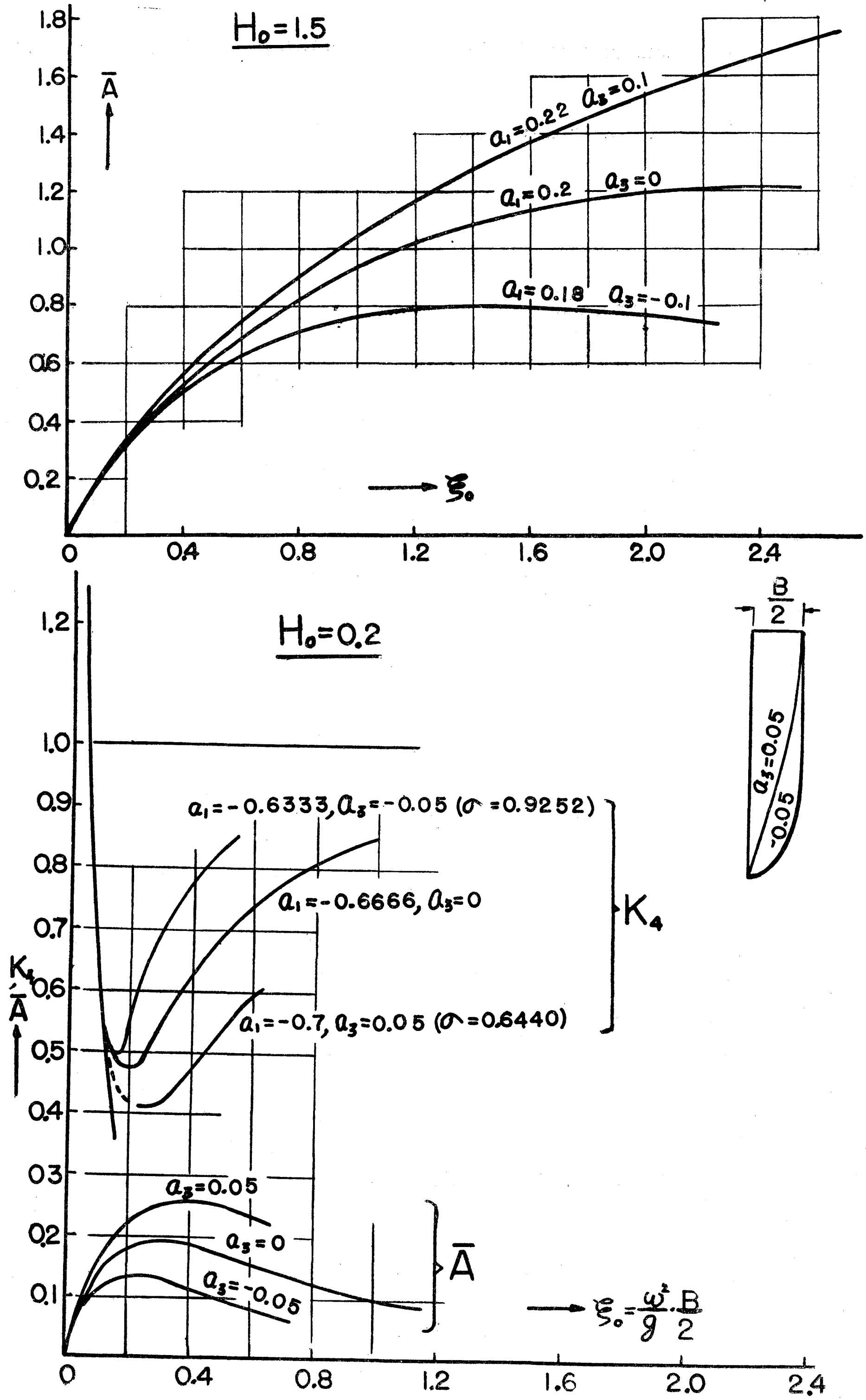


Fig. 6.


H₀=1.5
Ā
a₁=0.22 a₃=0.1
a₁=0.2 a₃=0
a₁=0.18 a₃=-0.1
ξ₀
H₀=0.2
a₁=-0.6333, a₃=-0.05 (σ=0.9252)
a₁=-0.6666, a₃=0
a₁=-0.7, a₃=0.05 (σ=0.6440)
K₄
K₄
Ā
a₃=0.05
a₃=0
a₃=-0.05
Ā
ξ₀ = ω²/g · B/2
B/2
a₃=0.05
-0.05

Figs. 7 and 8.

μ_z = added mass of ship for pure heaving

N_p = damping moment of ship due to unit angular velocity for pure pitching

N'_p = non-dimensional coefficient $\frac{N_p \cdot \sqrt{gL}}{\Delta \cdot L^2}$

μ_φ = added mass moment of inertia of ship for pure pitching

$H_0{}^*$ = B^*/T beam to draft ratio

From Havelock [1] the sectional damping force N is expressed as follows:

$$N = \frac{\rho g^2}{\omega^3} \cdot \bar{A}^2$$

Then we obtain N_h and N_p readily.

$$N_h = \int_{-\frac{L}{2}}^{\frac{L}{2}} N dx = \rho g^2 \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\bar{A}^2}{\omega^3} \cdot dx \tag{13}$$

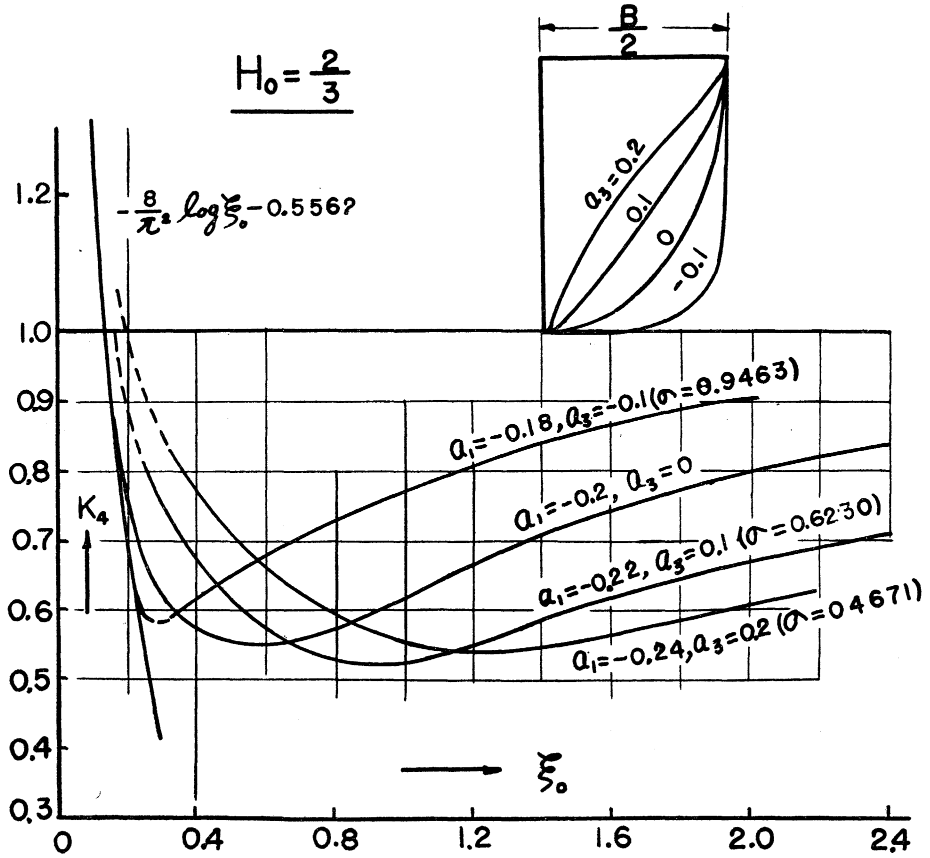


Fig. 9.

$$N_p = \int_{-\frac{L}{2}}^{\frac{L}{2}} N\cdot x^2\cdot dx = \rho g^2 \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\overline{A}^2}{\omega^3} x^2\cdot dx \tag{14}$$

Therefore N'_h and N'_p are give in the from

$$N'_h = \frac{1}{C_b\cdot T}\sqrt{\frac{B^*}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\overline{A}^2}{(\xi_1{}^*)^3}\cdot dx \tag{15}$$

$$N'_p = \frac{1}{C_b\cdot T\cdot L^2}\sqrt{\frac{B^*}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\overline{A}^2}{(\xi^*{}_1)^3}\cdot x^2\cdot dx \tag{16}$$

where c_b= block coefficient, $\xi_1{}^* = \omega\sqrt{\frac{B^*}{g}}$

ξ_0 is related to $\xi_1{}^*$ by the relation $\xi_0 = \frac{\omega^2}{g}\cdot\frac{B}{2} = \frac{1}{2}(\xi_1{}^*)^2\cdot\left(\frac{B}{B^*}\right)$

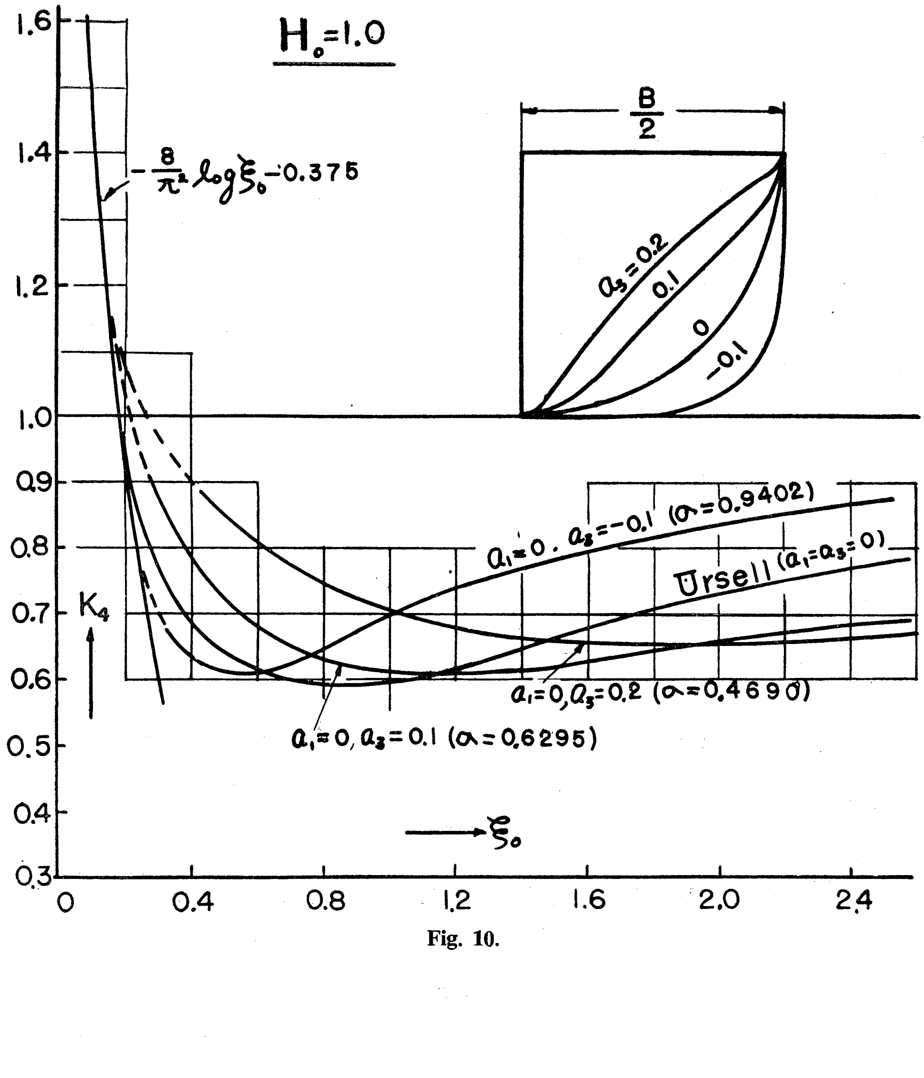


Fig. 10.

Using ω and B^* which are given previously, ξ_1^* is determined by the above mentioned relation. Then from ξ_1^* and $\frac{B}{B^*}$ we can obtain the ξ_0 for each ship section.

$\bar{A}$ for ξ_0, H_0 and σ of the section are obtained from figures shown in this paper, and consequently $\frac{\bar{A}^2}{(\xi_1^*)^3}$ due to ω can be determined for each section. In the next place, as the added mass of the cylinder is $\frac{1}{2}\rho\pi\left(\frac{B}{2}\right)^2\cdot C_0\cdot K_4$, μ_z is expressed as follows :

$$\mu_z=\frac{1}{2}\rho\pi\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(\frac{B}{2}\right)^2\cdot C_0\cdot K_4\cdot dx \tag{17}$$

Or using S, as used by K. Kroukovsky [5], μ_z is written into

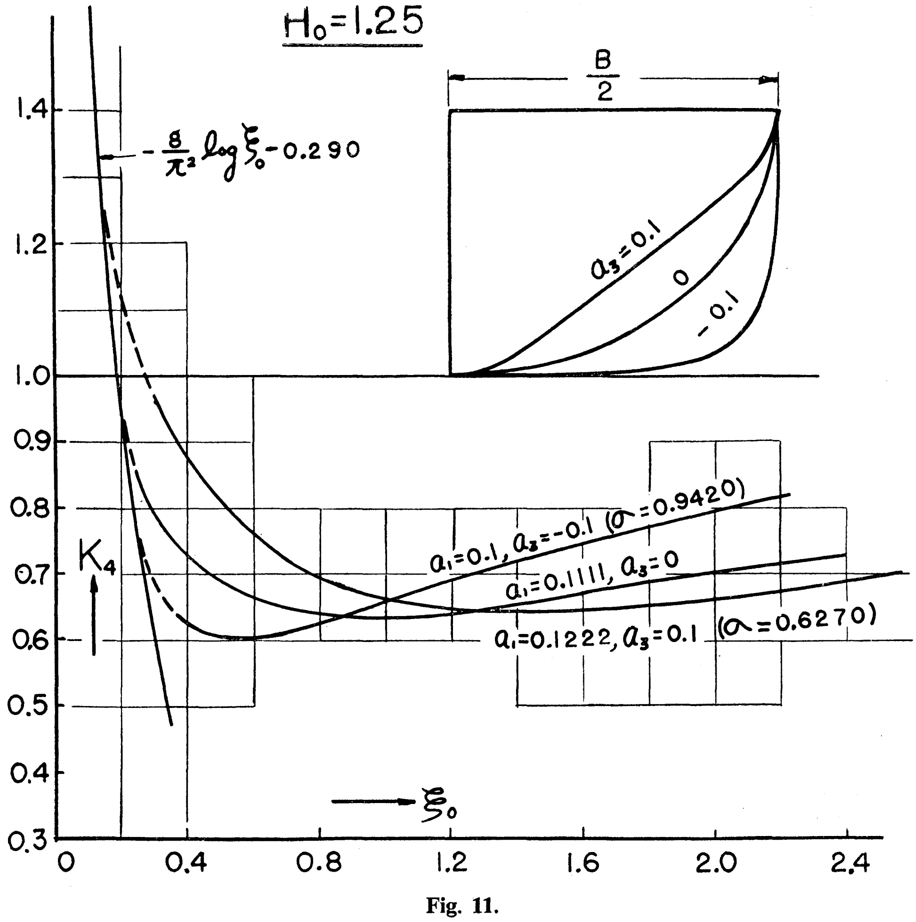


Fig. 11.

$$\mu_z = \rho \int_{-\frac{L}{2}}^{\frac{L}{2}} K_2 \cdot S \cdot K_4 \cdot dx \qquad (18)$$

where $K_2 = \dfrac{\text{added mass of the cylinder in case of } \omega \to \infty}{\text{mass of the cylinder}}$

We have $K_2 = \dfrac{(1+a_1)^2 + 3a_3^2}{1-a_1^2-3a_3^2}$ for Lewis forms, and generally K_2 will be obtained by means of Lewis-Prohaska method [11].

Similarly μ_φ is expressed as follows:

$$\mu_\varphi = \int_{-\frac{L}{2}}^{\frac{L}{2}} K_2 \cdot S \cdot K_4 \cdot x^2 \cdot dx \qquad (19)$$

The model used by Golovato was one of a family constructed to mathema-

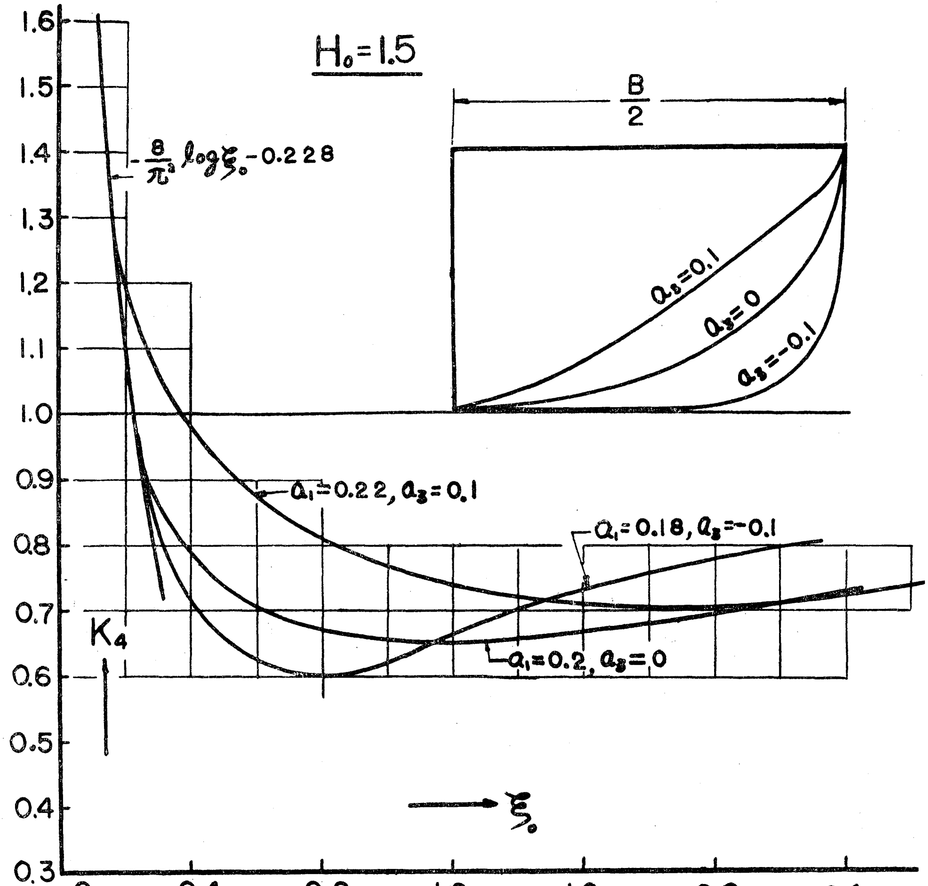


Fig. 12.

tical lines defined by Weinblum [12]. This model is wall-sided and symmetrical about y-axis. As heaving amplitude is comparatively small, it is suitable for the comparison of the theoretical calculation with the experimental results. Since $H_0^*=1.25$, we calculated the B, x, S, σ and K_2 of each section which corresponds to $H_0=1.25$, 1.0, 2/3 and 0.2, and then calculated ξ_0 for the given ξ_1^*. In the next place, we obtained the $\bar{A}$ and K_4 values from the figures in this paper, and then N_h and N_p were evaluated by means of numerical quadrature. Our results of calculation are plotted in Fig. 5 of Golovato [2]. These are shown in Fig. 13, where the experimental results for Froude number $=$ 0.09 and 0.36 are indicated. Our calculation gave better approximate values than the Grime's one. K_z values are also shown in Fig. 14, where the calculated values show good coincidence with the experimental results in the range of $\xi_1^*<2.5$. Also we calculated K_z using Ursell's K_4 for each section, which were indicated in it.

The ship model used in [8] by Gerritsuma was Todd 60-series parent form with a block coefficient $C_b=0.60$. In this case also, H_0^* equals 1.25, and therefore using the same method mentioned before we calculated N_h, N_p, μ_z and μ_φ. These are shown in Fig. 15–18. In regard to N_h, the calculated values were too small, however, the Source Method gave good results generally. On the other hand, the calculated values of N_p showed good coincidence with the experimental results, but Source Method differed considerably. For the differencce between our calculation and the experimental results, first of all, we must take into consideration the three dimensional effect. Havelock [13] and Vossers [14] calculated this effect, but Newmann [15] especially calculated the three-dimensional damping force for the same model with Gerritsuma's one when the speed of ship is zero. He used the three dimensional Source Method. According to his results, three-dimensianal value of N_h is about 20 percent higher than the two-dimensional value obtained by Strip Method. (See Fig. 1 of [15]) If we apply this three-dimensional correction to our results, they approach the experimental results considerably.

In addition to this, as the rear half of this model has wedge-shaped section, its damping force may differ from the theoretical calculation. And also, non-linear effect should be considered because of large heaving amplitude. The calculated values of μ_z and μ_φ are shown in Fig. 17 and 18. Both of these showed good coincidence whith the experimental results except for small ω. Throughout these comparison, the μ_z and μ_φ computed by means of the Strip Method gave good estimation. In view of these facts, as to the added mass and the added moment of inertia, it is supposed that the three-dimensional effect is very small except for samll ξ_1^* and could be negligible practically.

4. Conclusion

The following conclusions may be shown as a results of numerical calculation.

1. Our exact values of $\bar{A}$ are considered to give good estimation of the damping force than the Grim's values.

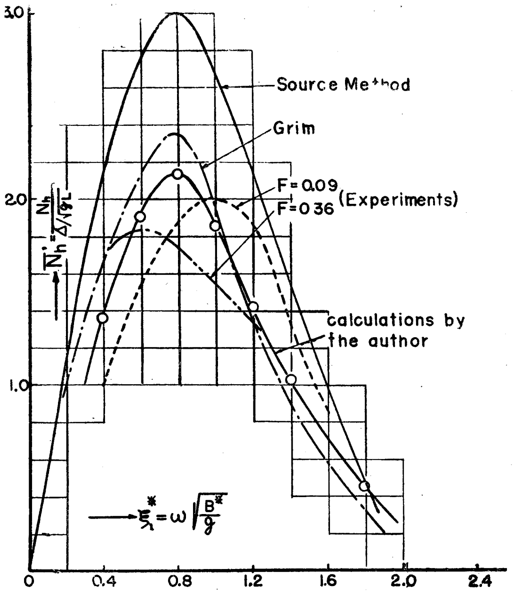


Fig. 13.

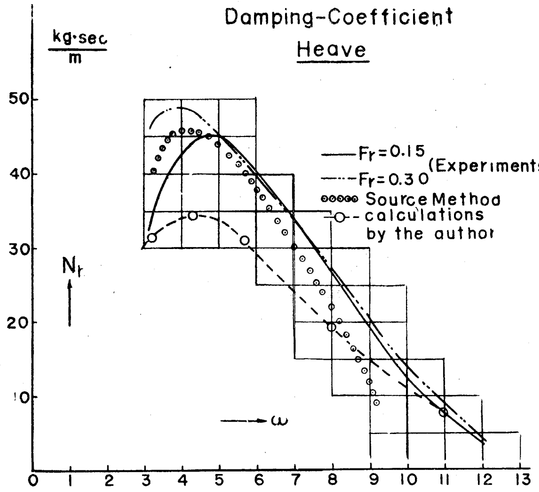


Fig. 15.

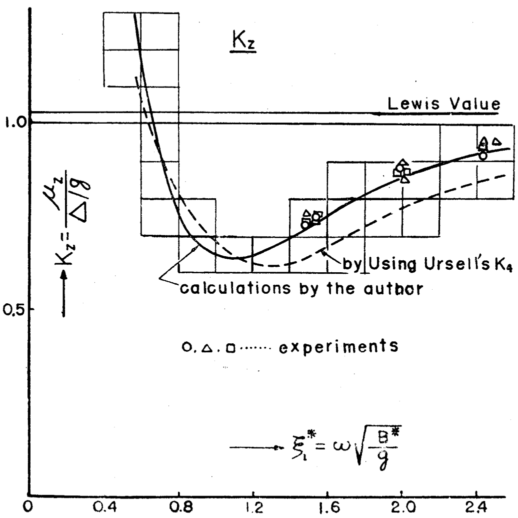


Fig. 14.

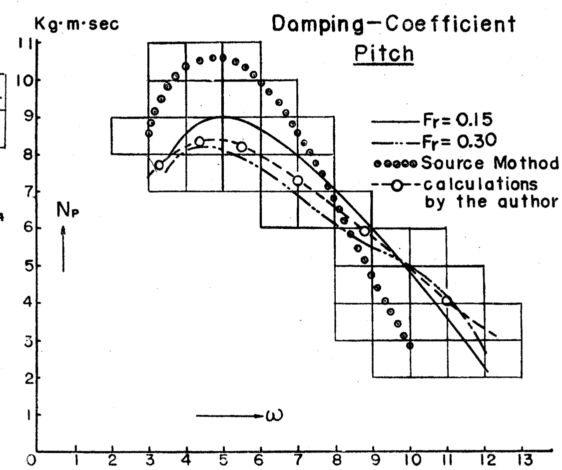


Fig 16.

2. On the evaluation of the damping force of an actual ship, it is necessary to estimate more closely the three-dimensional effect, non-linear effect and etc.
3. As to K_4, it varies considerably with the section contour.
4. The values of μ_z and μ_φ obtained by means of Strip Method give good results.

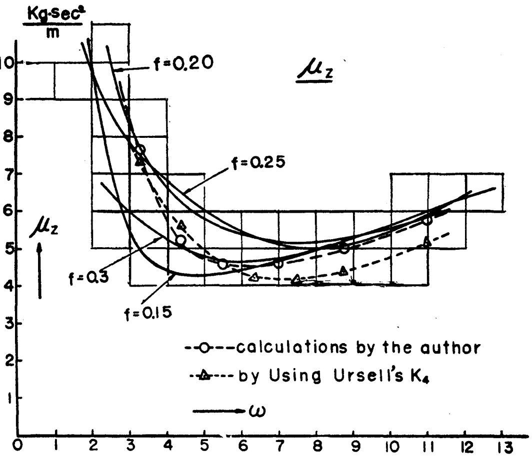


Fig. 17.

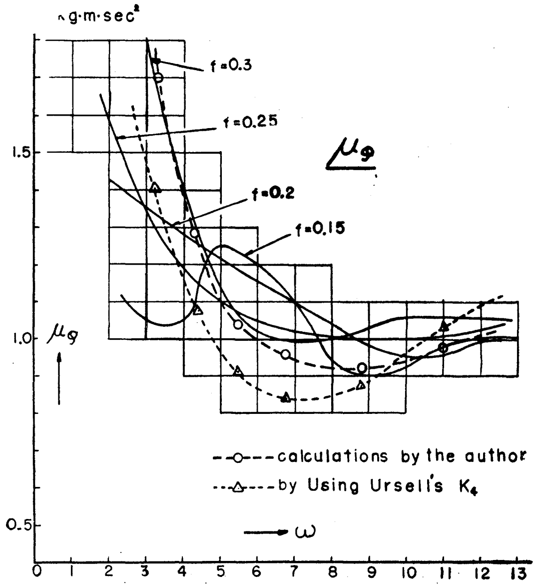


Fig. 18.

Acknowledgements

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APPENDIX

Here we describe the outline of our method of caluculation which follows after the method of F. Ursell [3].

In consequence of the transformation (5), free surface condition readuces to

$$\xi_0\phi\left(\frac{e^{\alpha}-a_1e^{-\alpha}-3a_3e^{-3\alpha}}{(1+a_1+a_3)}\right)\mp\frac{\partial\phi}{\partial\theta}=0 \qquad \text{at } \theta=\pm\frac{\pi}{2} \tag{20}$$

We take following set of velocity potentials which satisfy $\Delta^2\phi=0$, the surface condition (20) and the condition of symmetry about y-axis

$$\phi_{2m}=\Bigg[e^{-2m\alpha}\cos 2m\theta+\frac{\xi_0}{1+a_1+a_3}\Bigg\{\frac{e^{-(2m-1)\alpha}}{2m-1}\cos(2m-1)\theta+\frac{a_1e^{-(2m+1)\alpha}}{2m+1}\cos(2m+1)\theta-$$
$$-\frac{3a_3}{2m+3}e^{-(2m+3)\alpha}\cos(2m+3)\theta\Bigg\}\Bigg]\begin{matrix}\cos\omega t\\ \sin\omega t\end{matrix} \qquad (m=1,2,3\cdots\cdots) \tag{21}$$

The set of the conjugate stream functions is expressed as follows:

$$\psi_{2m}=\Bigg[e^{-2m\alpha}\sin 2m\theta+\frac{\xi_0}{1+a_1+a_3}\Bigg\{\frac{e^{-(2m-1)\alpha}}{2m-1}\sin(2m-1)\theta+\frac{a_1e^{-(2m+1)\alpha}}{2m+1}\sin(2m+1)-$$
$$-\frac{3a_3}{2m+3}e^{-(2m+3)\alpha}\sin(2m+3)\theta\Bigg\}\Bigg]\begin{matrix}\cos\omega t\\ \sin\omega t\end{matrix} \qquad (m=1,2,3\cdots\cdots) \tag{22}$$

These sets of functions tend to zero as α tends to infinity.

As a stream function representing such a train of waves at infinity we take the function describing a source at the origin. This is the same with F. Ursell [3]. It is expressed as follows:

$$\left.\begin{aligned}\Psi_0&=\frac{g\eta}{\pi\omega}\Big[\Psi_c(K,x,y,)\cos\omega t+\Psi_s(K,x,y)\sin\omega t\Big]\\ \Psi_c&=\pi e^{-ky}\sin Kx\\ \Psi_s&=\int_0^{\infty}\frac{e^{-kx}}{K^2+k^2}\{k\sin ky+K\cos ky\}\,dk-\pi e^{-ky}\cos Kx\end{aligned}\right\} \tag{23}$$

changingthe parameter

$$\Psi_0=\frac{g\eta}{\pi\omega}\Big[\Psi_c(\xi_0,a_1,a_3,\alpha,\theta)\cos\omega t+\Psi_s(\xi_0,a_1,a_3,\alpha,\theta)\sin\omega t\Big] \tag{24}$$

where η is the wave amplitude at infinity. Then, suppose that the stream function is expressed as follows:

$$\left(\frac{\pi\omega}{g\eta}\right)\psi=\Psi_c(\xi_0,a_1,a_3,\alpha,\theta)\cos\omega t+\Psi_s(\xi_0,a_1,a_3,\alpha,\theta)\sin\omega t$$

$$+\cos\omega t\sum_{m=1}^{\infty}p_{2m}(\xi_0)\Big[e^{-2m\alpha}\sin 2m\theta+\frac{\xi_0}{1+a_1+a_3}\Big\{\frac{e^{-(2m-1)\alpha}}{2m-1}\sin(2m-1)\theta$$

$$+ \frac{a_1 e^{-(2m+1)\alpha}}{2m+1} \sin(2m+1)\theta - \frac{3a_3}{2m+3} e^{-(2m+3)\alpha} \sin(2m+3)\theta \Big\} \Big]$$

$$+ \sin \omega t \sum_{m=1}^{\infty} q_{2m}(\xi_0) \Big[e^{-2m\alpha} \sin 2m\theta + \frac{\xi_0}{1+a_1+a_3} \Big\{ \frac{e^{-(2m-1)\alpha}}{2m-1} \sin(2m-1)\theta$$

$$+ \frac{a_1 e^{-(2m+1)\alpha}}{2m+1} \sin(2m+1)\theta - \frac{3a_3}{2m+3} e^{-(2m+3)\alpha} \sin(2m+3)\theta \Big\} \Big] \qquad (25)$$

We assume thate this series converges uniformly in the range of $\alpha \geqq 0$. This ψ must satisfy the condition (3) on the circumference of the cylinder. This condition reduces to

$$\left(-\frac{\partial \psi}{\partial \theta} \right)_{\alpha=o} = UM(\cos\theta + a_1 \cos\theta - 3a_3 \cos 3\theta) \qquad (26)$$

On the circumference of the cylinder, following relation are obtained from the Equations (25) & (26).

$$\left(\frac{\pi\omega}{g\eta} \right) \cdot \psi_{\alpha=0} = \Psi_{C_0}(\xi_0, a_1, a_3, \theta) \cos \omega t + \Psi_{S_0}(\xi_0, a_1, a_3, \theta) \sin \omega t$$

$$+ \cos \omega t \sum_{m=1}^{\infty} p_{2m}(\xi_0) \Big[\sin 2m\theta + \frac{\xi_0}{1+a_1+a_3} \Big\{ \frac{\sin(2m-1)\theta}{2m-1} + \frac{a_1 \sin(2m+1)\theta}{2m+1} - \frac{3a_3 \sin(2m+3)\theta}{2m+3} \Big\} \Big]$$

$$+ \sin \omega t \sum_{m=1}^{\infty} q_{2m}(\xi_0) \Big[\sin 2m\theta + \frac{\xi_0}{1+a_1+a_3} \Big\{ \frac{\sin(2m-1)\theta}{2m-1} + \frac{a_1 \sin(2m+1)\theta}{2m+1} - \frac{3a_3 \sin(2m+3)\theta}{2m+3} \Big\} \Big]$$

$$= -\left(\frac{\pi\omega}{g\eta} \right) UM(\sin\theta + a_1 \sin\theta - a_3 \sin 3\theta) \qquad (27)$$

Put $\theta = \frac{\pi}{2}$, then

$$\Psi_{C_0}\left(\xi_0, a_1, a_3, \frac{\pi}{2}\right) \cos \omega t + \Psi_{S_0}\left(\xi_0, a_1, a_3, \frac{\pi}{2}\right) \sin \omega t$$

$$+ \cos \omega t \sum_{m=1}^{\infty} p_{2m}(\xi_1) \frac{\xi_0}{1+a_1+a_3} (-1)^{m-1} \Big\{ \frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3} \Big\}$$

$$+ \sin \omega t \sum_{m=1}^{\infty} q_{2m}(\xi_0) \frac{\xi_0}{1+a_1+a_3} (-1)^{m-1} \Big\{ \frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3} \Big\}$$

$$= -\left(\frac{\pi\omega}{g\eta} \right) UM(1+a_1+a_3) \qquad (28)$$

Using (28) into the Equation (27) we obtain following relation

$$\Psi_{C_0}(\xi_0, a_1, a_3, \theta) - \frac{\sin\theta + a_1 \sin\theta - a_3 \sin 3\theta}{1+a_1+a_3} \Psi_{C_0}\left(\xi_0, a_1, a_3, \frac{\pi}{2}\right) = \sum_{m=1}^{\infty} p_{2m}(\xi_0) \cdot f_{2m}(\xi_0, a_1, a_3, \theta)$$

$$\Psi_{S_0}(\xi_0, a_1, a_3, \theta) - \frac{\sin\theta + a_1 \sin\theta - a_3 \sin 3\theta}{1+a_1+a_3} \Psi_{S_0}\left(\xi_0, a_1, a_3, \frac{\pi}{2}\right) = \sum_{m=1}^{\infty} q_{2m}(\xi_0) \cdot f_{2m}(\xi_0, a_1\ a_3, \theta) \qquad (29)$$

where

$$f_{2m}(\xi_0,\ a_1,\ a_3,\ \theta)$$
$$= -\Big[\sin 2m\,\theta + \frac{\xi_0}{1+a_1+a_3}\Big\{\frac{\sin(2m-1)\theta}{2m-1} + \frac{a_1\sin(2m+1)\theta}{2m+1} - \frac{3a_3\sin(2m+3)\theta}{2m+3}\Big\}$$
$$+\frac{\xi_0(-1)^m}{(1+a_1+a_3)^2}\Big\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\Big\}(\sin\theta + a_1\sin\theta - a_3\sin 3\theta)\Big] \tag{30}$$

From the Equation (30) we can determine $p_{2m}(\xi_0)$ and $q_{2m}(\xi_0)$.
Write

$$\Psi_{C_0}\Big(\xi_0,\ a_1,\ a_3,\ \frac{\pi}{2}\Big) + \sum_{m=1}^{\infty} p_{2m}(\xi_0)(-1)^{m-1}\frac{\xi_0}{1+a_1+a_3}\Big\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\Big\} = A_0(\xi_0)$$
$$\Psi_{S_0}\Big(\xi_0.\ a_1,\ a_3,\ \frac{\pi}{2}\Big) + \sum_{m=1}^{\infty} q_{2m}(\xi_0)(-1)^{m-1}\frac{\xi_0}{1+a_1+a_3}\Big\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\Big\} = B_0(\xi_0) \tag{31}$$

Then the Equation (28) reduces into

$$-\Big(\frac{\pi\omega}{g\eta}\Big)UM(1+a_1+a_3) = A_0(\xi_0)\cos\omega t + B_0(\xi_0)\sin\omega t$$

Finally using the Equations (2) and (8), it results

$$\overline{A} = \frac{\eta}{h} = \frac{\pi\omega^2}{g}\cdot\frac{B}{2}\cdot\frac{1}{\sqrt{A_0^2+B_0^2}} = \frac{\pi\xi_0}{\sqrt{A_0^2+B_0^2}} \tag{32}$$

In the Equation (29), (30), (31), put $a_1=a_3=0$, then these reduce into the equations that F. Ursell has given in [3].

In the next place, the velocity potential ϕ is readily derived from the Equation (25). It is given as follows:

$$\Big(\frac{\pi\omega}{g\eta}\Big)\phi = \Phi_C(\xi_0,\ a_1,\ a_3,\ \alpha,\ \theta)\cos\omega t + \Phi_S(\xi_0,\ a_1,\ a_3,\ \alpha,\ \theta)\sin\omega t$$
$$+\cos\omega t\sum_{m=1}^{\infty} p_{2m}(\xi_0)\Big[e^{-2m\alpha}\cos 2m\theta + \frac{\xi_0}{1+a_1+a_3}\Big\{\frac{e^{-(2m-1)\alpha}}{2m-1}\cos(2m-1)\theta$$
$$+\frac{a_1 e^{-(2m+1)\alpha}}{2m+1}\cos(2m+1)\theta - \frac{3a_3 e^{-(2m+3)\alpha}}{2m+3}\cos(2m+3)\theta\Big\}\Big]$$
$$+\sin\omega t\sum_{m=1}^{\infty} q_{2m}(\xi_0)\Big[e^{-2m\alpha}\cos 2m\theta + \frac{\xi_0}{1+a_1+a_3}\Big\{\frac{e^{-(2m-1)\alpha}}{2m-1}\cos(2m-1)\theta$$
$$\frac{a_1 e^{-(2m+1)\alpha}}{2m+1}\cos(2m+1)\theta - \frac{3a_3 e^{-(2m+3)\alpha}}{2m+3}\cos(2m+3)\theta\Big\}\Big] \tag{33}$$

where

$$\left.\begin{aligned}\Phi_C(\xi_0,\ a_1,\ a_3,\ \alpha,\ \theta) &= \pi e^{-ky}\cos Kx\\ \Phi_S(\xi_0,\ a_1,\ a_3,\ \alpha,\ \theta) &= \pi e^{-ky}\sin Kx - \int_0^\infty \frac{e^{-kx}}{K^2+k^2}\Big\{k\cos ky - K\sin ky\Big\}dk\end{aligned}\right\} \tag{34}$$

From the Equation (33) the pressure $p = -\rho\dfrac{\partial\phi}{\partial t}$ can be derived. The force per unit length acting on the cylinder in the direction of y-axis can be expressed as follows:

$$F=\left(\frac{g\eta}{\pi}\right)\cdot \rho\cdot B\left[\ M_0 \cos \omega t - N_0 \sin \omega t\ \right] \tag{35}$$

where

$$M_0=\int_0^{\frac{\pi}{2}} \Phi_{S_0}(\xi_0,\ a_1,\ a_3,\ \theta)\frac{\cos\theta+a_1\cos\theta-3a_3\cos 3\theta}{1+a_1+a_3}d\theta$$
$$+\frac{1}{1+a_1+a_3}\Big[\sum_{m=1}^{\infty}(-1)^{m-1}\cdot q_{2m}\left(\frac{1+a_1}{4m^2-1}+\frac{9a_3}{4m^2-9}\right)+$$
$$+\frac{\pi\xi_0}{4(1+a_1+a_3)}\left\{(1+a_1-a_1\ a_3)q_2-a_3\ q_4\right\}\Big]$$

$$N_0=\int_0^{\frac{\pi}{2}} \Psi_{C_0}(\xi_0,\ a_1,\ a_3,\ \theta)\frac{\cos\theta+a_1\cos\theta-3a_3\cos 3\theta}{1+a_1+a_3}d\theta$$
$$+\frac{1}{1+a_1+a_3}\Big[\sum_{m=1}^{\infty}(-1)^{m-1}\cdot p_{2m}\left(\frac{1+a_1}{4m^2-1}+\frac{9a_3}{4m_2-9}\right.+$$
$$+\frac{\pi\xi_0}{4(1+a_1+a_3)}\left\{(1+a_1-a_1\ a_3)p_2-a_3\ p_4\right\}\Big] \tag{36}$$

Combining the Equations (28) and (30), the acceleration of motion is

$$\frac{d^2y_h}{dt^2}=\left(\frac{2g\eta}{\pi B}\right)\left\{A_0(\xi_0)\sin\omega t-B_0(\xi_0)\cos\omega t\right\} \tag{37}$$

It follows that the force component in adverse phase with the acceleation is

$$\frac{g\eta}{\pi}\cdot\rho B\cdot\left(\frac{M_0B_0+N_0A_0}{A_0^2+B_0^2}\right)\left\{A_0(\xi_0)\sin\omega t-B_0(\xi_0)\cos\omega t\right\} \tag{38}$$

Taking their ratio, we obtain the added mass.

$$\text{Added Mass}=2\rho\cdot\left(\frac{B}{2}\right)^2\cdot\left(\frac{M_0B_0+N_0A_0}{A_0^2+B_0^2}\right) \tag{39}$$

On the other hand, added mass for Lewis forms in case of $\omega\to\infty$ was given by Lewis [9]. It is written as follows: $\frac{1}{2}\rho\pi C_0\cdot\left(\frac{B}{2}\right)^2$. The coefficient C_0 is, for the case of heaving, in the following form

$$C_0=\frac{(1+a_1)^2+3a_3^2}{(1+a_1+a_2)^2} \tag{40}$$

Using K_4, write

$$\text{Added Mass}=\frac{1}{2}\rho\pi\left(\frac{B}{2}\right)^2\cdot C_0\cdot K_4 \tag{41}$$

Finaly from the Equations (39), (40) and (41), K_4 can be obtained as follows:

$$K_4=\frac{4}{\pi}\cdot\frac{M_0B_0+N_0A_0}{A_0^2+B_0^2}\cdot\frac{(1+a_1+a_3)^2}{(1+a_1)^2+3a_3^2} \tag{42}$$

The work done by the cylinder per one cycle is $\frac{\rho g^2\eta^2}{\pi^2\omega}(M_0A_0-N_0B_0)$.

The energy transmitted by the waves at a distance from the cylinder in the same time is given by $\frac{1}{2}\cdot\frac{\rho g^2\eta^2}{\omega}$.

Both of these must be equal. And we obtain the following relation

$$M_0A_0-N_0B_0=\frac{\pi^2}{2}.$$

This relation is used to check our calculation.

From the Equation (30), write

$$\Psi_{C_0}(\xi_0,\ a_1,\ a_3,\ \theta)-\frac{\sin\theta+a_1\sin\theta-a_3\sin 3\theta}{1+a_1+a_3}\Psi_{C_0}\left(\xi_0,\ a_1,\ a_3,\ \frac{\pi}{2}\right)\equiv H(\theta)$$

Then we have readily $H(0)=H\left(\frac{\pi}{2}\right)=0$.

$H(\theta)$ is expanded in such a series of non-orthogonal polynomials throughout the range $0\leqq\theta\leqq\frac{\pi}{2}$.

$$H(\theta)=\sum_{m=1}^{\infty}p_{2m}(\xi_0)\cdot f_{2m}(\xi_0,\ a_1,\ a_3,\ \theta)$$

This expansion must converge uniformly throughout the range.

When $a_1=a_3=0$, that is, for the circular cylinder, its convergence was proved in [3], [16], [17] by F. Ursell. When $a_1\neq 0$, $a_3\neq 0$, it is very difficult to determine the range of convergence. Assuming the uniform convergence, numerical computation was done with terms of six polynomials. Even when $\xi_0=3.0$ the convergence was very fast. At the numerical computation we adopted the same method as F. Ursell used in [3]. That is, $H(\theta)$ was evaluated at 0°, 20°, 30°, ... 90° by numerical quadrature, with six ξ_0 values. We solved adout one hundred sets of six simultaneous linear equationsfor $p_{2m}(\xi_0)$, and similarly for $q_{2m}(\xi_0)$.

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