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AN IMPULSIVE MOTION OF A LIQUID CONTAINED IN A CIRCULAR PIPE

By Jun-ichi OKABE

The flow of a liquid contained in a long, circular pipe owing to an impulsive pressure which takes place suddenly on a section and spreads along the axis afterwards, is calculated by the operational method.

In APPENDIX I the tables of $J_0(j_{0,n}\cdot\sigma)$ are shown, J_0 being the Bessel function of the first kind, $j_{0,n}$ its nth zero, and σ a number between zero and one; n is varied from one to forty at the intervals of one, and σ, from zero to one at the intervals of one-fortieth.

In APPENDIX II the problem of the propagation of the flow pattern in a long, rigid, circular pipe filled with a still fluid is studied, and it is shown mathematically that the propagation with a finite velocity is impossible.

In APPENDIX III an approximate computation is suggested for the flow caused by an impulsive pressure traveling in a circular pipe.

1. We use a cylindrical coordinate system (x, r, θ), x and r being respectively the axial, and the radial, distances of the pipe whose radius is a, and θ the azimuth; see FIGURE 1. Assuming axial symmetry so that the velocity component in the direction of θ increasing and all derivatives with respect to θ vanish, and restricting ourselves to small disturbances so that the nonlinear inertia terms are all negligible, the Navier-Stokes equations of motion become the following:

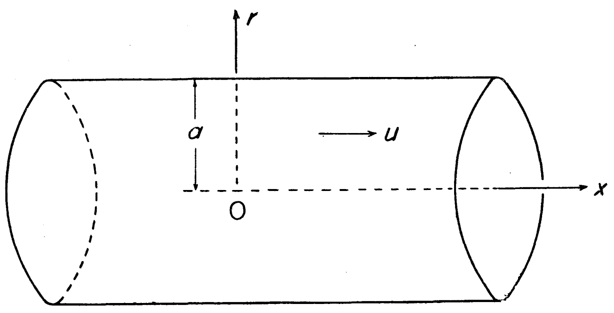

FIG. 1.

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial \omega}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2}\right), \tag{1.1}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial \omega}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial x^2}\right), \tag{1.2}$$

where ω is the pressure, ρ the constant density, ν the kinematic viscosity, of the liquid, t the time, and u and v are the velocity components in the axial, and the radial, directions, respectively. [1] The equation of continuity reads, on the other hand,

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv)=0\,, \tag{1.3}$$

from which we obtain

$$v=-\frac{1}{r}\int_0^r r\frac{\partial u}{\partial x}dr\,. \tag{1.4}$$

If r is very small (or more precisely, very small compared with the characteristic length with which x is measured), it will be concluded from (1.4) that

$$v\ll u\,, \tag{1.5}$$

provided that $\partial u/\partial x$ is not so large. So neglecting all terms containing v in (1.2), we shall readily arrive at the approximate result

$$\frac{\partial \omega}{\partial r}=0\,, \tag{1.6}$$

which means that the pressure may be regarded as a function of x and t only, being independent of r in the degree of our approximation. Further, from the same consideration, we see at once that $\partial^2 u/\partial x^2$ is much smaller than $\partial^2 u/\partial r^2$ or $\partial u/r\partial r$, so (1.1) reduces approximately to

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial \omega}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)\,. \tag{1.7}$$

Of course this single equation does not suffice to determine the two unknowns: u and ω. In what follows, therefore, assuming ω is given as an appropriate function of x and t, we shall solve the equation (1.7) with regard to u.

Suppose ω is expressed in the form

$$\frac{\omega}{\rho}=k\left\{1-\exp\left(-\frac{t}{\alpha^2x^2}\right)\right\}\,, \tag{1.8}$$

where k and α are the constants having the dimensions of (velocity)^2 and (time)$^{1/2}\times$ (length)$^{-1}$, respectively, then ω defined above has obviously the following properties:

$$\begin{array}{llll} \text{i) at } x=0, & \omega/\rho=k\,, & & \\ \text{ii) at } x\neq 0, & \omega/\rho=0\,, & \text{when } t=0\,, & \\ & \omega/\rho=k\,, & \text{when } t\to\infty\,. & \end{array} \tag{1.9}$$

and

Next, the differentiation yields

$$-\frac{\partial}{\partial x}\left(\frac{\omega}{\rho}\right)=\frac{2k}{\alpha^2}\frac{t}{x^3}\exp\left(-\frac{t}{\alpha^2x^2}\right)\,. \tag{1.10}$$

The behaviors of (1 8) and (1.10) when k and α have the numerical values *one*, namely

1) Goldstein, S., *Modern Developments in Fluid Dynamics,* Oxford at the Clarendon Press (1938), vol. I, pp. 103-104.

$$\omega/\rho = 1 - \exp\left(-t/x^2\right), \tag{1.11}$$

and

$$-\frac{\partial}{\partial x}\left(\frac{\omega}{\rho}\right) = \frac{2t}{x^3}\exp\left(-\frac{t}{x^2}\right), \tag{1.12}$$

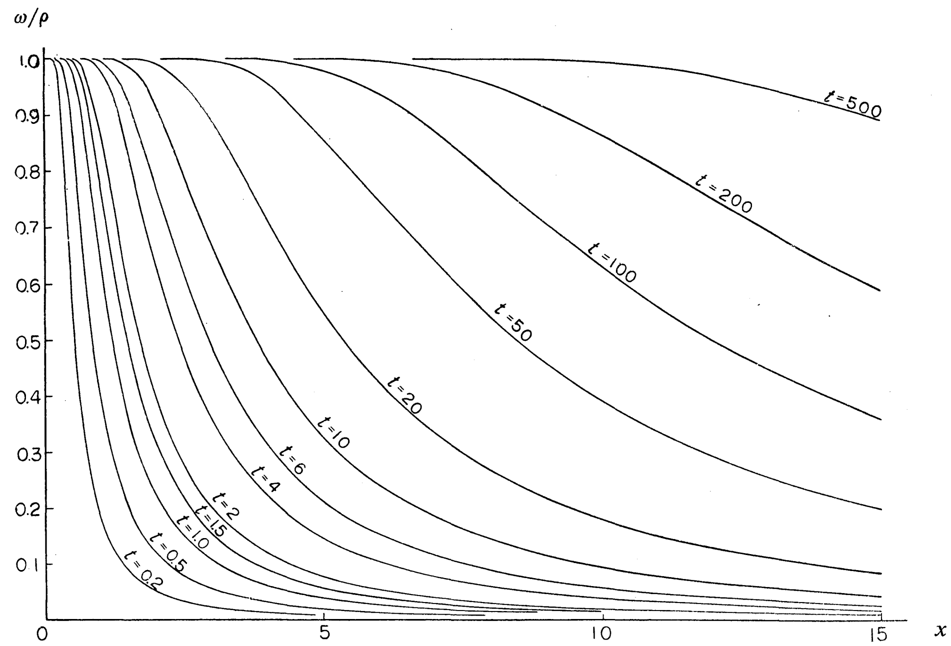

FIG. 2.

are shown in TABLES 1, 2 and FIGURES 2, 3 respectively. In generel, (1.8) can be regarded as typifying an impulsive pressure of a certain intensity, which makes its appearance suddenly at the instant $t=0$ over the section $x=0$ of a small pipe filled with a fluid and spreads to both directions $x>0$ and $x<0$ indefinitely.

Now, are there any real motions of a liquid corresponding to the pressure distribution illustrated by (1.8) ? To answer this question let us imagine a model which follows: A certain section of a long elastic pipe having a small radius and full of a still liquid is supplied continuously from outside with the new liquid of the same kind starting at a certain instant, for example a liquid syringed from a constant head into a rubber tube through its wall at the station $x=0$ steadily from the time $t=0$ on. Very roughly speaking, under suitable circumstances, the pressure at the injection point arises all at once to a higher level, k say, at $t=0$, preserving this value invariably afterwards, while owing to the elasticity of the wall this high tide of pressure diffuses gradually to both directions in the course of time, until at last ($t\to\infty$) the pressure within the pipe increases by the same constant k everywhere thus another state of equilibrium is again restored.

The mode of propagation of the pressure should be determined of course by

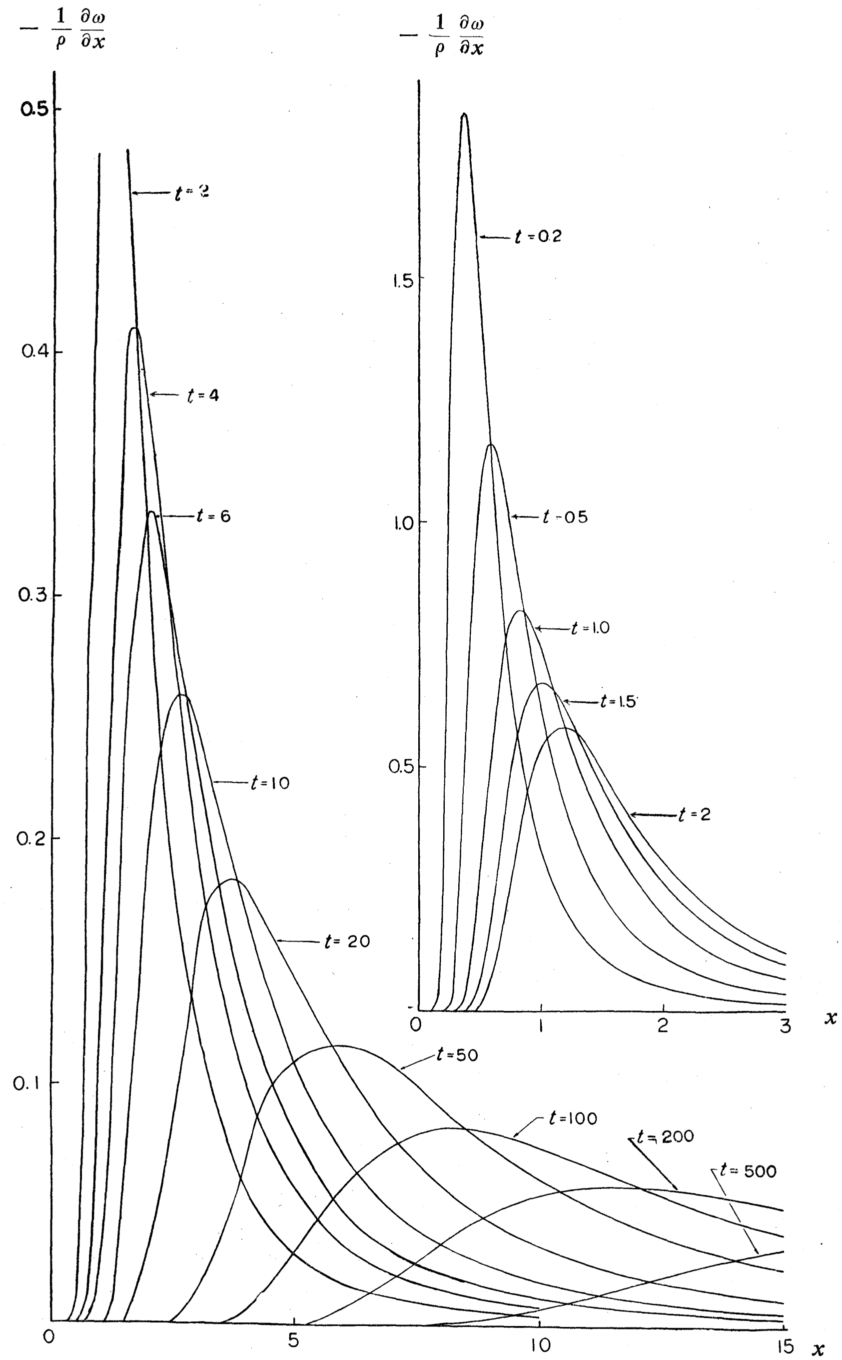
$-\frac{1}{\rho}\frac{\partial\omega}{\partial x}$
0.5
0.4
0.3
0.2
0.1
0
5
10
15
x
$t=2$
$t=4$
$t=6$
$t=10$
$t=20$
$t=50$
$t=100$
$t=200$
$t=500$
$-\frac{1}{\rho}\frac{\partial\omega}{\partial x}$
1.5
1.0
0.5
0
1
2
3
x
$t=0.2$
$t=0.5$
$t=1.0$
$t=1.5$
$t=2$

FIG. 3.

the elasticity of the wall. The problem is very difficult therefore, because we have to solve the equations of motion of hydrodynamics under the boundary conditions specified by the equations of motion of elasticity. For instance, the axial, and the radial, velocity components of the fluid do not vanish on the *elastic* wall as they do on the *rigid* one, but they should be put equal respectively with the axial, and the radial, time rates of the displacements of the wall itself.

But so long as the displacements of the wall are very small, it will not be unreasonable to assume in the first approximation that the flow velocity is negligible on the boundary. Then the motion of the liquid might be approximated very roughly by the flow pattern which is induced by the pressure of the fluid contained in the *rigid* pipe of the same radius propagating in just the same way as it be in the *elastic* pipe in question. The following note should be understood, therefore, as the tentative first approximation of our problem.[1)]

2. Our next step will be to solve the equation (1.7), substituting (1.10) for ω/ρ on its right-hand side. By doing this, we shall be enabled to find out the velocity distribution $u(x, r, t)$ corresponding to this impulsive pressure. In the first place, a bar is placed above a notation in order to denote its Laplace transform with respect to the time t; for example,

$$\bar{u}(x, r, p) \equiv \int_0^\infty u(x, r, t)\, e^{-pt}\, dt\,. \tag{2.1}$$

Since the motion starts from rest, we have

$$u = 0 \quad \text{at} \quad t = 0\,, \tag{2.2}$$

then by partial integration,

$$\overline{\frac{\partial u}{\partial t}} = p\bar{u}\,. \tag{2.3}$$

From (1.10) we obtain

1) Clear resemblance will be noticed between the preceding treatment and the approximation used in the boundary layer theory. Namely, in the first place the variation of pressure across the boundary layer is found negligibly small, and second the pressure distribution just outside the boundary layer cannot be derived from the theory, but it needs to be given beforehand by the potential flow theory or, to answer the purpose better, by the experiment. The pressure distribution (1.8) of our note is of some theoretical interest only, because it can be hardly ever realized in practice, however it is convenient to illustrate our method and might be compared with the typical pressure distributions appearing in the boundary layer theory, for example, the linearly increasing or decreasing pressure, etc. And by making use of the actual distributions of pressure with respect to x and t obtained experimentally for the right-hand side of our equation (1.7), we can calculate the velocity distributions (spatial and temporal) corresponding to this pressure; in other words, by measuring the pressure we can find out the velocity analytically in the circular tube. We wish our approximate method would be proved of some use toward this end in future.

In case that the liquid has a velocity *initially* along the x-axis, the treatment is subject to slight modifications: the pressure distribution is no longer symmetrical for the positive and negative values of x owing to the convection, and besides u at the time $t=0$ should be put equal not to zero but to that initial velocity profile, $U(r, x)$ say.

$$-\overline{\frac{\partial}{\partial x}\left(\frac{\omega}{\rho}\right)} = \frac{2k}{\alpha^2 x^3}\int_0^\infty t\,\exp\left\{-\left(p+\frac{1}{\alpha^2x^2}\right)t\right\}dt$$
$$= \frac{2k}{\alpha^2 x^3}\left(p+\frac{1}{\alpha^2x^2}\right)^{-2}. \tag{2.4}$$

The Laplace transform of the equation (1.7) is therefore as follows:

$$p\bar{u} = \frac{2k}{\alpha^2x^3}\left(p+\frac{1}{\alpha^2x^2}\right)^{-2} + \nu\left(\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}\right). \tag{2.5}$$

It should be noted that the derivative of $\bar{u}(x, r, p)$ with respect to x does not appear in the above equation, so in place of $\partial\bar{u}/\partial r$ we have written $d\bar{u}/dr$ regarding x as a parameter and not as an independent variable. The solution of (2.5) compatible with the boundary condition

$$u = 0 \quad \text{at} \quad r = a, \tag{2.6}$$

is readily found to be

$$\bar{u}(x, r, p) = \frac{2k}{\alpha^2x^3}\left\{J_0\left(i\sqrt{\frac{p}{\nu}}\,a\right)-J_0\left(i\sqrt{\frac{p}{\nu}}\,r\right)\right\}$$
$$\times\left\{p\left(p+\frac{1}{\alpha^2x^2}\right)^2 J_0\left(i\sqrt{\frac{p}{\nu}}\,a\right)\right\}^{-1}, \tag{2.7}$$

where J_0 denotes the Bessel function of the first kind.

The inverse transformation from $\bar{u}(x,r,p)$ to $u(x, r, t)$ is performed by the integral

$$u(x, r, t) = \frac{1}{2\pi i}\int_L e^{tp}\,\bar{u}(x, r, p)\,dp, \tag{2.8}$$

where L is the path of integration on the complex p-plane from $-i\infty$ to $i\infty$ such that all the singularities of the integrand are on its left, none being upon it; FIGURE 4. Substituting from (2.7) into (2.8), it follows that

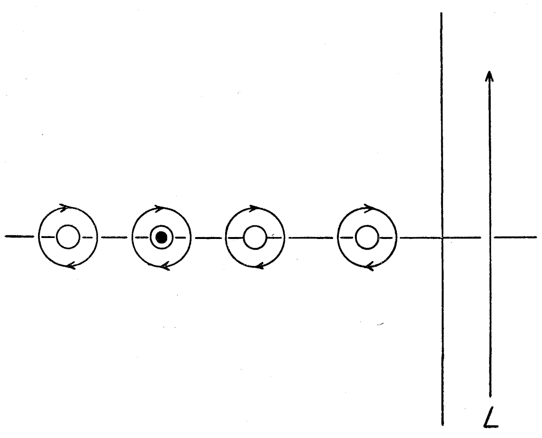

FIG. 4. p-plane.
○ $p=-\nu j_{0,n}^2/a^2$, $n=1, 2, 3,\cdots$
⊙ $p=-1/\alpha^2x^2$.

$$u(x,\ r,\ t) = \frac{1}{2\pi i}\int_L e^{tp}\frac{2k}{\alpha^2x^3}\left\{J_0\left(i\sqrt{\frac{p}{\nu}}a\right) - J_0\left(i\sqrt{\frac{p}{\nu}}r\right)\right\}$$

$$\times\left\{p\left(p+\frac{1}{\alpha^2x^2}\right)^2 J_0\left(i\sqrt{\frac{p}{\nu}}a\right)\right\}^{-1}dp\ . \tag{2.9}$$

The singular points of the function integrated are determined by the equations

$$p = 0\ ,\quad J_0\left(i\sqrt{\frac{p}{\nu}}a\right) = 0,\quad \text{and}\quad p + \frac{1}{\alpha^2x^2} = 0\ .$$

i) In the neighborhood of $p=0$, we have, however,

$$J_0\left(i\sqrt{\frac{p}{\nu}}a\right) = 1 + \frac{1}{4}\frac{p}{\nu}a^2 + \cdots\ , \tag{2.10}$$

and

$$J_0\left(i\sqrt{\frac{p}{\nu}}r\right) = 1 + \frac{1}{4}\frac{p}{\nu}r^2 + \cdots\ , \tag{2.11}$$

so

$$\bar{u}(x,\ r,\ p) = \frac{2k}{\alpha^2x^3}\left\{\frac{\alpha^4x^4}{4\nu}(a^2 - r^2) + \cdots\right\}\ , \tag{2.12}$$

from which we know that $p=0$ is *not* the pole of the integrand.

ii) In order to evaluate the contribution to the integral from the poles defined by the equation

$$J_0\left(i\sqrt{\frac{p}{\nu}}a\right) = 0\ , \tag{2.13}$$

let us write $j_{0,n}$ ($n=1,\ 2,\ \ldots$) for the positive zeros of the J_0-function, then these poles are given by

$$p = -\nu j_{0,n}{}^2\ /\ a^2\ . \tag{2.14}$$

According to the well-known formula, their contributions can be expressed in the form

$$u_1\ (x,\ r,\ t\geqq 0) = \frac{2k}{\alpha^2x^3}\sum_{n=1}^{\infty}\left[e^{tp}\left\{J_0\left(i\sqrt{\frac{p}{\nu}}a\right) - J_0\left(i\sqrt{\frac{p}{\nu}}r\right)\right\}\right.$$

$$\left.\times\left\{p\left(p+\frac{1}{\alpha^2x^2}\right)^2\frac{d}{dp}J_0\left(i\sqrt{\frac{p}{\nu}}a\right)\right\}^{-1}\right]_{p=-\nu j_{0,n}{}^2/a^2}\ , \tag{2.15}$$

and

$$u_1(x,\ r,\ t<0) = 0\ .$$

By virtue of the relation

$$\frac{d}{dp}J_0\left(i\sqrt{\frac{p}{\nu}}a\right) = -\frac{ia}{2\sqrt{p\nu}}J_1\left(i\sqrt{\frac{p}{\nu}}a\right)\ , \tag{2.16}$$

we can arrive at the result

$$u_1 = \frac{2k}{\alpha^2x^3}\ 2\sum_{n=1}^{\infty}\left[\exp\left(-j_{0,n}{}^2\frac{\nu t}{a^2}\right)J_0\left(j_{0,n}\frac{r}{a}\right)\right.$$

$$\left.\times\left\{j_{0,n}\left(\frac{1}{\alpha^2x^2} - j_{0,n}{}^2\frac{\nu}{a^2}\right)^2 J_1(j_{0,n})\right\}^{-1}\right],\quad \text{for } t\geqq 0\ , \tag{2.17}$$

and $$u_1 = 0\,, \text{ for } t<0\,.$$

If we introduce the non-dimensional variables τ, λ and μ such that

$$\begin{aligned} \tau &\equiv t \,/\, \alpha^2 x^2\,, \\ \lambda &\equiv r \,/\, \sqrt{\nu}\,\alpha x\,, \qquad (2.18) \\ \text{and} \quad \mu &\equiv a \,/\, \sqrt{\nu}\,\alpha x\,, \end{aligned}$$

then u_1 can be written

$$u_1(\tau \geqq 0,\ \lambda,\ \mu) = 2k \frac{a\alpha}{\sqrt{\nu}\,\mu}\ 2\sum_{n=1}^{\infty}\Big[\exp\Big(-j_{0,n}{}^2\frac{\tau}{\mu^2}\Big)J_0\Big(j_{0,n}\frac{\lambda}{\mu}\Big)\mu^4 \times \Big\{j_{0,n}\,(\mu^2-j_{0,n}{}^2)^2\,J_1(j_{0,n})\Big\}^{-1}\Big]\,, \qquad (2.19)$$

and $$u_1\,(\tau<0,\ \lambda,\ \mu) = 0\,.$$

iii) Letting u_2 denote the contribution to the integral (2.9) due to the double poles situated at

$$p = -1 \,/\, \alpha^2 x^2\,, \qquad (2.20)$$

we have

$$u_2(x,\ r,\ t \geqq 0) = \frac{2k}{\alpha^2 x^3}\frac{d}{dp}\Big[e^{tp}\Big\{J_0\Big(i\sqrt{\frac{p}{\nu}}\,a\Big) - J_0\Big(i\sqrt{\frac{p}{\nu}}\,r\Big)\Big\} \times \Big\{p\,J_0\Big(i\sqrt{\frac{p}{\nu}}\,a\Big)\Big\}^{-1}\Big]_{p=-1/\alpha^2x^2}\,, \qquad (2.21)$$

and $$u_2\,(x,\ r,\ t<0) = 0\,.$$

Performing the differentiation and making use of τ, λ and μ, it can be shown without difficulty that

$$u_2(\tau \geqq 0,\ \lambda,\ \mu) = -2k\frac{a\alpha}{\sqrt{\nu}\,\mu}\,e^{-\tau}\Big[\frac{\tau\{J_0(\mu)-J_0(\lambda)\} + \{\mu J_1(\mu)-\lambda J_1(\lambda)\}/2}{J_0(\mu)} + \frac{\{J_0(\mu)-J_0(\lambda)\}\,\{J_0(\mu)-\mu J_1(\mu)/2\}}{J_0(\mu)^2}\Big]\,, \qquad (2.22)$$

and $$u_2(\tau<0,\ \lambda,\ \mu) = 0\,.$$

Finally, by combining (2.19) with (2.22), we obtain for the velocity distribution the following expression:

$$\begin{aligned} u(\tau \geqq 0,\ \lambda,\ \mu) &\equiv u_1 + u_2 \\ &= 2k\frac{a\alpha}{\sqrt{\nu}\,\mu}\Big[2\sum_{n=1}^{\infty}\Big[\exp\Big(-j_{0,n}{}^2\frac{\tau}{\mu^2}\Big)J_0\Big(j_{0,n}\frac{\lambda}{\mu}\Big)\mu^4 \\ &\quad \times \Big\{j_{0,n}(\mu^2-j_{0,n}{}^2)^2\,J_1(j_{0,n})\Big\}^{-1}\Big] \end{aligned}$$

$$-e^{-\tau}\left[\frac{\tau\{J_0(\mu)-J_0(\lambda)\}+\{\mu J_1(\mu)-\lambda J_1(\lambda)\}/2}{J_0(\mu)}\right.$$

$$\left.+\frac{\{J_0(\mu)-J_0(\lambda)\}\{J_0(\mu)-\mu J_1(\mu)/2\}}{J_0(\mu)^2}\right]\Bigg], \tag{2.23}$$

and

$$u(\tau<0,\ \lambda,\ \mu)=0\,.$$

3. But when μ^2 coincides with any of $j_{0,n}{}^2$'s $(n=1, 2,\ldots)$, namely from (2.18) for the values of x such that

$$x=\pm a/\sqrt{\nu}\,\alpha\, j_{0,n}\,, \tag{3.1}$$

one of the denominators of the series appearing in (2.19) vanishes. To make a close investigation, we put

$$\mu=j_{0,n}+\delta\,, \tag{3.2}$$

supposing $\delta\ll 1$. (In what follows we shall consider only the case when μ approaches $j_{0,n}$; similar treatment holds good also when μ tends to $-j_{0,n}$.) Then neglecting the terms higher than the order of δ^2,

$$\mu^2=j_{0,n}{}^2+2j_{0,n}\delta+\delta^2\,, \tag{3.3}$$

and

$$\mu^4=j_{0,n}{}^4+4j_{0,n}{}^3\delta+6j_{0,n}{}^2\delta^2+\cdots, \tag{3.4}$$

while from (2.23) we have for $\tau\geqq 0$,

$$\lim_{\mu\to j_{0,n}} u=2k\frac{a\alpha}{\sqrt{\nu}\,j_{0,n}}\Bigg[2\lim_{\mu\to j_{0,n}}\sum_{m=1}^{\infty}{}'\left[\exp\left(-j_{0,m}{}^2\frac{\tau}{\mu^2}\right)J_0\left(j_{0,m}\frac{\lambda}{\mu}\right)\mu^4\right.$$

$$\left.\times\left\{j_{0,m}(\mu^2-j_{0,m}{}^2)^2J_1(j_{0,m})\right\}^{-1}\right]$$

$$+2\lim_{\mu\to j_{0,n}}\left[\exp\left(-j_{0,n}{}^2\frac{\tau}{\mu^2}\right)J_0\left(j_{0,n}\frac{\lambda}{\mu}\right)\mu^4\right.$$

$$\left.\times\left\{j_{0,n}(\mu^2-j_{0,n}{}^2)^2J_1(j_{0,n})\right\}^{-1}\right]$$

$$-\lim_{\mu\to j_{0,n}}e^{-\tau}\left[\frac{\tau\{J_0(\mu)-J_0(\lambda)\}+\{\mu J_1(\mu)-\lambda J_1(\lambda)\}/2}{J_0(\mu)}\right.$$

$$\left.+\frac{\{J_0(\mu)-J_0(\lambda)\}\{J_0(\mu)-\mu J_1(\mu)/2\}}{J_0(\mu)^2}\right]\Bigg], \tag{3.5}$$

where $\sum'$ signifies the summation of the series excluding the term $m=n$.

i) The first term on the right-hand side is obviously

$$u_{\mathrm{i}}=2\lim_{\mu\to j_{0,n}}\sum_{m=1}^{\infty}{}'\left[\exp\left(-j_{0,m}{}^2\frac{\tau}{\mu^2}\right)J_0\left(j_{0,m}\frac{\lambda}{\mu}\right)\mu^4\right.$$

$$\left.\times\left\{j_{0,m}(\mu^2-j_{0,m}{}^2)^2J_1(j_{0,m})\right\}^{-1}\right]$$

$$=2\sum_{m=1}^{\infty}{}'\left[\exp\left(-\frac{j_{0,m}{}^2}{j_{0,n}{}^2}\tau\right)J_0\left(\frac{j_{0,m}}{j_{0,n}}\lambda\right)j_{0,n}{}^4\right.$$

$$\times \left\{ j_{0,m}(j_{0,n}^2 - j_{0,m}^2)^2 \, J_1(j_{0,m}) \right\}^{-1} \Big]. \tag{3.6}$$

ii) In computing the second term, we make use of the following approximations:

$$\frac{1}{\mu^2} = \frac{1}{j_{0,n}^2}\left(1 - 2\frac{\delta}{j_{0,n}} + 3\frac{\delta^2}{j_{0,n}^2} - \cdots\right), \tag{3.7}$$

$$\exp\left(-j_{0,n}^2 \frac{\tau}{\mu^2}\right) = e^{-\tau}\left\{1 + 2\frac{\delta}{j_{0,n}}\tau + \frac{\delta^2}{j_{0,n}^2}(2\tau^2 - 3\tau) + \cdots\right\}, \tag{3.8}$$

$$J_0\left(j_{0,n}\frac{\lambda}{\mu}\right) = J_0(\lambda) - \frac{\delta}{j_{0,n}}\lambda J_0'(\lambda) + \frac{\delta^2}{j_{0,n}^2}\left\{\lambda J_0'(\lambda) + \frac{\lambda^2}{2}J_0''(\lambda)\right\} + \cdots, \tag{3.9}$$

where

$$J_0'(\lambda) = dJ_0(\lambda)/d\lambda, \quad J_0''(\lambda) = d^2J_0(\lambda)/d\lambda^2, \quad \text{etc.}, \tag{3.10}$$

and

$$\frac{1}{(\mu^2 - j_{0,n}^2)^2} = \frac{1}{4\, j_{0,n}^2\delta^2}\left(1 - \frac{\delta}{j_{0,n}} + \frac{3}{4}\frac{\delta^2}{j_{0,n}^2} - \cdots\right). \tag{3.11}$$

The limiting form of the second term is therefore

$$\begin{aligned} u_{\text{ii}} \equiv\ & 2 \lim_{\mu\to j_{0,n}}\Big[\exp\left(-j_{0,n}^2\frac{\tau}{\mu^2}\right) J_0\left(j_{0,n}\frac{\lambda}{\mu}\right)\mu^4 \\ & \times\left\{j_{0,n}(\mu^2 - j_{0,n}^2)^2 J_1(j_{0,n})\right\}^{-1}\Big] \\ =\ & e^{-\tau}\frac{J_0(\lambda) j_{0,n}}{2J_1(j_{0,n})\delta^2}\Big[1 + \frac{\delta}{j_{0,n}}\left\{2\tau - \lambda\frac{J_0'(\lambda)}{J_0(\lambda)} + 3\right\} \\ & + \frac{\delta^2}{j_{0,n}^2}\Big\{-2\tau\lambda\frac{J_0'(\lambda)}{J_0(\lambda)} + 6\tau - 3\lambda\frac{J_0'(\lambda)}{J_0(\lambda)} + 2\tau^2 - 3\tau \\ & + \frac{\lambda J_0'(\lambda) + \lambda^2 J_0''(\lambda)/2}{J_0(\lambda)} + \frac{11}{4}\Big\} + \cdots\Big], \quad (\delta\to 0). \end{aligned} \tag{3.12}$$

iii) In the third term we take into consideration the expansions

$$J_0(\mu) = \delta J_0'(j_{0,n})\left\{1 + \frac{\delta}{2}\frac{J_0''(j_{0,n})}{J_0'(j_{0,n})} + \frac{\delta^2}{6}\frac{J_0'''(j_{0,n})}{J_0'(j_{0,n})} + \cdots\right\}, \tag{3.13}$$

and

$$\begin{aligned} \mu J_1(\mu) = j_{0,n} J_1(j_{0,n})\Big[& 1 + \delta\left\{\frac{1}{j_{0,n}} + \frac{J_1'(j_{0,n})}{J_1(j_{0,n})}\right\} \\ & + \delta^2\left\{\frac{1}{j_{0,n}}\frac{J_1'(j_{0,n})}{J_1(j_{0,n})} + \frac{1}{2}\frac{J_1''(j_{0,n})}{J_1(j_{0,n})}\right\} + \cdots\Big], \end{aligned} \tag{3.14}$$

so we obtain, as δ tends to zero,

$$\begin{aligned} u_{\text{iii}} \equiv & -\lim_{\mu\to j_{0,n}}\left[e^{-\tau}\frac{\tau\{J_0(\mu) - J_0(\lambda)\} + \{\mu J_1(\mu) - \lambda J_1(\lambda)\}/2}{J_0(\mu)}\right] \\ = & -\frac{e^{-\tau}}{\delta J_0'(j_{0,n})}\Big(-\tau J_0(\lambda) + \frac{1}{2}\left\{j_{0,n}J_1(j_{0,n}) - \lambda J_1(\lambda)\right\} \\ & + \delta\Big[\tau J_0'(j_{0,n}) + \frac{1}{2}j_{0,n}J_1(j_{0,n})\left\{\frac{1}{j_{0,n}} + \frac{J_1'(j_{0,n})}{J_1(j_{0,n})}\right\} \end{aligned}$$

$$- \frac{1}{2} \frac{J_0''(j_{0,n})}{J_0'(j_{0,n})} \left\{ -\tau J_0(\lambda) + \frac{1}{2} \{ j_{0,n} J_1(j_{0,n}) - \lambda J_1(\lambda) \} \right\} \Big] + \cdots \Big],$$

$$(\delta \to 0). \qquad (3.15)$$

iv) Finally, for the last term we have

$$\frac{1}{J_0(\mu)^2} = \frac{1}{\delta^2 J_0'(j_{0,n})^2} \Big[1 - \delta \frac{J_0''(j_{0,n})}{J_0'(j_{0,n})} + \delta^2 \Big\{ -\frac{1}{3} \frac{J_0'''(j_{0,n})}{J_0'(j_{0,n})}$$

$$+ \frac{3}{4} \frac{J_0''(j_{0,n})^2}{J_0'(j_{0,n})^2} \Big\} + \cdots \Big], \qquad (3.16)$$

$$J_0(\mu) - \frac{\mu}{2} J_1(\mu) = - \frac{j_{0,n} J_1(j_{0,n})}{2} + \delta \Big[J_0'(j_{0,n}) - \frac{1}{2} \Big\{ J_1(j_{0,n})$$

$$+ j_{0,n} J_1'(j_{0,n}) \Big\} \Big] + \delta^2 \Big[\frac{J_0''(j_{0,n})}{2} - \frac{1}{2} \Big\{ J_1'(j_{0,n}) + \frac{1}{2} j_{0,n} J_1''(j_{0,n}) \Big\} \Big]$$

$$+ \cdots. \qquad (3.17)$$

Accordingly,

$$u_{\mathrm{iv}} \equiv - \lim_{\mu \to j_{0,n}} \Big[e^{-\tau} \frac{\{J_0(\mu) - J_0(\lambda)\} \{J_0(\mu) - \mu J_1(\mu)/2\}}{J_0(\mu)^2} \Big].$$

$$= \frac{e^{-\tau} J_0(\lambda)}{\delta^2 J_0'(j_{0,n})^2} \Big[- \frac{j_{0,n} J_1(j_{0,n})}{2} + \delta \Big[\frac{j_{0,n} J_1(j_{0,n})}{2} \Big\{ \frac{J_0'(j_{0,n})}{J_0(\lambda)} + \frac{J_0''(j_{0,n})}{J_0'(j_{0,n})} \Big\}$$

$$+ J_0'(j_{0,n}) - \frac{1}{2} \Big\{ J_1(j_{0,n}) + j_{0,n} J_1'(j_{0,n}) \Big\} \Big]$$

$$+ \delta^2 \Big[- \frac{j_{0,n} J_1(j_{0,n}) J_0''(j_{0,n})}{2 J_0(\lambda)} - \frac{J_0'(j_{0,n})}{J_0(\lambda)} \Big\{ J_0'(j_{0,n}) - \frac{1}{2} \{ J_1(j_{0,n})$$

$$+ j_{0,n} J_1'(j_{0,n}) \} \Big\} - \frac{J_0''(j_{0,n})}{J_0'(j_{0,n})} \Big\{ J_0'(j_{0,n}) - \frac{1}{2} \{ J_1(j_{0,n})$$

$$+ j_{0,n} J_1'(j_{0,n}) \} \Big\}$$

$$- \frac{j_{0,n} J_1(j_{0,n})}{2} \Big\{ - \frac{1}{2} \frac{J_0''(j_{0,n})}{J_0(\lambda)} + \frac{3}{4} \frac{J_0''(j_{0,n})^2}{J_0'(j_{0,n})^2} - \frac{1}{3} \frac{J_0'''(j_{0,n})}{J_0'(j_{0,n})} \Big\}$$

$$+ \frac{J_0''(j_{0,n})}{2} - \frac{1}{2} \Big\{ J_1'(j_{0,n}) + \frac{1}{2} j_{0,n} J_1''(j_{0,n}) \Big\} \Big] + \cdots \Big], \quad (\delta \to 0). \quad (3.18)$$

v) Summing up (3.6), (3.12), (315), and (3.18), both the terms of δ^{-2} and δ^{-1} drop out and we are led up to the following conclusion:

$$\lim_{\mu \to j_{0,n}} u = 2k \frac{a\alpha}{\sqrt{\nu}\, j_{0,n}} \Big[2 \sum_{m=1}^{\infty}{}' \Big[\exp\Big(- \frac{j_{0,m}^2}{j_{0,n}^2} \tau \Big) J_0\Big(\frac{j_{0,m}}{j_{0,n}} \lambda \Big) j_{0,n}^4$$

$$\times \Big\{ j_{0,m} (j_{0,n}^2 - j_{0,m}^2)^2 J_1(j_{0,m}) \Big\}^{-1} \Big]$$

$$+ \frac{e^{-\tau}}{J_1(j_{0,n})} \Big[\Big(2\tau^2 + 2\tau - \frac{\lambda^2}{2} - \frac{1}{3} + \frac{j_{0,n}^2}{6} \Big) \frac{J_0(\lambda)}{2\, j_{0,n}}$$

$$+ (\tau + 1) \Big\{ \frac{\lambda J_1(\lambda)}{j_{0,n}} - J_1(j_{0,n}) \Big\} \Big] \Big]. \qquad (3.19)$$

In deriving this result, the formulas given below are useful.

$$J_0(j_{0,n}) = 0\,, \tag{3.20}$$

$$J_1(j_{0,n}) + j_{0,n}\, J_1'(j_{0,n}) = 0\,, \tag{3.21}$$

$$J_0''(j_{0,n}) - \left\{ J_1'(j_{0,n}) + \frac{1}{2} j_{0,n}\, J_1''(j_{0,n}) \right\}$$

$$= \frac{1}{2}\left(j_{0,n} - \frac{2}{j_{0,n}} \right) J_1(j_{0,n}), \tag{3.22}$$

$$\frac{\lambda^2}{2} J_0''(\lambda) + \lambda J_0'(\lambda) = \frac{\lambda}{2}\left\{ J_0'(\lambda) - \lambda J_0(\lambda) \right\}\,,$$

$$J_0'''(j_{0,n}) = \left(1 - \frac{2}{j_{0,n}^{\,2}} \right) J_1(j_{0,n})\,. \tag{3.23}$$

4. With a view to knowing the general properties of the solution given by (2.23), we take up the following examples:

$$\mu = 2, \quad 10, \quad 30, \quad \text{and} \quad 50\,.$$

Before entering the details, however, we must add a few words about the nature of the solution at $\tau = 0$. From the condition (2.2) imposed upon the Laplace transform $\bar{u}$, the solution should satisfy the requirement

$$u(\tau = 0\,, \lambda\,, \mu) = 0\,.$$

Namely, putting $\tau = 0$ on the right-hand side of (2.23), we have to prove the identity

$$2 \sum_{n=1}^{\infty} \frac{J_0(j_{0,n} \cdot \sigma)\, \mu^3}{j_{0,n}(\mu^2 - j_{0,n}^{\,2})^2\, J_1(j_{0,n})}$$

$$= \frac{1}{\mu}\left[\frac{\mu J_1(\mu) - \lambda J_1(\lambda)}{2\, J_0(\mu)} + \frac{\{J_0(\mu) - J_0(\lambda)\}\, \{J_0(\mu) - \mu J_1(\mu)/2\}}{J_0(\mu)^2} \right], \tag{4.1}$$

at least for the range $\mu \geqq \lambda \geqq 0$, where for shortness we write

$$\sigma \equiv \lambda \,/\, \mu = r/a\,. \tag{4.2}$$

Because the analytical proof seems difficult, let us take an indirect and, at the same time, an empirical method. That is to say, fixing the values of σ and μ at $\sigma = 0$, 0.4, and $\mu = 2$, 10, 30, 50, we shall calculate and compare the values of the both hand sides of (4.1).

i) At the point $\sigma = 0$, namely from (4.2) $r = 0$ (the axis of the pipe), the left-hand side of (4.1) reduces to

$$\sum_{n=1}^{\infty} 2\, \mu^3 \left\{ j_{0,n}(\mu^2 - j_{0,n}^{\,2})^2\, J_1(j_{0,n}) \right\}^{-1}. \tag{4.3}$$

As it will become evident by numerical computations, this series converges very slowly for large values of μ. However, it is not difficult to know the asymptotic form of each term for a very large value of n. For we may put approximately

$$j_{0,n} \sim (n - 1/4)\pi\,, \tag{4.4}$$

as n tends to infinity,[1] and again by the formula of the asymptotic expansion we obtain[2]

$$J_1(j_{0,n}) \sim \sqrt{\frac{2}{\pi j_{0,n}}} \cos\left(j_{0,n} - \frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$\sim (-)^{n+1} \sqrt{2} / \pi \sqrt{n - 1/4}\,. \tag{4.5}$$

Neglecting $1/4$ against n, and μ^2 against $j_{0,n}{}^2$, it can be proved that

$$2\mu^3 \{ j_{0,n}(\mu^2 - j_{0,n}{}^2)^2 J_1(j_{0,n}) \}^{-1} \sim (-)^{n+1} 2^{1/2}\mu^3 \pi^{-4} n^{-9/2}\,, \tag{4.6}$$

approximately. For example, in the case

$$n = 40,$$

the exact, and the approximate, values represented by the both hand sides of (4.6) are respectively

μ	exact values	approximata values
2	$-0.0000\ 00007$	$-0.0000\ 00007$
10	$-0.0000\ 00934$	$-0.0000\ 00897$
30	$-0.0000\ 2805$	$-0.0000\ 2421$
50	$-0.0001\ 6351$	$-0.0001\ 1209$

By making a partial sum for $n = 2m$ and $n = 2m + 1$, we get

$$\frac{\sqrt{2}\,\mu^3}{\pi^4}\left\{ - \frac{1}{(2m)^{9/2}} + \frac{1}{(2m+1)^{9/2}} \right\} \sim - \frac{9}{64}\,\frac{\mu^3}{\pi^4 m^{11/2}}\,. \tag{4.7}$$

The remainder of the series for the values of $n \geqq 40$ is approximated, therefore, by

$$R_r \equiv \sum_{n=40}^{\infty} \frac{2\mu^3}{j_{0,n}(\mu^2 - j_{0,n}{}^2)^2 J_1(j_{0,n})} \sim - \frac{9}{64}\,\frac{\mu^3}{\pi^4} \sum_{m=20}^{\infty} m^{-11/2}$$

$$\sim - \int_{20}^{\infty} \frac{9}{64}\,\frac{\mu^3}{\pi^4}\,\frac{dm}{m^{11/2}} = -0.4483 \times 10^{-9} \times \mu^3\,. \tag{4.8}$$

Adding the exact value of the sum from $n = 1$ to 39 we find the left-hand side of (4.1), while the right-hand side of which can be evaluated with no difficulty:

μ	left-hand side	right-hand side
2	4.0193 942	4.0193 953
10	0.8659 8181	0.8659 8178
30	-7.5405 139	-7.5405 174
50	-15.9902 12	-15.9902 43

1) Watson, G. N., *A Treatise on the Theory of Bessel Functions*, Cambridge at the University Press (1922), pp. 503–505.

2) Watson, G. N., *ibid.*, pp. 194–196.

The small differences between the both hand sides are probably due to the accumulation of minor errors accompanying the rounding off of the numbers.

ii) We next turn to the case $\tau=0$ and $\sigma=0.4$; by virtue of the formulas (4.4), (4.5) together with

$$J_0(0.4\, j_{0,n}) \sim \sqrt{\frac{2}{0.4\pi\, j_{0,n}}} \cos\left(0.4\, j_{0,n} - \frac{\pi}{4}\right), ^{1)} \tag{4.9}$$

the series on the left-hand side of (4.1) is found to become asymptotically

$$\frac{2\mu^3 J_0(0.4\, j_{0,n})}{j_{0,n}(\mu^2 - j_{0,n}{}^2)^2 J_1(j_{0,n})} \sim (-)^{n+1} \frac{\sqrt{10}\,\mu^3}{\pi^5 n^5} \cos\left(\frac{2}{5}n\pi - \frac{7}{20}\pi\right), \tag{4.10}$$

as n tends to infinity. Putting $n=10m+1$, $10m+2, \dots$, $10m+9$, $10m+10$, and writing s_m for the partial sum consisting of $n=10m+1$, $10m+2, \dots$, $10m+10$, we have

$$\begin{aligned} s_m &\equiv \sum_{n=10m+1}^{10m+10} (-)^{n+1} \frac{\sqrt{10}\,\mu^3}{\pi^5 n^5} \cos\left(\frac{2}{5}n\pi - \frac{7}{20}\pi\right) \\ &= \frac{\sqrt{10}\,\mu^3}{\pi^5} \left\{ \frac{1}{(10m+1)^5} \cos\left(\frac{1}{20}\pi\right) - \frac{1}{(10m+2)^5} \cos\left(\frac{9}{20}\pi\right) \right. \\ &\quad + \frac{1}{(10m+3)^5} \cos\left(\frac{17}{20}\pi\right) - \frac{1}{(10m+4)^5} \cos\left(\frac{25}{20}\pi\right) \\ &\quad + \frac{1}{(10m+5)^5} \cos\left(\frac{33}{20}\pi\right) - \frac{1}{(10m+6)^5} \cos\left(\frac{41}{20}\pi\right) \\ &\quad + \frac{1}{(10m+7)^5} \cos\left(\frac{49}{20}\pi\right) - \frac{1}{(10m+8)^5} \cos\left(\frac{57}{20}\pi\right) \\ &\quad \left. + \frac{1}{(10m+9)^5} \cos\left(\frac{65}{20}\pi\right) - \frac{1}{(10m+10)^5} \cos\left(\frac{73}{20}\pi\right) \right\} \\ &\sim \frac{\sqrt{10}\,\mu^3}{\pi^5} \frac{25}{(10m)^6} \left\{ \cos\left(\frac{1}{20}\pi\right) - \cos\left(\frac{9}{20}\pi\right) + \cos\left(\frac{17}{20}\pi\right) \right. \\ &\qquad \left. - \cos\left(\frac{25}{20}\pi\right) + \cos\left(\frac{33}{20}\pi\right) \right\}. \end{aligned} \tag{4.11}$$

So the remainder of the series (4.1) for $n \geqq 41$ is given approximately by

$$\begin{aligned} R_r &\equiv \sum_{n=41}^{\infty} \frac{2\mu^3 J_0(0.4\, j_{0,n})}{j_{0,n}(\mu^2 - j_{0,n}{}^2)^2 J_1(j_{0,n})} = \sum_{m=4}^{\infty} s_m \\ &\sim \int_4^{\infty} s_m\, dm = 5.557 \times 10^{-11} \times \mu^3, \end{aligned} \tag{4.12}$$

and finally we obtain the following result for (4.1):

1) Watson, G. N., *ibid.*, pp. 194–196.

μ	left-hand side	right hand side
2	3.1489 3524	3.1489 3585
10	−0.2579 17354	−0.2579 17320
30	−0.8452 8942	−0.8452 8877
50	−2.8936 0105	−2.8935 9762

From these two tables we think we can infer almost certainly that the identity (4.1) should be valid for any values of λ and μ ($\mu \geqq \lambda \geqq 0$).

5. Finally, by means of (2.23) we shall calculate $u(\tau, \lambda, \mu)$ for the values of

$$\tau = 0.1, \quad 1, \quad 10, \qquad\qquad (5.1)$$

and

$$\mu = 2, \quad 10, \quad 30, \quad 50,$$

at various λ ranging from zero to μ (or in terms of σ, from zero to one). For the convenience of our subsequent discussions we rewrite the formula in the form

$$u(\tau, \lambda, \mu) \equiv 2k \frac{a\alpha}{\sqrt{\nu}} u^* \equiv 2k \frac{a\alpha}{\sqrt{\nu}} (u_1^* + u_2^*) , \qquad (5.2)$$

where

$$u_1^* = \sum_{n=1}^{\infty} 2\Big[\exp\Big(-j_{0,n}{}^2 \frac{\tau}{\mu^2} \Big) \mu^3 \Big\{ j_{0,n}(\mu^2 - j_{0,n}{}^2)^2 J_1(j_{0,n}) \Big\}^{-1} \times J_0(j_{0,n} \cdot \sigma) \Big]$$

$$\equiv \sum_{n=1}^{\infty} c_n J_0(j_{0,n} \cdot \sigma) , \qquad (5.3)$$

and

$$u_2^* = -\frac{e^{-\tau}}{\mu} \Big[\frac{\tau\{J_0(\mu) - J_0(\lambda)\} + \{\mu J_1(\mu) - \lambda J_1(\lambda)\}/2}{J_0(\mu)} + \frac{\{J_0(\mu) - J_0(\lambda)\}\{J_0(\mu) - \mu J_1(\mu)/2\}}{J_0(\mu)^2} \Big] . \qquad (5.4)$$

The coefficients c_n' s, which depend upon τ, μ and n, are shown in TABLES 3 ($\tau=0$), 4($\tau=0.1$), 5($\tau=1$), and 6($\tau=10$). As seen in these tables, for large values of μ (small x in other words,) $|c_n|$ decreases very slowly with increasing n, especially when τ or t is small The series u^* converges only very slowly, therefore, for large μ and small τ; for example when $\mu=30$ or 50 and $\tau=0.1$, the first 40 terms are found insufficient to compute the value of u_1^* with satisfactory accuracy. And in fact as we know later from the numerical calculations, in the case when τ is small, u_1^* and u_2^* have nearly the same value with opposite signs, so u^* should be evaluated as the small difference between two large numbers. Thus in order to find u^* precisely u_1^* has to be computed with much accuracy, which is unfortunately very difficult.

$J_0(j_{o,n} \cdot \sigma)$ in (5.3) was calculated for σ and n varying from 0 to 1 at the intervals of 0.025 and from 0 to 40 at the intervals of 1, respectively. TABLES 16 in

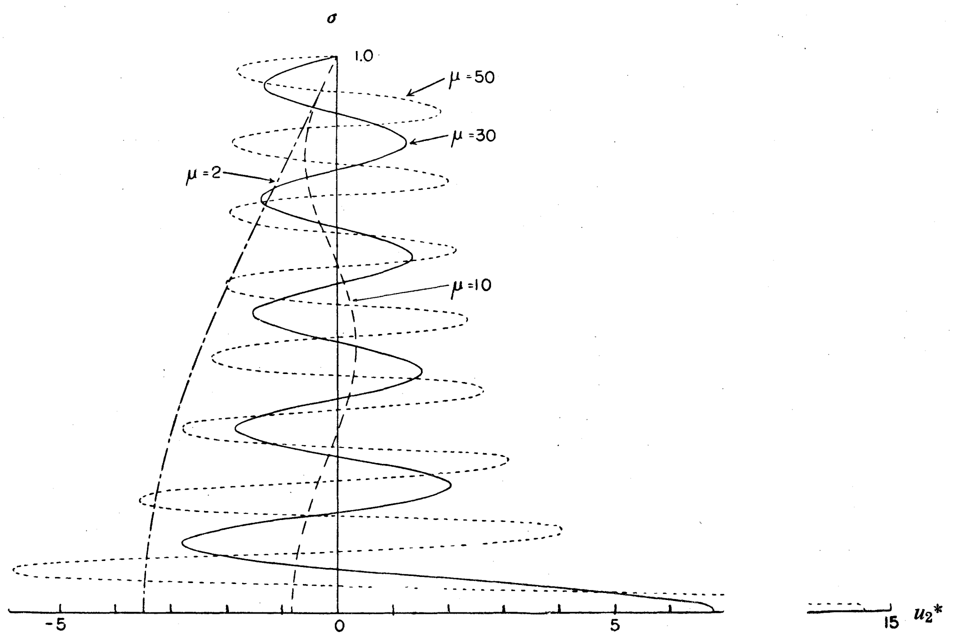

FIG. 5. $u_2^*(\mu=2,\ 10,\ 30,\ 50;\ \tau=0.1)$

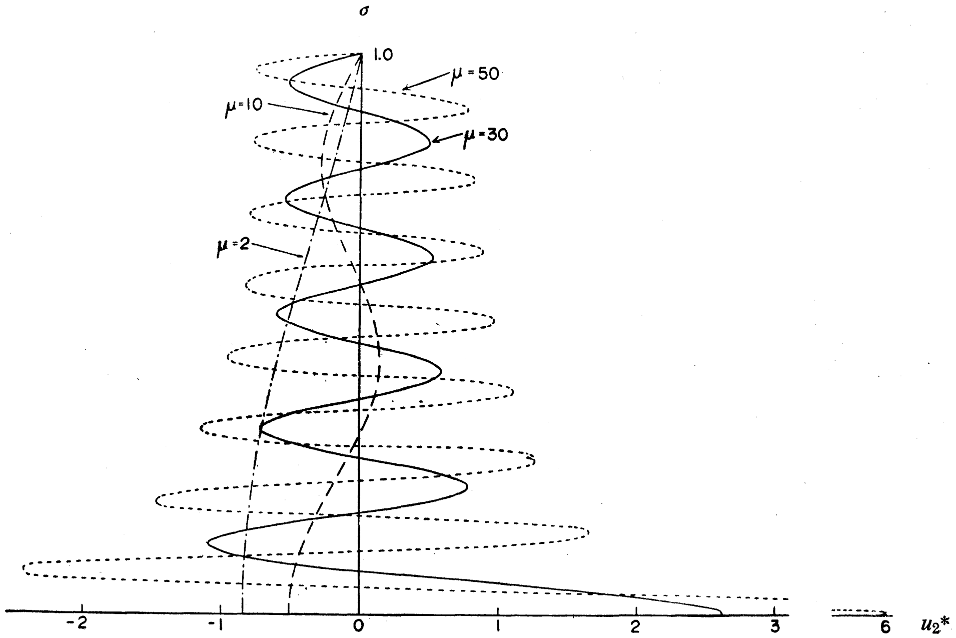

FIG. 6. u_2^* $(\mu=2,\ 10,\ 20,\ 50;\ \tau=1)$

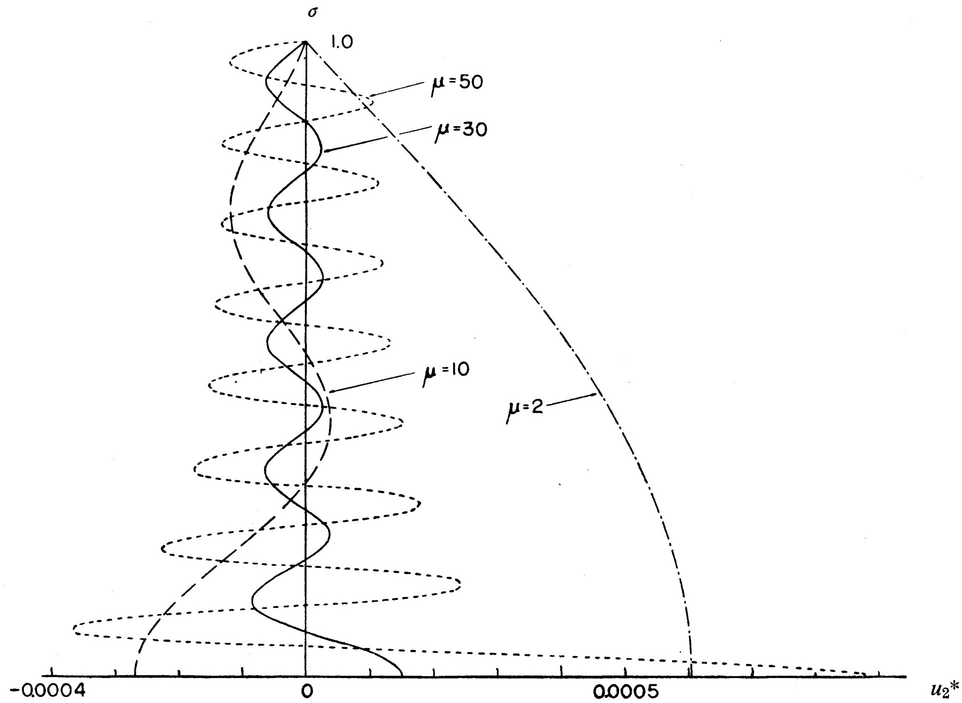

FIG. 7. $u_2^*(\mu=2, 10, 30, 50;\ \tau=10)$

the APPENDIX I embody the result collectively.[1] u_1^* and u_2^* are shown in TABLES 7($\tau=0.1$), 8($\tau=1$), 9($\tau=10$), and in 10($\tau=0.1$), 11($\tau=1$), 12($\tau=10$), respectively. In FIGURES 5, 6, 7 are shown the behaviors of u_2^* which depends upon τ, μ and σ. The feature of u_1^* is almost similar to that of u_2^* and its figure is omitted. And in TABLES 13($\tau=0.1$), 14($\tau=1$), 15($\tau=10$), and FIGURES 8 to 19 at the end of this note, our final result u^* is shown.

As clearly seen in TABLE 13 and FIGURE 17, the calculated values of u^* for

1) J_0-functions of arguments less than 100 were computed with the aid of linear interpolations. Use was made of the following tables:

Hayashi, K., *Bessel Kansu Hyô*, Iwanami (1949), enlarged edition.

British Association for the Advancement of Science, *Bessel Functions, Part* 1, *Functions of Orders Zero and Unity* (*Mathematical Tables*, Vol. VI), Cambridge at the University Press (1937).

Staff of the Computation Laboratory of Harvard University, *Tables of the Bessel Functions of the First Kind of Orders Zero and One*, Harvard University Press (1947).

The most excellent book last mentioned became available to us only at the final period of our work. J_0' s whose arguments exceed 100 were evaluated approximately by means of the following formula (Watson, G. N., *loc. cit.*, pp. 194-196):

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right).$$

$\tau=0.1$, $\mu=50$ are markedly dispersed in both of the regions $\sigma\sim0$ and $\sigma\sim1$. For intermediate values of σ, the points are observed much more well-behaved. The scattering in the neighborhood of $\sigma\sim1$ is due most likely to the errors introduced by the asymptotic expansions of J_0-functions whose arguments exceed 100, while the irregularity appearing when σ is small should be ascribed to the slow convergency of the $u_1{}^*$-series (*the cut-off effect*, in other words). We suppose that the terms of several times of *forty* would be necessary to know the values of $u_1{}^*$ with sufficient accuracy at the point $\sigma=0$ and its neighborhood.

With a view to estimating the error caused by cutting-off the $u_1{}^*$–series at $n=40$, we shall evaluate the remainder of the series for the case $\tau=0.1$, $\mu=50$ at the point $\sigma=0$ (the axis of the pipe). Making use of the approximation procedure explained in **4**, especially in (4.4) and (4.6), we can write the remainder in the form

$$R_r = \frac{\sqrt{2}\,\mu^3}{\pi^4}\sum_{n=41}^{\infty}\left[(-)^{n+1}\,n^{-9/2}\exp\left\{-\frac{\pi^2}{\mu^2}\,\tau\left(n-\frac{1}{4}\right)^2\right\}\right]. \tag{5.5}$$

In our case $\pi^2\tau\mu^{-2}$ being about the order of $1/2500$, the exponential function may be expanded in the form

$$1-n^2\pi^2\tau/\mu^2\,, \tag{5.6}$$

since as n tends to infinity the error of this expansion will become negligible by virtue of the factor $n^{-9/2}$. Thus

$$R_r \sim \frac{\sqrt{2}\,\mu^3}{\pi^4}\sum_{n=41}^{\infty}\left\{(-)^{n+1}\,n^{-9/2}+(-)^n\pi^2\tau\,\mu^{-2}n^{-5/2}\right\}. \tag{5.7}$$

Substituting the integration for the summation, we can readily obtain

$$\begin{aligned} R_r &\sim \frac{\sqrt{2}\,\mu^3}{\pi^4}\left(2^{-10}\times10^{-9/2}-\pi^2\tau\mu^{-2}\times2^{-6}\times10^{-5/2}\right)\\ &= 0.0000\ 2064\,. \end{aligned} \tag{5.8}$$

Adding this value to $u^*\doteqdot0.0000\ 5309$, which is the summation of the series of the first 40 terms, we get

$$u^*=0.0000\ 7373\,, \tag{5.9}$$

as the corrected value of u^* ($\tau=0.1$; $\mu=50$, $\sigma=0$). The point obtained in this way is indicated by the black circle in FIGURE 17.

Throughout these works including the appendices, the writer is very grateful to Professor Yamada of our institute for the advices and the critical discussions. His thanks are due also to Messrs. Inoue and Aoki for their help in numerical calculations as well as in preparing the manuscript.

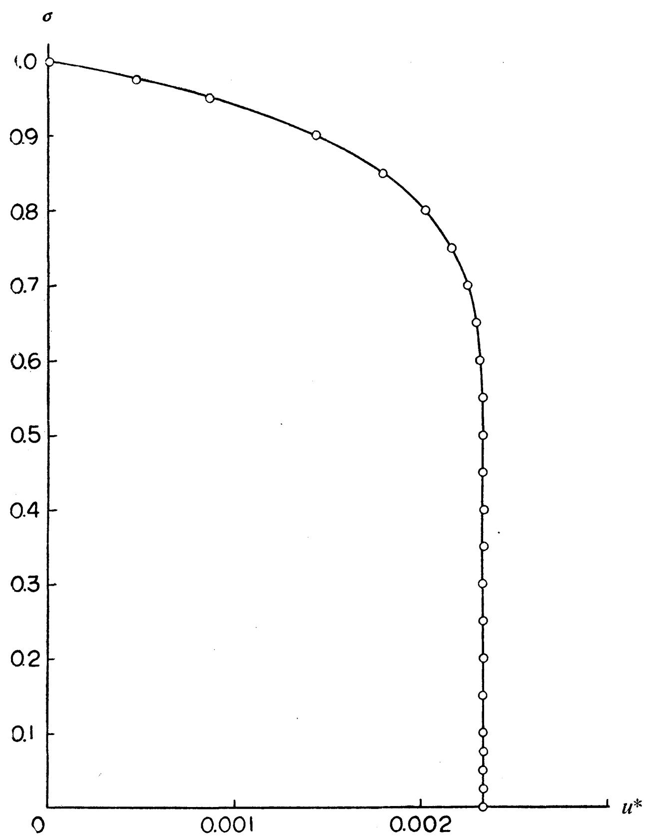

FIG. 8. $u^*(\mu=2,\ \tau=0.1)$

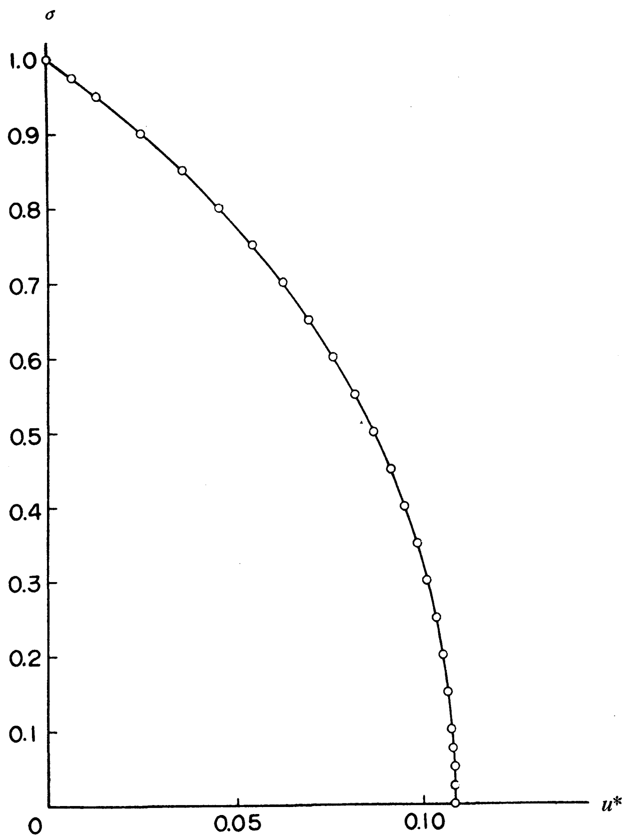

FIG. 9. u^* $(\mu=2,\ \tau=1)$

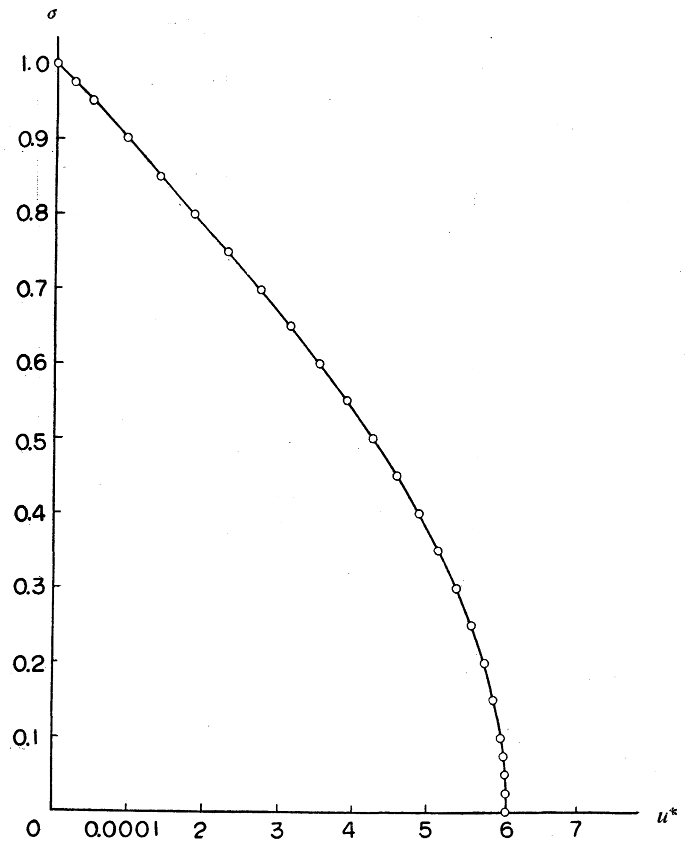

FIG. 10. $u^*(\mu=2,\ \tau=10)$

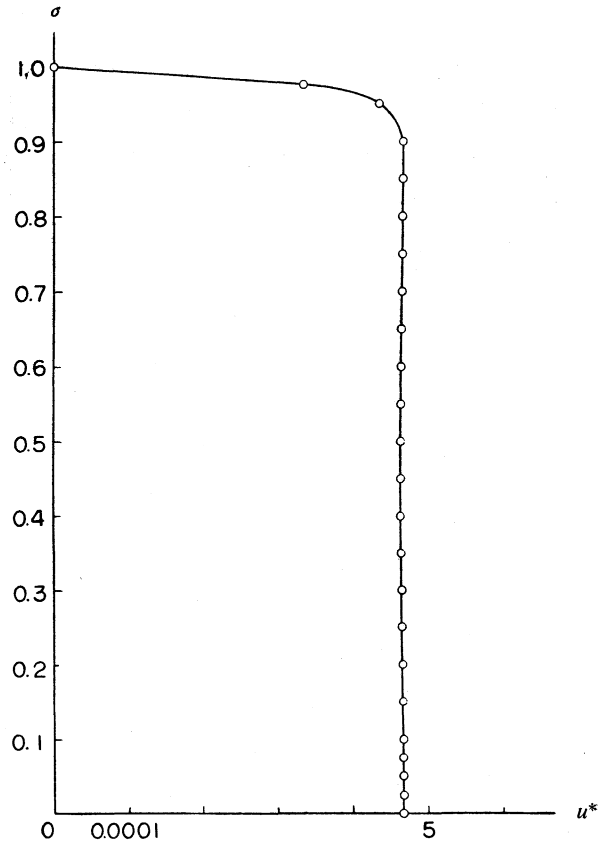

FIG. 11. $u^*(\mu=10,\ \tau=0.1)$

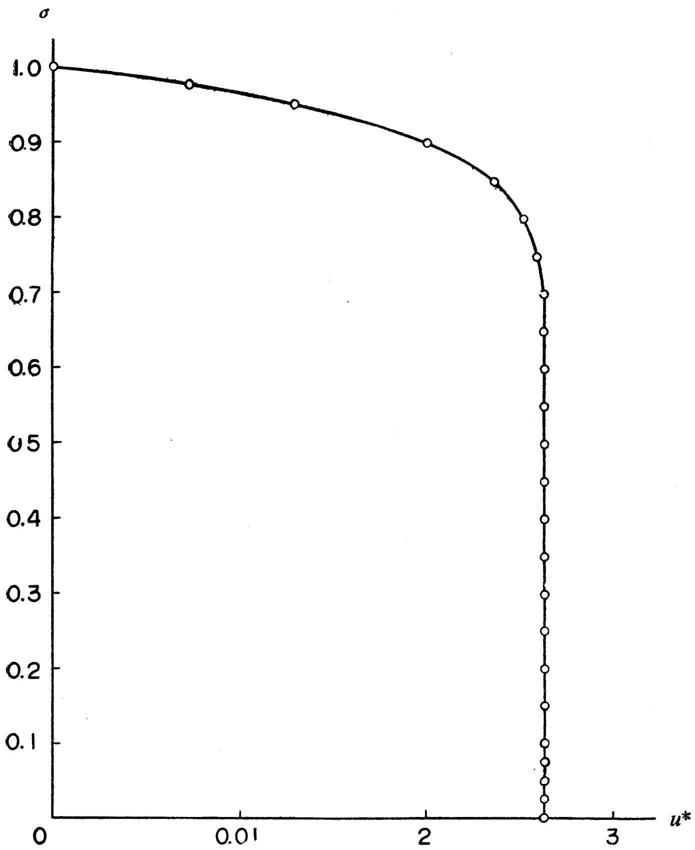

FIG. 12. $u^*(\mu=10, \ \tau=1)$

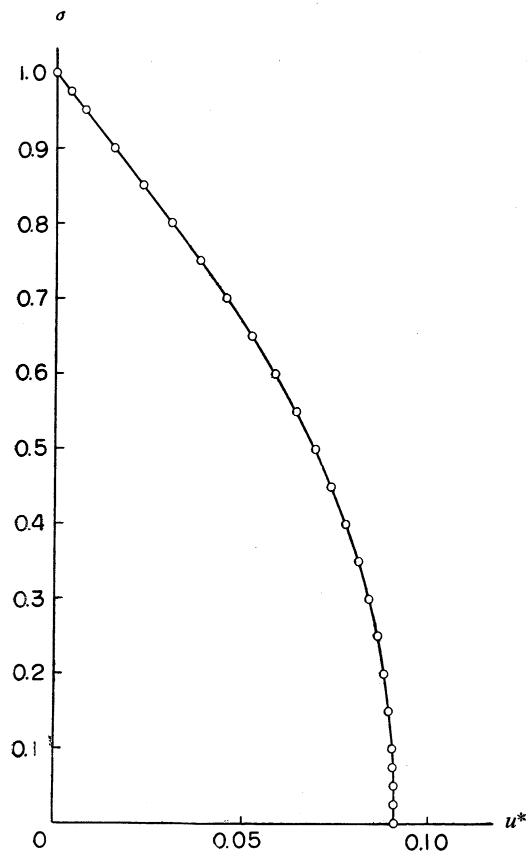

FIG. 13. $u^*(\mu=10, \ \tau=10)$

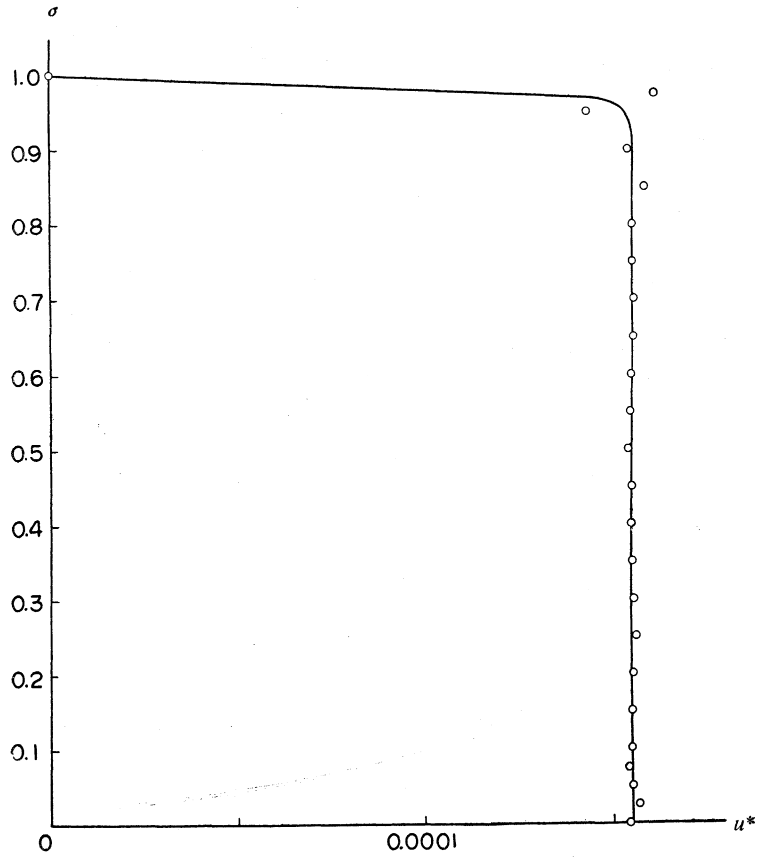

FIG. 14. $u^*(\mu=30,\ \tau=0.1)$

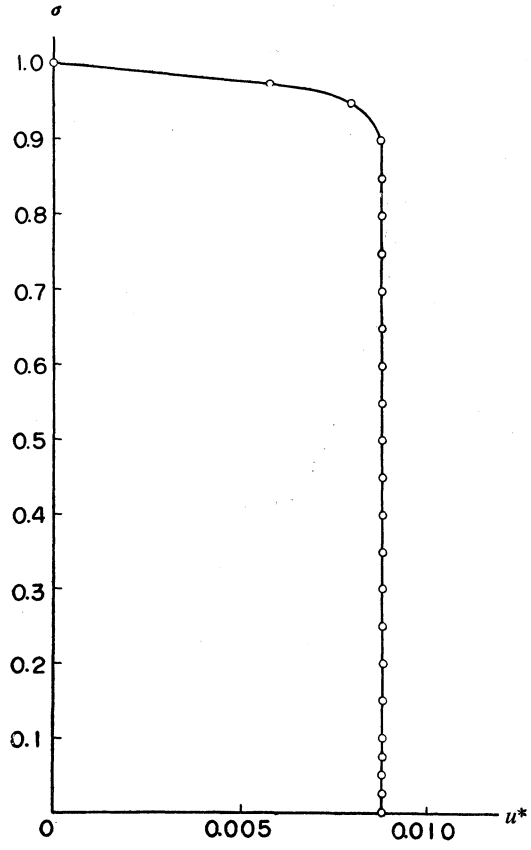

FIG. 15. $u^*(\mu=30,\ \tau=1)$

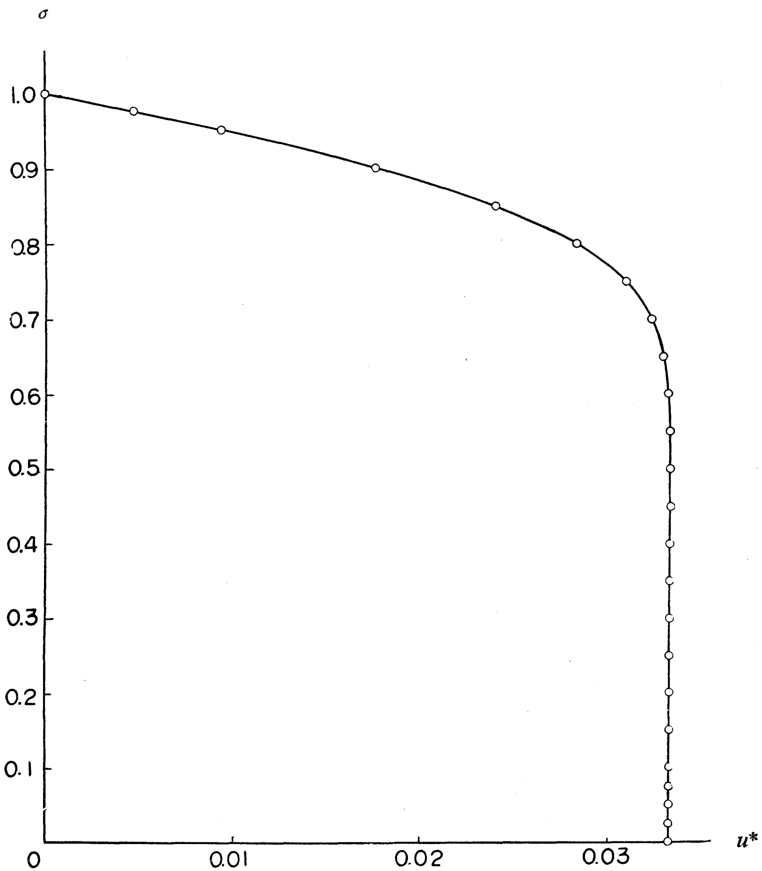

FIG. 16. $u^*(\mu=30, \quad \tau=10)$

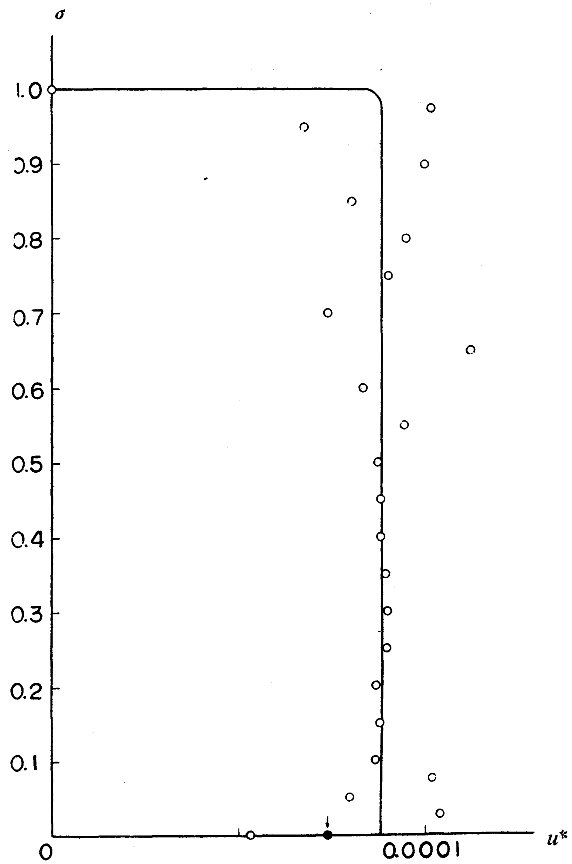

FIG. 17. $u^*(\mu=50, \quad \tau=0.1)$

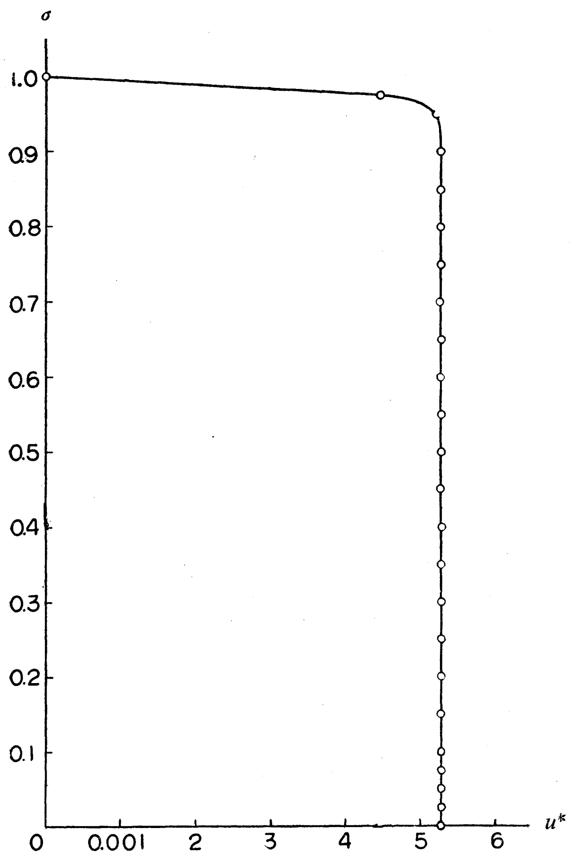

FIG. 18. $u^*(\mu=50, \ \tau=1)$

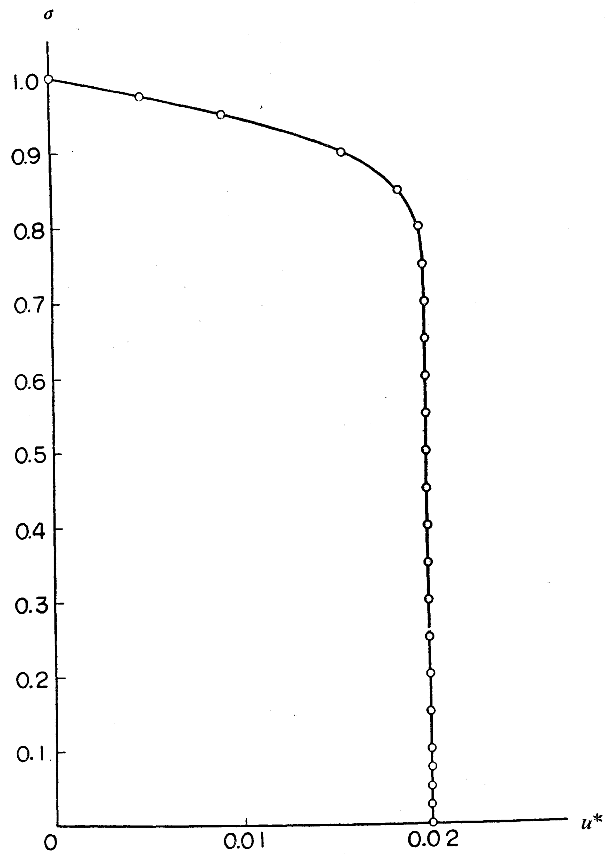

FIG. 19. $u^*(\mu=50, \ \tau=10)$

TABLE 1. $1-\exp(-t/x^2)$.

t \ x	0.2	0.3	0.4	0.5	0.7	0.85	1.0	1.5	2	3	4	6	8	10	12	15	20	30
0.2	0.993	0.892	0.714	0.551	0.335	0.242	0.181	0.085	0.049	0.022	0.012	0.006	0.003	0.002	0.001	0.001	0.001	0.000
0.5	1.000	0.996	0.956	0.865	0.640	0.499	0.393	0.199	0.118	0.054	0.031	0.014	0.008	0.005	0.003	0.002	0.001	0.001
1.0	1.000	1.000	0.998	0.982	0.870	0.749	0.632	0.359	0.221	0.105	0.061	0.027	0.015	0.010	0.007	0.004	0.003	0.001
1.5	1.000	1.000	1.000	0.998	0.953	0.875	0.777	0.487	0.313	0.154	0.090	0.041	0.023	0.015	0.010	0.007	0.004	0.002
2	1.000	1.000	1.000	1.000	0.983	0.937	0.865	0.589	0.393	0.199	0.118	0.054	0.031	0.020	0.014	0.009	0.005	0.002
4	1.000	1.000	1.000	1.000	1.000	0.996	0.982	0.831	0.632	0.359	0.221	0.105	0.061	0.039	0.027	0.018	0.010	0.004
6	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.931	0.777	0.487	0.313	0.154	0.090	0.058	0.041	0.026	0.015	0.007
10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.988	0.918	0.671	0.465	0.243	0.145	0.095	0.067	0.043	0.025	0.011
20	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.993	0.892	0.714	0.426	0.268	0.181	0.130	0.085	0.049	0.022
50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.956	0.751	0.542	0.393	0.293	0.199	0.118	0.054
100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.938	0.790	0.632	0.501	0.359	0.221	0.105
200	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.956	0.865	0.751	0.589	0.393	0.199
500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.993	0.969	0.892	0.714	0.426

TABLE 2. $\frac{2t}{x^3}\exp\left(-\frac{t}{x^2}\right)$.

t \ x	0.2	0.3	0.4	0.5	0.7	0.85	1.0	1.5	2	3	4	6	8	10	12	15	20	30
0.2	0.337	1.605	1.791	1.438	0.775	0.494	0.327	0.108	0.048	0.014	0.006	0.002	0.001	0.000				
0.5		0.143	0.687	1.083	1.051	0.815	0.607	0.237	0.110	0.035	0.015	0.005	0.002	0.001				
1.0			0.060	0.293	0.758	0.816	0.736	0.380	0.195	0.066	0.029	0.009	0.004	0.002				
1.5			0.004	0.060	0.410	0.613	0.669	0.456	0.258	0.094	0.043	0.013	0.006	0.003				
2				0.011	0.197	0.409	0.541	0.487	0.303	0.119	0.055	0.018	0.008	0.004				
4					0.007	0.051	0.147	0.401	0.368	0.190	0.097	0.033	0.015	0.008				
6						0.005	0.030	0.247	0.335	0.228	0.129	0.047	0.021	0.011	0.007	0.003	0.001	0.000
10								0.070	0.205	0.244	0.167	0.070	0.033	0.018	0.011	0.006	0.002	0.001
20								0.002	0.034	0.161	0.179	0.106	0.057	0.033	0.020	0.011	0.005	0.001
50										0.014	0.069	0.115	0.089	0.061	0.041	0.024	0.011	0.004
100											0.006	0.058	0.082	0.074	0.058	0.038	0.019	0.007
200												0.007	0.034	0.054	0.058	0.049	0.030	0.012
500													0.001	0.007	0.018	0.032	0.036	0.021

(the value of t; the value of x for the maximum, the maximum value of the function)$\equiv$(0.2; 0.365, 1.833), (0.5; 0.577, 1.159), (1.0; 0.817, 0.820), (1.5; 1.000, 0.669), (2; 1.155, 0.580), (4; 1.633, 0.410), (6; 2.000, 0.335), (10; 2.582, 0.259), (20; 3.651, 0.183), (50; 5.774, 0.116), (100; 8.165, 0.082), (200; 11.547, 0.058), (500; 18.257, 0.037).

TABLE 3.

n	$c_n(\tau=0;\mu=2)$	$c_n(\tau=0;\mu=10)$	$c_n(\tau=0;\mu=30)$	$c_n(\tau=0;\mu=50)$
1	4.0304 37968	0.1804 67424	0.0540 9209	0.0321 8824
2	−0.0121 56497	−0.2202 61731	−0.0380 2451	−0.0218 2477
3	0.0013 55470	1.3500 07793	0.0337 6526	0.0180 9586
4	−0.0003 20092	−0.4787 24697	−0.0340 2140	−0.0163 6231
5	0.0001 08242	0.0429 13482	0.0381 9666	0.0156 3450
6	−0.0000 45329	−0.0114 85133	−0.0484 0701	−0.0156 0024
7	0. 21892	0.0044 43975	0.0725 3593	0.0161 8515
8	−0. 11710	−0.0020 89313	−0.1455 3956	−0.0174 5826
9	0. 06764	0.0011 11157	0.6214 7429	0.0196 4275
10	−0. 04149	−0.0006 44125	−8.2578 7900	−0.0232 1503
11	0.0000 02670	0.0003 98129	0.2008 1320	0.0291 8168
12	−0. 1787	−0. 2 58665	−0.0519 8738	−0.0398 7931
13	0. 1237	0. 1 74920	0.0215 3322	0.0617 5571
14	−0. 0880	−0. 1 22249	−0.0110 2921	−0.1186 7802
15	0. 0641	0. 0 87830	0.0063 8811	0.3704 2444
16	−0.0000 00477	−0.0000 64604	−0.0040 1216	−16.8124 8866
17	0. 361	0. 48495	0. 26 6992	0.5956 0028
18	−0. 278	−0. 37053	−0. 18 5592	−0.1128 3522
19	0. 217	0. 28756	0. 13 3501	0.0433 8463
20	−0. 172	−0. 22628	−0. 09 8726	−0.0218 2488
21	0.0000 00138	0.0000 18028	0.0007 4703	0.0126 7536
22	−0. 111	−0. 14524	−0. 5 7629	−0. 80 5543
23	0. 091	0. 11820	0. 4 5201	0. 54 4728
24	−0. 075	−0. 09708	−0. 3 5967	−0. 38 5535
25	0. 062	0. 08040	0. 2 8983	0. 28 2586
26	−0.0000 00052	−0.0000 06710	−0.0002 3618	−0.0021 2969
27	0. 44	0. 05640	0. 1 9440	0. 16 4186
28	−0. 37	−0. 04772	−0. 1 6145	−0. 12 8993
29	0. 32	0. 04062	0. 1 3518	0. 10 2981
30	−0. 27	−0. 03477	−0. 1 1402	−0. 08 3356
31	0.0000 00023	0.0000 02992	0.0000 9683	0.0006 8285
32	−0. 20	−0. 02587	−0. 8275	−0. 5 6533
33	0. 18	0. 02247	0. 7112	0. 4 7244
34	−0. 15	−0. 01961	−0. 6145	−0. 3 9813
35	0. 14	0. 01718	0. 5335	0. 3 3806
36	−0.0000 00012	−0.0000 01510	−0.0000 4654	−0.0002 8901
37	0. 11	0. 01333	0. 4077	0. 2 4863
38	−0. 09	−0. 01180	−0. 3586	−0. 2 1511
39	0. 08	0. 01048	0. 3166	0. 1 8709
40	−0. 07	−0. 00934	−0. 2805	−0. 1 6351

TABLE 4.

n	$c_n(\tau=0.1;\ \mu=2)$	$c_n(\tau=0.1;\ \mu=10)$	$c_n(\tau=0.1;\ \mu=30)$	$c_n(\tau=0.1;\ \mu=50)$
1	3.4878 8447	0.1794 2676	0.0540 5734	0.0321 8079
2	—0.0056 7507	—0.2136 5130	—0.0378 9599	—0.0217 9819
3	0.0002 0846	1.2526 0245	0.0334 8547	0.0180 4174
4	—0.0000 0990	—0.4165 8287	—0.0334 9985	—0.0162 7156
5	0.0000 0041	0.0343 3804	0.0372 6214	0.0154 9570
6	—0.0000 0001	—0.0082 8536	—0.0466 8205	—0.0153 9779
7	0. 0000	0. 28 3379	0.0689 9882	0.0158 9647
8		—0. 11 5464	—0.1362 5856	—0.0170 4899
9		0. 05 2179	0.5714 0978	0.0190 5773
10		—0. 02 5200	—7.4401 5971	—0.0223 5971
11		0.0001 2723	0.1769 0607	0.0278 7998
12		—0. 0 6620	—0.0446 8200	—0.0377 6349
13		0. 0 3515	0.0180 1670	0.0579 1633
14		—0. 0 1891	—0.0089 6375	—0.1101 4143
15		0. 0 1026	0.0050 3205	0.3399 3324
16		—0.0000 0558	—0.0030 5650	—15.2439 2146
17		0. 0304	0. 19 6276	0.5331 4750
18		—0. 0165	—0. 13 1370	—0.0996 3732
19		0. 0089	0. 09 0790	0.0377 6202
20		—0. 0048	—0. 06 4365	—0.0187 0985
21		0.0000 0026	0.0004 6587	0.0106 9387
22		—0. 14	—0. 3 4303	—0.0066 8308
23		0. 07	0. 2 5624	0.0044 4056
24		—0. 04	—0. 1 9376	—0.0030 8567
25		0. 02	0. 1 4805	0.0022 1881
26		—0.0000 0001	—0.0001 1414	—0.0016 3919
27		0. 0000	0. 0 8869	0. 12 3779
28			—0. 0 6939	—0. 09 5177
29			0. 0 5461	0. 07 4308
30			—0. 0 4320	—0. 05 8774
31			0.0000 3433	0.0004 7011
32			—0. 2739	—0. 3 7972
33			0. 2194	0. 3 0935
34			—0. 1762	—0. 2 5394
35			0. 1419	0. 2 0987
36			—0. 1146	—0.0001 7449
37			0. 0927	0. 1 4588
38			—0. 0751	—0. 1 2255
39			0. 0610	0. 1 0342
40			—0. 0496	—0. 0 8763

TABLE 5.

n	$c_n(\tau=1;\ \mu=2)$	$c_n(\tau=1;\ \mu=10)$	$c_n(\tau=1;\ \mu=30)$	$c_n(\tau=1;\ \mu=50)$
1	0.9494 0336	0.1703 2671	0.0537 4562	0.0321 1387
2	—0.0000 0598	—0.1624 0673	—0.0367 5867	—0.0215 6037
3	0.0000 0000	0.6384 1948	0.0310 6944	0.0175 6184
4		—0.1191 9047	—0.0291 5133	—0.0154 7714
5		0.0046 1754	0.0298 1601	0.0143 0068
6		—0.0004 3842	—0 0336 7637	—0.0136 8994
7		0. 0 4940	0.0439 9852	0.0135 1933
8		—0. 0 0555	—0.0753 0235	—0.0137 7142
9		0. 0 0058	0.2683 2937	0.0145 1746
10		—0. 0 0005	—2.9107 5701	—0.0159 4919
11		0.0000 0000	0.0565 3234	0.0184 8979
12			—0.0114 3523	—0.0231 2025
13			0.0036 2054	0.0325 0245
14			—0.0013 8676	—0.0562 5671
15			0.0005 8762	0.1569 0605
16			—0.0002 6414	—6.3136 2816
17			0. 1 2308	0.1967 3418
18			—0. 0 5860	—0.0325 2509
19			0. 0 2825	0.0108 2753
20			—0. 0 1370	—0.0046 7882
21			0.0000 0665	0.0023 1583
22			—0. 322	—0. 12 4442
23			0. 155	0. 07 0593
24			—0. 074	—0. 04 1583
25			0. 035	0. 02 5168
26			—0.0000 0016	—0.0001 5539
27			0. 08	0. 0 9737
28			—0. 03	—0. 0 6169
29			0. 02	0. 0 3940
30			—0. 01	—0. 0 2532
31			0.0000 0000	0.0000 1633
32				—0. 1057
33				0. 0684
34				—0. 0444
35				0. 0287
36				—0.0000 0186
37				0. 120
38				—0. 078
39				0. 050
40				—0. 032

TABLE 6.

n	$c_n(\tau=10;\ \mu=2)$	$c_n(\tau=10;\ \mu=10)$	$c_n(\tau=10;\ \mu=30)$	$c_n(\tau=10;\ \mu=50)$
1	0.0000 0214	0.1012 1345	0.0507 2558	0.0314 5218
2	0. 0000	—0.0104 6138	—0.0271 0346	—0.0193 2038
3		0.0007 5515	0.0146 9276	0.0134 1180
4		—0.0000 0044	—0.0072 5795	—0.0093 8225
5		0.0000 0000	0.0032 0824	0.0064 0930
6			—0.0012 8552	—0.0042 2506
7			0. 04 8910	0. 26 7610
8			—0. 02 0010	—0. 16 2847
9			0. 01 3992	0. 09 5518
10			—0. 02 4435	—0. 05 4383
11			0.0000 0063	0.0003 0432
12			—0. 0001	—0. 1 7108
13			0. 0000	0. 1 0071
14				—0. 0 6798
15				0. 0 6888
16				—0.0009 3780
17				0. 0 0921
18				—0. 0 0045
19				0. 0 0004
20				—0. 0 0000

TABLE 7.

σ	$u_1^*(\tau=0.1;\ \mu=2)$	$u_1^*(\tau=0.1;\ \mu=10)$	$u_1^*(\tau=0.1;\ \mu=30)$	$u_1^*(\tau=0.1;\ \mu=50)$
0.000	3.482 408	0.829 881	—6.784 850	—14.499 131
0.025	3.479 282	0.824 225	—5.813 491	—9.461 437
0.050	3.469 911	0.807 255	—3.310 271	0.521 001
0.075	3.454 321	0.778 978	—0.315 430	5.827 857
0.100	3.432 552	0.739 454	1.984 022	2.864 074
0.150	3.370 724	0.627 529	2.037 296	—4.011 466
0.200	3.285 077	0.475 600	—1.283 662	3.520 278
0.250	3.176 512	0.293 297	—1.605 501	—1.777 271
0.300	3.046 168	0.097 311	1.034 747	—0.272 254
0.350	2.895 418	—0.088 997	1.502 280	1.980 708
0.400	2.725 844	—0.238 469	—0.760 012	—2.621 754
0.450	2.539 224	—0.325 232	—1.342 812	2.207 678
0.500	2.337 518	—0.330 929	0.667 512	—0.869 664
0.550	2.122 837	—0.250 452	1.355 685	—0.643 893
0.600	1.897 427	—0.095 436	—0.466 466	1.851 537
0.650	1.663 639	0.105 886	—1.257 208	—2.149 398
0.700	1.423 910	0.312 992	0.411 393	1.612 048
0.750	1.180 728	0.480 996	1.304 629	—0.365 835
0.800	0.936 601	0.570 591	—0.228 793	—0.904 300
0.850	0.694 033	0.557 676	—1.225 178	1.815 976
0.900	0.455 495	0.440 210	0.181 285	—1.868 394
0.950	0.223 384	0.240 249	1.281 066	1.211 853
0.975	0.110 461	0.121 977	0.869 795	1.807 334
1.000	0	0	0	0

TABLE 8.

σ	$u_1^*(\tau=1\,;\ \mu=2)$	$u_1^*(\tau=1\,;\ \mu=10)$	$u_1^*(\tau=1\,;\ \mu=30)$	$u_1^*(\tau=1\,;\ \mu=50)$
0.000	0.949 397	0.531 372	—2.610 949	—6.001 667
0.025	0.948 540	0.526 978	—2.233 372	—3.911 503
0.050	0.945 969	0.513 890	—1.260 671	0.229 436
0.075	0.941 692	0.492 401	—0.097 889	2.428 920
0.100	0.935 721	0.462 997	0.793 194	1.197 388
0.150	0.918 764	0.383 382	0.807 124	—1.650 669
0.200	0.895 282	0.282 849	—0.482 867	1.472 288
0.250	0.865 528	0.172 075	—0.598 933	—0.728 141
0.300	0.829 824	0.063 897	0.428 935	—0.097 132
0.350	0.788 553	—0.028 015	0.600 323	0.829 397
0.400	0.742 159	—0.091 078	—0.283 123	—1.073 860
0.450	0.691 138	—0.116 040	—0.498 693	0.928 601
0.500	0.636 035	—0.099 112	0.289 354	—0.353 127
0.550	0.577 433	—0.043 407	0.545 865	—0.249 799
0.600	0.515 954	0.040 812	—0.171 575	0.774 897
0.650	0.452 241	0.137 370	—0.468 503	—0.877 896
0.700	0.386 960	0 226 866	0.191 713	0.682 331
0.750	0.320 789	0.290 366	0.529 556	—0.145 375
0.800	0.254 404	0.313 352	—0.080 423	—0.356 663
0.850	0.188 481	0.288 981	—0.459 931	0.759 551
0.900	0.123 684	0.219 805	0.102 743	—0.761 510
0.950	0.060 652	0.117 419	0.523 622	0.517 108
0.975	0.029 991	0.059 298	0.351 849	0.754 925
1.000	0	0	0	0

TABLE 9.

σ	$u_1^*(\tau=10\,;\ \mu=2)$	$u_1^*(\tau=10\,;\ \mu=10)$	$u_1^*(\tau=10\,;\ \mu=30)$	$u_1^*(\tau=10\,;\ \mu=50)$
0.000	0.0000 0214	0.0915 068	0.0331 648	0.0191 104
0.025	0. 0214	0.0914 563	0.0331 900	0.0194 202
0.050	0. 0213	0.0913 047	0.0332 544	0.0200 329
0.075	0. 0212	0.0910 514	0.0333 302	0.0203 560
0.100	0.0000 0211	0.0906 958	0.0333 860	0.0201 713
0.150	0. 0207	0.0896 728	0.0333 782	0.0197 548
0.200	0. 0202	0.0882 228	0.0332 934	0.0202 152
0.250	0. 0195	0.0863 280	0.0332 974	0.0198 861
0.300	0.0000 0187	0.0839 655	0.0333 678	0.0199 876
0.350	0. 0178	0.0811 087	0.0333 658	0.0201 150
0.400	0. 0167	0.0777 292	0.0333 011	0.0198 406
0.450	0. 0156	0.0737 997	0.0332 983	0.0201 358
0.500	0.0000 0143	0.0692 985	0.0333 501	0.0199 399
0.550	0. 0130	0.0642 136	0.0333 213	0.0199 669
0.600	0. 0116	0.0585 482	0.0331 656	0.0201 058
0.650	0. 0102	0.0523 247	0.0328 990	0.0198 690
0.700	0.0000 0087	0.0455 893	0.0323 287	0.0200 939
0.750	0. 0072	0.0384 132	0.0309 850	0.0199 181
0.800	0. 0057	0.0308 929	0.0283 429	0.0196 490
0.850	0. 0042	0.0231 480	0.0239 762	0.0187 614
0.900	0.0000 0028	0.0153 167	0.0175 985	0.0154 783
0.950	0. 0014	0.0075 493	0.0093 175	0.0092 716
0.975	0. 0007	0.0037 377	0.0047 027	0.0048 989
1.000	0	0	0	0

TABLE 10.

σ	$u_2^*(\tau=0.1;\ \mu=2)$	$u_2^*(\tau=0.1;\ \mu=10)$	$u_2^*(\tau=0.1;\ \mu=30)$	$u_2^*(\tau=0.1;\ \mu=50)$
0.000	−3.480 070	−0.829 413	6.785 004	14.499 184
025	−3.476 944	−0.823 757	5.813 648	9.461 541
050	−3.467 573	−0.806 787	3.310 426	−0.520 921
075	−3.451 983	−0.778 510	0.315 585	−5.827 755
0.100	−3.430 214	−0.738 986	−1.983 867	−2.863 988
125	−3.402 324	−0.688 383	−2.791 795	2.850 320
150	−3.368 386	−0.627 061	−2.037 141	4.011 553
175	−3.328 489	−0.555 650	−0.351 547	−0.017 595
0.200	−3.282 739	−0.475 132	1.283 818	−3.520 191
225	−3.231 255	−0.386 907	2.035 061	−2.182 603
250	−3.174 173	−0.292 829	1.605 658	1.777 361
275	−3.111 644	−0.195 221	0.333 709	3.067 828
0.300	−3.043 830	−0.096 843	−1.034 591	0.272 344
325	−2.970 912	−0.000 820	−1.761 658	−2.688 909
350	−2.893 080	0.089 465	−1.502 125	−1.980 619
375	−2.810 540	0.170 523	−0.458 309	1.233 812
0.400	−2.723 507	0.238 937	0.760 167	2.621 841
425	−2.632 210	0.291 555	1.483 542	0.479 094
450	−2.536 889	0.325 700	1.342 967	−2.207 590
475	−2.437 794	0.339 367	0.453 957	−1.894 004
0.500	−2.335 186	0.331 396	−0.667 357	0.869 751
525	−2.229 332	0.301 613	−1.399 394	2.345 994
550	−2.120 511	0.250 920	−1.355 530	0.643 987
575	−2.009 008	0.181 324	−0.585 382	−1.871 770
0.600	−1.895 114	0.095 904	0.466 621	−1.851 454
625	−1.779 128	−0.001 291	1.215 243	0.592 674
650	−1.661 353	−0.105 418	1.257 364	2.149 510
675	−1.542 097	−0.211 063	0.587 017	0.783 136
0.700	−1.421 671	−0.312 524	−0.411 236	−1.611 975
725	−1.300 389	−0.404 122	−1.181 537	−1.828 894
750	−1.178 569	−0.480 529	−1.304 474	0.365 924
775	−1.056 526	−0.537 089	−0.723 928	1.996 182
0.800	−0.934 580	−0.570 123	0.228 949	0.904 395
825	−0.813 047	−0.577 175	1.023 877	−1.397 353
850	−0.692 242	−0.557 208	1.225 338	−1.815 896
875	−0.572 479	−0.510 715	0.730 004	0.171 654
0.900	−0.454 069	−0.439 743	−0.181 130	1.868 493
925	−0.337 317	−0.347 824	−1.001 796	1.012 155
950	−0.222 526	−0.239 813	−1.280 923	−1.211 786
975	−0.109 990	−0.121 643	−0.869 634	−1.807 232
1.000	0	0	0	0

TABLE 11.

σ	$u_2^*(\tau=1;\ \mu=2)$	$u_2^*(\tau=1;\ \mu=10)$	$u_2^*(\tau=1;\ \mu=30)$	$u_2^*(\tau=1;\ \mu=50)$
0.000	−0.841 033	−0.504 948	2.619 757	6.006 951
025	−0.840 224	−0.500 554	2.242 181	3.916 787
050	−0.837 799	−0.487 466	1.269 479	−0.224 153
075	−0.833 767	−0.465 977	0.106 697	−2.423 635
0.100	−0.828 138	−0.436 573	−0.784 386	−1.192 103
125	−0.820 932	−0.399 940	−1.094 802	1.177 514
150	−0.812 168	−0.356 958	−0.798 316	1.655 954
175	−0.801 876	−0.308 700	−0.142 071	−0.016 854
0.200	−0.790 085	−0.256 425	0.491 675	−1.467 004
225	−0.776 833	−0.201 554	0.779 370	−0.908 555
250	−0.762 160	−0.145 651	0.607 742	0.733 426
275	−0.746 111	−0.090 383	0.110 684	1.264 276
0.300	−0.728 735	−0.037 473	−0.420 127	0.102 417
325	−0.710 086	0.011 354	−0.698 224	−1.122 593
350	−0.690 221	0.054 439	−0.591 515	−0.824 115
375	−0.669 202	0.090 260	−0.181 697	0.508 739
0.400	−0.647 093	0.117 502	0.291 931	1.079 156
425	−0.623 962	0.135 132	0.568 757	0.187 375
450	−0.599 881	0.142 464	0.507 501	−0.923 327
475	−0.574 925	0.139 213	0.156 523	−0.787 675
0.500	−0.549 171	0.125 536	−0.280 546	0.358 414
525	−0.522 699	0.102 049	−0.561 217	0.964 683
550	−0.495 590	0.069 829	−0.537 059	0.255 086
575	−0.467 929	0.030 383	−0.231 067	−0.784 417
0.600	−0.439 803	−0.014 398	0.180 384	−0.769 614
625	−0.411 300	−0.062 321	0.468 257	0.244 118
650	−0.382 507	−0.110 984	0.477 315	0.883 192
675	−0.353 517	−0.157 889	0.209 044	0.312 198
0.700	−0.324 419	−0.200 421	−0.182 907	−0.677 051
725	−0.295 306	−0.236 723	−0.480 203	−0.759 923
750	−0.266 269	−0.264 334	−0.520 754	0.150 661
775	−0.237 400	−0.281 795	−0.286 495	0.819 652
0.800	−0.208 791	−0.288 012	0.089 232	0.361 952
825	−0.180 534	−0.282 473	0.397 306	−0.588 423
850	−0.152 719	−0.265 299	0.468 740	−0.754 263
875	−0.125 434	−0.237 257	0.266 918	0.070 651
0.900	−0.098 769	−0.199 734	−0.093 974	0.766 792
925	−0.072 809	−0.154 678	−0.413 425	0.406 157
950	−0.047 640	−0.104 502	−0.515 683	−0.511 883
975	−0.023 344	−0.051 961	−0.346 078	−0.750 456
1.000	0	0	0	0

TABLE 12.

σ	$u_2^*(\tau=10;\ \mu=2)$	$u_2^*(\tau=10;\ \mu=10)$	$u_2^*(\tau=10;\ \mu=30)$	$u_2^*(\tau=10;\ \mu=50)$
0.000	0.0006 0441	—0.0002 6932	0.0001 5199	0.0008 7956
025	0. 6 0394	—0. 2 6619	0. 1 2680	0. 5 6977
050	0. 6 0252	—0. 2 5694	0. 0 6233	—0. 0 4292
075	0. 6 0018	—0. 2 4195	—0. 0 1350	—0. 3 6605
0.100	0.0005 9689	—0.0002 2187	—0.0000 6941	—0.0001 8129
125	0. 5 9268	—0. 1 9753	—0. 8543	0. 1 6835
150	0. 5 8755	—0. 1 6995	—0. 6159	0. 2 3519
175	0. 5 8151	—0. 1 4027	—0. 1647	—0. 0 1405
0.200	0.0005 7456	—0.0001 0970	0.0000 2330	—0.0002 2522
225	0. 5 6671	—0. 0 7948	0. 3692	—0. 1 3826
250	0. 5 5799	—0. 0 5080	0. 1938	0. 1 0385
275	0. 5 4841	—0. 0 2474	—0. 1718	0. 1 7699
0.300	0.0005 3797	—0.0000 0228	—0.0000 5122	0.0000 0238
325	0. 5 2670	0. 1582	—0. 6394	—0. 1 7478
350	0. 5 1461	0. 2901	—0. 4930	—0. 1 2497
375	0. 5 0173	0. 3697	—0. 1670	0. 0 7156
0.400	0.0004 8808	0.0000 3962	0.0000 1489	0.0001 4946
425	0. 4 7367	0. 3716	0. 2773	0. 0 1398
450	0. 4 5853	0. 2998	0. 1511	—0. 1 4577
475	0. 4 4269	0. 1871	—0. 1529	—0. 1 1896
0.500	0.0004 2617	0.0000 0414	—0.0000 4600	0.0000 5016
525	0. 4 0899	—0. 1280	—0. 5972	0. 1 3247
550	0. 3 9119	—0. 3111	—0. 4893	0. 0 2316
575	0. 3 7280	—0. 4973	—0. 1996	—0. 1 2570
0.600	0.0003 5383	—0.0000 6767	0.0000 1075	—0.0001 1580
625	0. 3 3433	—0. 0 8395	0. 2592	0. 0 3402
650	0. 3 1432	—0. 0 9777	0. 1708	0. 1 2043
675	0. 2 9384	—0. 1 0844	—0. 1075	0. 0 3088
0.700	0.0002 7292	—0.0001 1547	—0.0000 4196	—0.0001 1030
725	0. 2 5159	—0. 1 1859	—0. 5905	—0. 1 1396
750	0. 2 2988	—0. 1 1773	—0. 5241	0. 0 2092
775	0. 2 0784	—0. 1 1306	—0. 2569	0. 1 1111
0.800	0.0001 8549	—0.0001 0492	0.0000 0626	0.0000 3758
825	0. 1 6287	—0. 0 9386	0. 2562	—0. 9769
850	0. 1 4001	—0. 0 8057	0. 2151	—0. 1 1280
875	0. 1 1696	—0. 0 6583	—0. 0393	0. 0 0978
0.900	0.0000 9375	—0.0000 5050	—0.0000 3669	0.0001 0342
925	0. 7041	—0. 3546	—0. 5858	0. 0 4352
950	0. 4698	—0. 2154	—0. 5734	—0. 0 8688
975	0. 2350	—0. 0950	—0. 3347	—0. 1 1198
1.000	0	0	0	0

TABLE 13.

σ	$u^*(\tau=0.1;\ \mu=2)$	$u^*(\tau=0.1;\ \mu=10)$	$u^*(\tau=0.1;\ \mu=30)$	$u^*(\tau=0.1;\ \mu=50)$
0.000	0.002 338	0.000 468	0.000 154	0.000 053*
0.025	0. 2 338	0. 468	0. 157	0. 104
0.050	0. 2 338	0. 468	0. 155	0. 080
0.075	0. 2 338	0. 468	0. 154	0. 102
0.100	0.002 338	0.000 468	0.000 155	0.000 086
0.150	0. 2 338	0. 468	0. 155	0. 088
0.200	0. 2 339	0. 468	0. 156	0. 086
0.250	0. 2 338	0. 468	0. 157	0. 089
0.300	0.002 338	0.000 468	0.000 156	0.000 090
0.350	0. 2 338	0. 468	0. 156	0. 089
0.400	0. 2 337	0. 467	0. 155	0. 088
0.450	0. 2 335	0. 468	0. 156	0. 088
0.500	0.002 332	0.000 468	0.000 155	0.000 087
0.550	0. 2 326	0. 468	0. 155	0. 094
0.600	0. 2 313	0. 468	0. 155	0. 083
0.650	0. 2 286	0. 468	0. 156	0. 112
0.700	0.002 240	0.000 468	0.000 156	0.000 073
0.750	0. 2 159	0. 468	0. 156	0. 090
0.800	0. 2 021	0. 468	0. 156	0. 095
0.850	0. 1 791	0. 468	0. 159	0. 080
0.900	0.001 426	0.000 467	0.000 154	0.000 099
0.950	0. 0 858	0. 436	0. 144	0. 067
0.975	0. 0 471	0. 334	0. 161	0. 101
1.000	0	0	0	0

* 0.000 074 (corrected value)

TABLE 14.

σ	$u^*(\tau=1;\ \mu=2)$	$u^*(\tau=1;\ \mu=10)$	$u^*(\tau=1;\ \mu=30)$	$u^*(\tau=1;\ \mu=50)$
0.000	0.108 365	0.026 424	0.008 808	0.00 528
0.025	0.108 316	0.026 424	0. 8 808	0. 528
0.050	0.108 170	0.026 424	0. 8 808	0. 528
0.075	0.107 926	0.026 424	0. 8 807	0. 528
0.100	0.107 583	0.026 424	0.008 808	0.00 528
0.150	0.106 596	0.026 424	0. 8 808	0. 528
0.200	0.105 197	0.026 424	0. 8 808	0. 528
0.250	0.103 369	0.026 424	0. 8 809	0. 529
0.300	0.101 089	0.026 424	0.008 808	0.00 529
0.350	0.098 332	0.026 424	0. 8 808	0. 528
0.400	0.095 067	0.026 424	0. 8 808	0. 530
0.450	0.091 257	0.026 424	0. 8 808	0. 527
0.500	0.086 863	0.026 424	0.008 808	0.00 529
0.550	0.081 843	0.026 422	0. 8 805	0. 529
0.600	0.076 150	0.026 414	0. 8 810	0. 528
0.650	0.069 733	0.026 386	0. 8 813	0. 530
0.700	0.062 541	0.026 445	0.008 806	0.00 528
0.750	0.054 520	0.026 033	0. 8 802	0. 529
0.800	0.045 613	0.025 340	0. 8 810	0. 529
0.850	0.035 763	0.023 681	0. 8 809	0. 529
0.900	0.024 914	0.020 071	0.008 770	0.00 528
0.950	0.013 012	0.012 917	0. 7 939	0. 522
0.975	0.006 648	0.007 338	0. 5 771	0. 447
1.000	0	0	0	0

TABLE 15.

σ	$u^*(\tau=10;\ \mu=2)$	$u^*(\tau=10;\mu=10)$	$u^*(\tau=10;\mu=30)$	$u^*(\tau=10;\mu=50)$
0.000	0.0006 066	0.091 237	0.033 317	0.019 990
0.025	0. 6 061	0.091 190	0.033 317	0.019 990
0.050	0. 6 047	0.091 048	0.033 317	0.019 990
0.075	0. 6 023	0.090 809	0.033 317	0.019 990
0.100	0 0005 990	0.090 474	0.033 317	0.019 990
0.150	0. 5 896	0.089 503	0.033 317	0.019 990
0.200	0. 5 766	0.088 113	0.033 317	0.019 990
0.250	0. 5 599	0.086 277	0.033 317	0.019 990
0.300	0.0005 398	0.083 963	0.033 317	0.019 990
0.350	0. 5 164	0.081 138	0.033 316	0.019 990
0.400	0. 4 898	0.077 769	0.033 316	0.019 990
0.450	0. 4 601	0.073 830	0.033 313	0.019 990
0.500	0.0004 276	0.069 303	0.033 304	0.019 990
0.550	0. 3 925	0.064 183	0.033 272	0.019 990
0.600	0. 3 550	0.058 480	0.033 176	0.019 990
0.650	0. 3 153	0.052 227	0.032 916	0.019 989
0.700	0.0002 738	0.045 474	0.032 287	0.019 984
0.750	0. 2 306	0.038 295	0.030 933	0.019 939
0.800	0. 1 861	0.030 788	0.028 349	0.019 687
0.850	0. 1 404	0.023 067	0.023 998	0.018 649
0.900	0.0000 940	0.015 266	0.017 562	0.015 582
0.950	0. 471	0.007 528	0.009 260	0.009 185
0.975	0. 236	0.003 728	0.004 669	0.004 787
1.000	0	0	0	0

APPENDIX I TABLE 16.

(1)

σ	$J_0(j_{0,1}\cdot\sigma)$	$J_0(j_{0,2}\cdot\sigma)$	$J_0(j_{0,3}\cdot\sigma)$	$J_0(j_{0,4}\cdot\sigma)$	$J_0(j_{0,5}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	0.999 097	0.995 245	0.988 333	0.978 393	0.965 469
050	0.996 389	0.981 046	0.953 741	0.914 970	0.865 446
075	0.991 884	0.957 607	0.897 430	0.813 827	0.710 232
0.100	0.985 594	0.925 260	0.821 365	0.681 464	0.515 679
125	0.977 537	0.884 467	0.728 183	0.526 318	0.301 377
150	0.967 733	0.835 805	0.621 099	0.358 160	0.088 441
175	0.956 210	0.779 963	0.503 778	0.187 405	−0.102 861
0.200	0.942 999	0.717 729	0.380 195	0.024 378	−0.255 332
225	0.928 135	0.649 974	0.254 486	−0.121 404	−0.356 687
250	0.911 659	0.577 647	0.130 785	−0.241 879	−0.400 771
275	0.893 614	0.501 751	0.013 072	−0.330 973	−0.388 043
0.300	0.874 050	0.423 331	−0.094 975	−0.384 980	−0.325 273
325	0.853 020	0.343 461	−0.190 116	−0.402 759	−0.224 502
350	0.830 579	0.263 220	−0.269 663	−0.385 745	−0.101 414
375	0.806 789	0.183 684	−0.331 576	−0.337 761	0.026 668
0.400	0.781 712	0.105 902	−0.374 527	−0.264 665	0.142 925
425	0.755 416	0.030 884	−0.397 935	−0.173 855	0.233 039
450	0.727 971	−0.040 416	−0.401 978	−0.073 662	0.286 814
475	0.699 450	−0.107 112	−0.387 563	0.027 298	0.299 239
0.500	0.669 930	−0.168 402	−0.356 278	0.120 783	0.270 862
525	0.639 488	−0.223 575	−0.310 312	0.199 537	0.207 469
550	0.608 205	−0.272 029	−0.252 350	0.257 809	0.119 112
575	0.576 163	−0.313 270	−0.185 460	0.291 740	0.018 640
0.600	0.543 448	−0.346 925	−0.112 954	0.299 589	−0.080 067
625	0.510 144	−0.372 743	−0.038 248	0.281 795	−0.163 942
650	0.476 340	−0.390 598	0.035 276	0.240 862	−0.222 387
675	0.442 122	−0.400 493	0.104 411	0.181 083	−0.248 573
0.700	0.407 580	−0.402 549	0.166 251	0.108 139	−0.240 192
725	0 372 803	−0.397 013	0.218 306	0.028 588	−0.199 601
750	0.337 882	−0.384 243	0.258 588	−0.050 691	−0.133 361
775	0.302 904	−0.364 705	0.285 691	−0.123 068	−0.051 226
0.800	0 267 960	−0.338 962	0.298 824	−0.182 698	0.035 245
825	0.233 137	−0.307 665	0.297 835	−0.224 971	0.114 275
850	0.198 525	−0.271 538	0.283 195	−0.246 862	0.175 441
875	0.164 209	−0.231 371	0.255 967	−0.247 128	0.211 021
0.900	0.130 274	−0.187 997	0.217 740	−0.226 365	0.216 943
925	0.096 805	−0.142 287	0.170 547	−0.186 908	0.193 232
950	0.063 884	−0.095 125	0.116 767	−0.132 593	0.143 895
975	0.031 589	−0.047 403	0.059 011	−0.068 397	0 076 278
1.000	0	0	0	0	0

$j_{0,1}=2.4048\ 2556$ $j_{0,2}=5.5200\ 7811$ $j_{0,3}=8.6537\ 2791$
$j_{0,4}=11.7915\ 3444$ $j_{0,5}=14.9309\ 1771$

(2)

σ	$J_0(j_{0,6}\cdot\sigma)$	$J_0(j_{0,7}\cdot\sigma)$	$J_0(j_{0,8}\cdot\sigma)$	$J_0(j_{0,9}\cdot\sigma)$	$J_0(j_{0,10}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	0.949 622	0.930 924	0.909 462	0.885 334	0.858 652
050	0.806 079	0.737 954	0.662 312	0.580 522	0.494 056
075	0.590 879	0.460 600	0.324 609	0.188 267	0.056 839
0.100	0.335 854	0.154 604	−0.015 692	−0.163 864	−0.280 810
125	0.077 885	−0.120 657	−0.274 626	−0.370 590	−0.402 758
150	−0.147 139	−0.315 350	−0.396 232	−0.386 084	−0.297 369
175	−0.310 089	−0.399 848	−0.367 360	−0.237 023	−0.054 673
0.200	−0.393 084	−0.370 383	−0.218 959	−0.007 289	0.184 100
225	−0.391 971	−0.248 531	−0.013 921	0.198 471	0.297 842
250	−0.316 395	−0.074 797	0.174 236	0.297 065	0.241 873
275	−0.187 520	0.101 947	0.284 771	0.257 678	0.062 508
0.300	−0.033 853	0.236 499	0.287 738	0.109 720	−0.134 650
325	0.114 031	0.297 879	0.191 155	−0.074 247	−0.243 795
350	0.228 908	0.275 881	0.035 556	−0.212 091	−0.213 686
375	0.291 490	0.182 207	−0.121 645	−0.247 390	−0.070 396
0.400	0.293 411	0.046 129	−0.226 379	−0.171 651	0.101 700
425	0.238 107	−0.093 987	−0.245 855	−0.024 894	0.208 271
450	0.139 541	−0.201 006	−0.178 372	0.124 458	0.195 736
475	0.019 056	−0.248 504	−0.052 159	0.210 727	0.077 502
0.500	−0.098 974	−0.227 042	0.085 845	0.199 308	−0.076 845
525	−0.191 865	−0.145 773	0.187 013	0.100 126	−0.181 786
550	−0.242 824	−0.029 168	0.217 823	−0.038 861	−0.182 671
575	−0.243 869	0.090 088	0.170 717	−0.154 411	−0.083 758
0.600	−0.196 970	0.180 045	0.065 566	−0.196 402	0.056 730
625	−0.113 264	0.217 877	−0.058 375	−0.149 296	0.160 444
650	−0.010 557	0.195 545	−0.156 886	−0.037 560	0.172 238
675	0.090 351	0.121 350	−0.196 360	0.086 695	0.089 207
0.700	0.169 788	0.017 124	−0.165 113	0.167 773	−0.039 720
725	0.212 962	−0.088 066	−0.076 665	0.171 144	−0.142 328
750	0.212 661	−0.165 907	0.035 869	0.097 991	−0.163 317
775	0.170 395	−0.196 367	0.132 033	−0.016 538	−0.093 902
0.800	0.095 819	−0.172 774	0.178 448	−0.120 038	0.024 918
825	0.004 607	−0.103 176	0.160 292	−0.166 676	0.126 38
850	−0.084 793	−0.007 736	0.085 934	−0.137 354	0.155 29
875	−0.154 833	0.087 031	−0.016 704	−0.047 406	0.097 89
0.900	−0.192 267	0.155 558	−0.110 531	0.061 234	−0.011 79
925	−0.190 622	0.180 061	−0.162 611	0.139 47	−0.111 97
950	−0.151 280	0.155 108	−0.155 644	0.153 12	−0.147 78
975	−0.082 998	0.088 749	−0.093 652	0.097 78	−0.101 20
1.000	0	0	0	0	0

$j_{0,6}=18.0710\ 6397$ $j_{0,7}=21.2116\ 3663$ $j_{0\ 8}=24,3524\ 7153$
$j_{0,9}=27.4934\ 7913$ $j_{0,10}=30.6346\ 0647$

(3)

σ	$J_0(j_{0,11}\cdot\sigma)$	$J_0(j_{0,12}\cdot\sigma)$	$J_0(j_{0,13}\cdot\sigma)$	$J_0(j_{0,14}\cdot\sigma)$	$J_0(j_{0,15}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	0.829 537	0.798 124	0.764 554	0.728 984	0.691 572
050	0.404 457	0.313 310	0.222 207	0.132 716	0.046 356
075	−0.064 743	−0.172 086	−0.261 534	−0.330 315	−0.376 649
0.100	−0.360 173	−0.398 770	−0.396 757	−0.357 513	−0.287 247
125	−0.373 427	−0.292 357	−0.175 202	−0.041 219	0.089 412
150	−0.155 624	0.005 940	0.153 063	0.256 956	0.299 492
175	0.125 202	0.253 796	0.300 112	0.257 789	0.145 108
0.200	0.289 868	0.280 377	0.168 899	0.003 741	−0.151 200
225	0.251 672	0.094 318	−0.093 548	−0.224 076	−0.241 802
250	0.057 223	−0.143 044	−0.246 544	−0.203 209	−0.048 024
275	−0.155 309	−0.249 648	−0.168 330	0.020 448	0.182 133
0.300	−0.249 498	−0.157 484	0.051 596	0.203 939	0.183 305
325	−0.175 190	0.047 255	0.208 223	0.165 620	−0.026 653
350	0.006 964	0.200 712	0.168 313	−0.038 564	−0.190 267
375	0.172 029	0.189 836	−0.018 711	−0.189 271	−0.122 308
0.400	0.214 585	0.033 925	−0.176 962	−0.136 515	0.083 184
425	0.114 763	−0.138 628	−0.167 379	0.053 051	0.179 500
450	−0.054 249	−0.194 492	−0.008 413	0.177 275	0.061 248
475	−0.178 661	−0.097 231	0.149 504	0.112 364	−0.122 056
0.500	−0.179 747	0.070 186	0.165 003	−0.065 002	−0.153 377
525	−0.062 977	0.175 120	0.031 318	−0.166 647	−0.003 777
550	0.089 680	0.139 504	−0.124 351	−0.091 429	0.143 28
575	0.176 648	−0.002 718	−0.160 995	0.074 983	0.115 86
0.600	0.143 778	−0.136 776	−0.050 801	0.156 70	−0.046 34
625	0.017 642	−0.159 093	0.100 78	0.072 76	−0.147 49
650	−0.115 165	−0.056 829	0.155 31	−0.083 32	−0.071 33
675	−0.167 060	0.085 798	0.067 31	−0.147 07	0.085 85
0.700	−0.106 893	0.156 43	−0.078 43	−0.055 81	0.136 28
725	0.021 703	0.102 69	−0.148 01	0.090 24	0.024 44
750	0.131 61	−0.029 29	−0.081 09	0.137 52	−0.112 41
775	0.150 96	−0.134 09	0.057 17	0.040 22	−0.112 23
0.800	0.069 91	−0.130 91	0.139 17	−0.095 84	0.020 34
825	−0.054 88	−0.025 60	0.092 32	−0.127 94	0.124 82
850	−0.139 63	0.096 56	−0.036 98	−0.025 81	0.078 79
875	−0.129 50	0.139 71	−0.128 93	0.100 26	−0.058 99
0.900	−0.033 89	0.072 40	−0.101 10	0.118 25	−0.123 17
925	0.081 55	−0.049 70	0.017 90	0.012 43	−0.039 98
950	0.139 86	−0.129 65	0.117 43	−0.103 53	0.088 28
975	0.103 94	−0.106 31	0.107 53	−0.108 43	0.108 75
1.000	0	0	0	0	0

$j_{0,11}=33.7758\ 2021$ $j_{0,12}=36.9170\ 9835$ $j_{0,13}=40.0584\ 2576$
$j_{0,14}=43.1997\ 9171$ $j_{0,15}=46.3411\ 8837$

(4)

σ	$J_0(j_{0,16}\cdot\sigma)$	$J_0(j_{0,17}\cdot\sigma)$	$J_0(j_{0,18}\cdot\sigma)$	$J_0(j_{0,19}\cdot\sigma)$	$J_0(j_{0,20}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	0.652 491	0.611 916	0.570 029	0.527 020	0.483 080
050	−0.035 442	−0.111 359	−0.180 210	−0.240 970	−0.292 792
075	−0.399 807	−0.400 121	−0.378 942	−0.338 554	−0.282 040
0.100	−0.194 388	−0.088 799	0.019 108	0.119 281	0.202 873
125	0.198 304	0.271 224	0.299 813	0.282 433	0.224 070
150	0.276 076	0.195 759	0.078 657	−0.048 706	−0.159 108
175	−0.001 555	−0.138 744	−0.228 264	−0.247 434	−0.194 568
0.200	−0.240 102	−0.234 622	−0.142 422	−0.002 358	0.131 306
225	−0.146 267	0.011 334	0.154 520	0.217 830	0.176 163
250	0.124 196	0.215 284	0.178 610	0.042 168	−0.111 248
275	0.209 672	0.093 005	−0.080 225	−0.189 244	−0.163 345
0.300	0.017 820	−0.151 719	−0.189 945	−0.073 744	0.095 646
325	−0.181 686	−0.158 965	0.009 879	0.160 291	0.153 760
350	−0.131 004	0.063 299	0.179 199	0.098 386	−0.082 894
375	0.086 353	0.178 810	0.051 497	−0.130 687	−0.146 219
0.400	0.177 253	0.028 056	−0.150 144	−0.116 731	0.072 099
425	0.031 415	−0.154 245	−0.099 471	0.100 65	0.140 05
450	−0.147 137	−0.102 037	0.107 64	0.129 16	−0.062 72
475	−0.123 457	0.095 818	0.130 86	−0.070 64	−0.134 84
0.500	0.060 818	0.143 90	−0.057 35	−0.135 99	0.054 41
525	0.156 29	−0.020 30	−0.144 12	0.041 25	0.130 32
550	0.041 98	−0.147 22	0.005 25	0.137 55	−0.046 94
575	−0.122 46	−0.053 24	0.139 52	−0.013 10	−0.126 31
0.600	−0.118 69	0.114 66	0.042 94	−0.134 20	0.040 14
625	0.041 31	0.107 74	−0.119 05	−0.013 16	0.122 68
650	0.140 56	−0.056 85	−0.082 25	0.126 41	−0.033 89
675	0.050 58	−0.131 83	0.086 21	0.036 94	−0.119 34
0.700	−0.102 92	−0.010 54	0.108 99	−0.114 72	0.028 09
725	−0.115 03	0.121 97	−0.045 56	−0.057 70	0.116 23
750	0.025 36	0.070 70	−0.121 03	0.099 72	−0.022 68
775	0.127 61	−0.082 87	0.002 24	0.075 01	−0.113 29
0.800	0.057 70	−0.109 62	0.117 96	−0.082 10	0.017 61
825	−0.086 46	0.025 84	0.038 64	−0.088 53	0.110 48
850	−0.111 77	0.119 20	−0.101 07	0.062 58	−0.012 84
875	0.011 80	0.034 20	−0.072 51	0.098 04	−0.107 77
0.900	0.116 24	−0.098 82	0.073 11	−0.041 89	0.008 33
925	0.063 62	−0.082 47	0.095 86	−0.103 45	0.105 14
950	−0.072 02	0.055 11	−0.037 92	0.020 78	−0.004 06
975	−0.108 53	0.107 79	−0.106 52	0.104 78	−0.102 56
1.000	0	0	0	0	0

$j_{0,16}=49.4826\ 0990$ $j_{0,17}=52.6240\ 5184$ $j_{0,18}=55.7655\ 1076$
$j_{0,19}=58.9069\ 8393$ $j_{0,20}=62.0484\ 6919$

(5)

σ	$J_0(j_{0,21}\cdot\sigma)$	$J_0(j_{0,22}\cdot\sigma)$	$J_0(j_{0,23}\cdot\sigma)$	$J_0(j_{0,24}\cdot\sigma)$	$J_0(j_{0,25}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	0.438 406	0.393 195	0.347 646	0.301 960	0.256 334
050	−0.335 023	−0.367 209	−0.389 108	−0.400 689	−0.402 129
075	−0.213 114	−0.135 931	−0.054 875	0.025 662	0.101 488
0.100	0.262 966	0.295 091	0.297 532	0.271 367	0.220 263
125	0.135 314	0.030 618	−0.073 929	−0.162 964	−0.224 026
150	−0.230 224	−0.248 981	−0.213 834	−0.134 584	−0.029 866
175	−0.088 378	0.038 250	0.148 165	0.210 619	0.209 697
0.200	0.209 444	0.205 776	0.125 430	0.001 656	−0.117 632
225	0.054 212	−0.087 369	−0.181 654	−0.186 700	−0.103 909
250	−0.193 518	−0.161 208	−0.038 028	0.101 652	0.177 246
275	−0.027 015	0.121 395	0.179 925	0.112 490	−0.029 412
0.300	0.179 751	0.115 173	−0.039 331	−0.155 457	−0.141 260
325	0.004 316	−0.142 119	−0.149 281	−0.016 622	0.126 11
350	−0.166 919	−0.068 976	0.098 20	0.153 24	0.042 10
375	0.015 111	0.150 63	0.098 24	−0.070 59	−0.147 10
0.400	0.154 41	0.024 45	−0.132 70	−0.103 63	0.064 52
425	−0.031 94	−0.148 01	−0.037 02	0.124 73	0.092 51
450	−0.141 91	0.016 45	0.140 55	0.027 26	−0.126 41
475	0.046 55	0.135 61	−0.023 69	−0.133 41	0.002 23
0.500	0.129 26	−0.051 89	−0.123 44	0.049 68	0.118 34
525	−0.059 19	−0.115 09	0.074 22	0.098 73	−0.086 35
550	−0.116 41	0.080 33	0.086 59	−0.103 41	−0.051 30
575	0.070 01	0.088 42	−0.107 25	−0.035 59	0.119 11
0.600	0.103 36	−0.100 66	−0.037 87	0.119 24	−0.035 91
625	−0.079 12	−0.057 75	0.118 78	−0.033 73	−0.088 59
650	−0.090 15	0.112 25	−0.013 58	−0.095 35	0.097 84
675	0.086 57	0.025 35	−0.108 58	0.086 81	0.015 23
0.700	0.076 83	−0.115 00	0.058 82	0.042 37	−0.105 19
725	−0.092 45	0.006 54	0.079 99	−0.107 95	0.060 67
750	−0.063 50	0.109 34	−0.090 55	0.020 71	0.058 13
775	0.096 79	−0.035 82	−0.039 16	0.092 58	−0.100 38
0.800	0.050 24	−0.096 19	0.104 21	−0.072 97	0.015 75
825	−0.099 65	0.060 70	−0.006 05	−0.048 00	0.085 98
850	−0.037 16	0.076 92	−0.098 54	0.098 25	−0.077 04
875	0.101 08	−0.079 73	0.047 52	−0.009 64	−0.028 12
0.900	0.024 36	−0.053 20	0.075 67	−0.089 97	0.095 09
925	−0.101 14	0.091 94	−0.078 22	0.060 91	−0.041 04
950	−0.011 94	0.026 90	−0.040 56	0.052 66	−0.063 03
975	0.099 90	−0.096 83	0.093 36	−0.089 52	0.085 33
1.000	0	0	0	0	0

$j_{0,21}=65.1899\ 6480$ $j_{0,22}=68.3314\ 6933$ $j_{0,23}=71.4729\ 8160$
$j_{0,24}=74.6145\ 0064$ $j_{0,25}=77.7560\ 2563$

(6)

σ	$J_0(j_{0,26}\cdot\sigma)$	$J_0(j_{0,27}\cdot\sigma)$	$J_0(j_{0,28}\cdot\sigma)$	$J_0(j_{0,29}\cdot\sigma)$	$J_0(j_{0,30}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	0.210 967	0.166 052	0.121 780	0.078 339	0.035 907
050	−0.393 805	−0.376 285	−0.350 313	−0.316 786	−0.276 738
075	0.168 813	0.224 421	0.265 813	0.291 310	0.300 111
0.100	0.150 048	0.068 099	−0.017 387	−0.098 175	−0.166 753
125	−0.249 235	−0.236 260	−0.188 450	−0.114 148	−0.025 305
150	0.077 085	0.163 476	0.211 712	0.212 904	0.168 434
175	0.148 228	0.046 244	−0.065 377	−0.154 118	−0.195 176
0.200	−0.188 150	−0.185 476	−0.113 350	−0.001 245	0.107 495
225	0.024 413	0.136 369	0.180 063	0.137 448	0.031 547
250	0.148 065	0.034 905	−0.094 174	−0.164 490	−0.137 687
275	−0.145 479	−0.157 062	−0.060 115	0.074 803	0.153 56
0.300	−0.013 462	0.120 35	0.151 85	0.059 25	−0.078 20
325	0.145 13	0.027 45	−0.111 65	−0.141 08	−0.037 07
350	−0.109 70	−0.138 20	−0.017 49	0.117 71	0.121 94
375	−0.042 54	0.109 55	0.123 28	−0.013 00	−0.128 85
0.400	0.138 60	0.021 95	−0.120 20	−0.094 13	0.058 91
425	−0.077 47	−0.124 46	0.017 60	0.128 12	0.041 78
450	−0.064 93	0.101 67	0.093 76	−0.069 27	−0.111 85
475	0.128 47	0.017 61	−0.121 04	−0.035 62	0.111 41
0.500	−0.047 74	−0.113 82	0.046 01	0.109 78	−0.044 45
525	−0.081 52	0.095 62	0.063 78	−0.102 08	−0.045 84
550	0.115 06	0.014 01	−0.115 14	0.021 89	0.104 47
575	−0.020 36	−0,105 24	0.067 87	0.070 37	−0.097 85
0.600	−0.092 76	0.090 80	0.034 25	−0.108 38	0.032 78
625	0.098 88	0.010 95	−0.103 48	0.067 26	0.049 33
650	0.004 47	−0.098 10	0.083 15	0.020 44	−0.098 56
675	−0.098 98	0.086 84	0.006 24	−0.090 17	0.086 61
0.700	0.080 55	0.008 27	−0.087 14	0.092 67	−0.022 93
725	0.026 41	−0.092 02	0.091 91	−0.028 54	−0.052 33
750	−0.100 48	0.083 52	−0.019 17	−0.053 92	0.093 48
775	0.060 76	0.005 90	−0.067 33	0.094 92	−0.076 87
0.800	0.045 09	−0.086 73	0.094 35	−0.066 33	0.014 38
825	−0.097 65	0.080 67	−0.041 01	−0.009 11	0.054 88
850	0.040 24	0.003 75	−0.045 33	0.075 74	−0.088 90
875	0.060 19	−0.082 08	0.090 92	−0.085 88	0.068 18
0.900	−0.090 92	0.078 21	−0.058 48	0.033 84	−0.006 80
925	0.019 75	0.001 80	−0.022 45	0.041 17	−0.057 01
950	0.071 48	−0.077 92	0.082 26	−0.084 48	0.084 61
975	−0.080 83	0.076 04	−0.071 00	0.065 72	−0.060 25
1.000	0	0	0	0	0

$j_{0,26}=80.8975\ 5587$ $j_{0,27}=84.0390\ 9078$ $j_{0,28}=87.1806\ 2984$

$j_{0,29}=90.3221\ 7264$ $j_{0,30}=93.4637\ 1878$

(7)

σ	$J_0(j_{0,31}\cdot\sigma)$	$J_0(j_{0,32}\cdot\sigma)$	$J_0(j_{0,33}\cdot\sigma)$	$J_0(j_{0,34}\cdot\sigma)$	$J_0(j_{0,35}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	−0.005 339	−0.045 232	−0.083 614	−0.120 334	−0.155 250
050	−0.231 312	−0.181 735	−0.129 289	−0.075 285	−0.021 032
075	0.292 310	0.268 863	0.231 515	0.182 689	0.125 340
0.100	−0.217 010	−0.244 755	−0.248 051	−0.227 331	−0.185 287
125	0.064 390	0.141 567	0.195 133	0.217 825	0.207 175
150	0.089 323	−0.006 417	−0.097 811	−0.165 454	−0.195 580
175	−0.178 302	−0.110 257	−0.012 328	0.086 233	0.156 872
0.200	0.172 256	0.170 197	0.104 197	0.000 979	−0.099 594
225	−0.085 720	−0.158 813	−0.154 714	−0.077 712	0.033 738
250	−0.032 442	0.088 135	0.154 14	0.129 22	0.030 44
275	0.123 82	0.009 61	−0.107 70	−0.147 10	−0.083 72
0.300	−0.147 70	−0.095 14	0.033 01	0.130 50	0.119 12
325	0.098 32	0.136 98	0.045 64	−0.085 89	−0.132 74
350	−0.004 94	−0.122 45	−0.104 66	0.025 17	0.124 13
375	−0.084 40	0.061 17	0.127 86	0.037 06	−0.096 19
0.400	0.126 82	0.020 09	−0.110 67	−0.086 84	0.054 55
425	−0.104 71	−0.088 54	0.060 77	0.113 82	−0.006 64
450	0.032 58	0.118 29	0.004 86	−0.113 24	−0.039 62
475	0.051 63	−0.099 89	−0.065 44	0.086 77	0.076 94
0.500	−0.106 15	0.043 04	0.102 85	−0.041 76	−0 099 84
525	0.105 79	0.028 02	−0.106 84	−0.010 66	0.105 37
550	−0.053 33	−0.084 70	0.077 78	0.058 18	−0.093 45
575	−0.023 10	0.105 34	−0.026 05	−0.090 19	0.066 79
0.600	0.084 87	−0.083 36	−0.031 51	0.100 04	−0.030 35
625	−0.102 18	0.029 22	0.076 99	−0.086 39	−0.009 58
650	0.068 26	0.034 40	−0.096 90	0.053 31	0.046 36
675	−0.001 84	−0.081 95	0.086 11	−0.009 15	−0.074 16
0.700	−0.063 08	0.095 18	−0.049 05	−0.035 51	0.088 77
725	0.094 45	−0.070 08	−0.001 89	0.070 39	−0.088 28
750	−0.077 90	0.017 93	0.050 52	−0.087 77	0.073 29
775	0.023 22	0.039 66	−0.081 89	0.084 21	−0.046 72
0.800	0.041 25	−0.079 61	0.086 85	−0.061 23	0.013 31
825	−0.083 25	0.086 58	−0.064 71	0.024 73	0.021 26
850	0.082 60	−0.058 79	0.023 13	0.016 43	−0.051 30
875	−0.040 88	0.008 34	0.024 44	−0.052 57	0.072 01
0.900	−0.020 00	0.044 02	−0.063 07	0.075 47	−0.080 25
925	0.069 26	−0.077 36	0.081 02	−0.080 18	0.075 0
950	−0.082 71	0.078 89	−0.073 29	0.066 2	−0.057 6
975	0.054 61	−0.048 83	0.043 0	−0.037 1	0.031 1
1.000	0	0	0	0	0

$j_{0,31}=96.6052\ 6795$ $j_{0,32}=99.7468\ 1986$ $j_{0,33}=102.8883\ 743$
$j_{0,34}=106.0299\ 309$ $j_{0,35}=109.1714\ 896$

(8)

σ	$J_0(j_{0,36}\cdot\sigma)$	$J_0(j_{0,37}\cdot\sigma)$	$J_0(j_{0,38}\cdot\sigma)$	$J_0(j_{0,39}\cdot\sigma)$	$J_0(j_{0,40}\cdot\sigma)$
0.000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
025	−0.188 232	−0.219 160	−0.247 924	−0.274 428	−0.298 585
050	0.032 193	0.083 168	0.130 754	0.173 918	0.211 753
075	0.062 790	−0.001 457	−0.063 899	−0.121 224	−0.170 474
0.100	−0.126 555	−0.057 226	0.015 769	0.085 332	0.144 875
125	0.165 732	0.100 549	0.022 013	−0.057 778	−0.126 850
150	−0.182 820	−0.131 107	−0.052 586	0.035 196	0.113 166
175	0.179 927	0.150 047	0.077 339	−0.015 964	−0.102 245
0.200	−0.159 810	−0.158 162	−0.096 954	−0.000 796	0.093 212
225	0.126 09	0.156 31	0.111 81	0.015 60	−0.085 54
250	−0.083 13	−0.145 53	−0.122 15	−0.028 76	0.078 88
275	0.035 73	0.127 17	0.128 19	0.040 48	−0.073 02
0.300	0.011 25	−0.102 79	−0.130 16	−0.050 90	0.067 77
325	−0.053 29	0.074 16	0.128 31	0.060 11	−0.063 03
350	0.086 64	−0.043 12	−0.122 99	−0.068 17	0.058 70
375	−0.108 57	0.011 59	0.114 40	0.075 13	−0.054 73
0.400	0.117 61	0.018 63	−0.103 10	−0.081 02	0.051 04
425	−0.113 64	−0.045 87	0.089 47	0.085 89	−0.047 61
450	0.097 81	0.068 70	−0.073 99	−0.089 74	0.044 39
475	−0.072 42	−0.086 02	0.057 16	0.092 62	−0.041 36
0.500	0.040 58	0.097 09	−0.039 50	−0.094 55	0.038 50
525	−0.005 96	−0.101 54	0.021 53	0.095 55	−0.035 79
550	−0.027 69	0.099 42	−0.003 75	−0.095 66	0.033 21
575	0.056 88	−0.091 15	−0.013 36	0.094 92	−0.030 75
0.600	−0.078 70	0.077 49	0.029 33	−0.093 36	0.028 39
625	0.091 16	−0.059 51	−0.043 77	0.091 02	−0.026 13
650	−0.093 31	0.038 48	0.056 33	−0.087 96	0.023 96
675	0.085 30	−0.015 81	−0.066 71	0.084 21	−0.021 87
0.700	−0.068 38	−0.007 03	0.074 70	−0.079 83	0.019 86
725	0.044 66	0.028 63	−0.080 15	0.074 88	−0.017 92
750	−0.016 90	−0.047 68	0.082 99	−0.069 40	0.016 04
775	−0.011 77	0.063 08	−0.083 21	0.063 47	−0.014 22
0.800	0.038 25	−0.073 99	0.080 90	−0.057 14	0.012 45
825	−0.059 78	0.079 86	−0.076 20	0.050 6	−0.010 8
850	0.074 17	−0.080 48	0.069 4	−0.043 6	0.009 2
875	−0.080 10	0.076 0	−0.060 6	0.036 5	−0.007 5
0.900	0.077 2	−0.066 8	0.050 2	−0.029 2	0.006 0
925	−0.066 0	0.053 6	−0.038 6	0.021 9	−0.004 4
950	0.047 9	−0.037 3	0.026 1	−0.014 5	0.002 9
975	−0.025 1	0.019 1	−0.013 1	0.007 3	−0.001 5
1.000	0	0	0	0	0

$j_{0,36}=112.3130\ 503$ $j_{0,37}=115.4546\ 127$ $j_{0,38}=118.5961\ 766$
$j_{0,39}=121.7377\ 421$ $j_{0,40}=124.8793\ 089$

Appendix II

Is It Possible For The Flow Pattern To Propagate With A Finite Velocity In A Long, Rigid, Circular Pipe Filled With A Still Liquid ?

Physical intuition suggests the nagative answer to this question. To the present writer, however, it seems convenient to describe briefly the mathematical proof of this problem for the future development to an elastic pipe which he intends to study in detail.

When small motions may be assumed, the equations of motion and of continuity read (1.1), (1.2) and (1.3). Writing V for the propagation velocity of the flow pattern which we are going to investigate, let us put

$$\omega \equiv \omega(x-Vt,\ r)\ ,$$
$$u \equiv u\,(x-Vt,\ r)\ , \tag{1}$$
and
$$v \equiv v\ (x-Vt,\ r)\ .$$

If we introduce non-dimensional quantities defined by

$$\vartheta = \frac{x-Vt}{a} \quad \text{and} \quad \sigma = \frac{r}{a}\ , \tag{2}$$

then ω, u and v should all become the functions of ϑ and σ. Taking into account that

$$\frac{\partial}{\partial x} = \frac{1}{a}\frac{\partial}{\partial \vartheta}\ , \quad \frac{\partial^2}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2}{\partial \vartheta^2}\ ,$$
$$\frac{\partial}{\partial t} = -\frac{V}{a}\frac{\partial}{\partial \vartheta}\ , \tag{3}$$
$$\frac{\partial}{\partial r} = \frac{1}{a}\frac{\partial}{\partial \sigma}\ , \quad \text{and} \quad \frac{\partial^2}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2}{\partial \sigma^2}\ ,$$

(1.1), (1.2) and (1.3) are transformed into

$$\frac{\partial u}{\partial \vartheta} - \frac{1}{\rho V}\frac{\partial \omega}{\partial \vartheta} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial u}{\partial \sigma}\right) = 0\ , \tag{4}$$

$$\frac{\partial v}{\partial \vartheta} - \frac{1}{\rho V}\frac{\partial \omega}{\partial \sigma} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial \vartheta^2} + \frac{\partial^2 v}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial v}{\partial \sigma} - \frac{v}{\sigma^2}\right) = 0\ , \tag{5}$$

and
$$\frac{\partial u}{\partial \vartheta} + \frac{1}{\sigma}\frac{\partial}{\partial \sigma}(\sigma v) = 0\ , \tag{6}$$

respectively, where we write for convenience

$$R \equiv V a/\nu\ . \tag{7}$$

Next we attach the mark $\sim$ to the quantity to signify its Fourier transform with respect to ϑ: for example,

$$\tilde{u}(p, \sigma) \equiv \int_{-\infty}^{\infty} u(\vartheta, \sigma)\, e^{-ip\vartheta}\, d\vartheta \,. \tag{8}$$

Then making use of the conditions

$$\omega,\ u,\ v,\ \partial u/\partial\vartheta,\ \partial v/\partial\vartheta \to 0 \quad \text{as} \quad \vartheta \to \pm\infty \,, \tag{9}$$

we obtain from the equations (4), (5) and (6)

$$ip\tilde{u} - \frac{ip}{\rho V}\tilde{\omega} + \frac{1}{R}\left(\frac{d^2\tilde{u}}{d\sigma^2} + \frac{1}{\sigma}\frac{d\tilde{u}}{d\sigma} - p^2\tilde{u}\right) = 0 \,, \tag{10}$$

$$ip\tilde{v} - \frac{1}{\rho V}\frac{d\tilde{\omega}}{d\sigma} + \frac{1}{R}\left(\frac{d^2\tilde{v}}{d\sigma^2} + \frac{1}{\sigma}\frac{d\tilde{v}}{d\sigma} - \frac{\tilde{v}}{\sigma^2} - p^2\tilde{v}\right) = 0, \tag{11}$$

and
$$ip\tilde{u} + \frac{1}{\sigma}\frac{d}{d\sigma}(\sigma\tilde{v}) = 0 \,, \tag{12}$$

respectively. Further, if we substitute for p another variable ζ defined by

$$ip = \zeta \,, \tag{13}$$

and transform the dependent and independent variables by the equations

$$(\zeta^2 + R\zeta)\,\tilde{u}/\zeta = \varphi \,, \tag{14}$$

$$\sqrt{\zeta^2 + R\zeta}\ \tilde{v} = \psi \,, \tag{15}$$

$$R\tilde{\omega}/\rho V = \chi \,, \tag{16}$$

and
$$\sqrt{\zeta^2 + R\zeta}\ \sigma = \alpha \,, \tag{17}$$

then (10), (11) and (12) become respectively

$$\frac{d^2\varphi}{d\alpha^2} + \frac{1}{\alpha}\frac{d\varphi}{d\alpha} + \varphi = \chi \,, \tag{18}$$

$$\frac{d^2\psi}{d\alpha^2} + \frac{1}{\alpha}\frac{d\psi}{d\alpha} + \left(1 - \frac{1}{\alpha^2}\right)\psi = \frac{d\chi}{d\alpha} \,, \tag{19}$$

and
$$\varphi = -\frac{\zeta^2 + R\zeta}{\zeta^2}\frac{1}{\alpha}\frac{d}{d\alpha}(\alpha\psi) \,. \tag{20}$$

Elimination of χ between (18) and (19) yields

$$\left\{\frac{d^2}{d\alpha^2} + \frac{1}{\alpha}\frac{d}{d\alpha} + \left(1 - \frac{1}{\alpha^2}\right)\right\}\left(\frac{d\varphi}{d\alpha} - \psi\right) = 0 \,, \tag{21}$$

it follows therefore that

$$\frac{d\varphi}{d\alpha} - \psi = AJ_1(\alpha) \,, \tag{22}$$

where A is an arbitrary constant. It should be noted here that the boundary conditions

$$\frac{d\varphi}{d\alpha}\,, \quad \psi = 0 \quad \text{at} \quad \alpha = 0 \tag{23}$$

are automatically satisfied by the solution (22). On the other hand, we have from (20)

$$\frac{d\varphi}{d\alpha} = -\frac{\zeta^2 + R\zeta}{\zeta^2}\left(\frac{d^2\psi}{d\alpha^2} + \frac{1}{\alpha}\frac{d\psi}{d\alpha} - \frac{\psi}{\alpha^2}\right). \tag{24}$$

So combining (22) and (24), the equation defining ψ is found to be

$$\psi + A J_1(\alpha) = -\frac{\zeta^2 + R\zeta}{\zeta^2}\left(\frac{d^2\psi}{d\alpha^2} + \frac{1}{\alpha}\frac{d\psi}{d\alpha} - \frac{\psi}{\alpha^2}\right). \tag{25}$$

With a view to obtaining a particular solution of this equation, we assume the form

$$\psi = B J_1(\alpha), \tag{26}$$

and by substituting this expression for ψ in (25), we find that

$$B = \zeta A/R. \tag{27}$$

To obtain the complementary function we write again

$$\zeta\alpha/\sqrt{\zeta^2 + R\zeta} = \beta, \tag{28}$$

then the homogeneous equation derived from (25) is readily shown to be

$$\frac{d^2\psi}{d\beta^2} + \frac{1}{\beta}\frac{d\psi}{d\beta} + \left(1 - \frac{1}{\beta^2}\right)\psi = 0. \tag{29}$$

Thus with the aid of another undetermined constant, C say, the solution of the equation (25) can be expressed in the form

$$\psi = \zeta A J_1(\alpha)/R + C J_1(\beta). \tag{30}$$

If we substitute from (30) into (20) and take into consideration the formula[1)]

$$\frac{1}{z}\frac{d}{dz}\left\{z J_1(z)\right\} = J_0(z), \tag{31}$$

we get

$$\varphi = -\frac{\zeta^2 + R\zeta}{R\zeta} A J_0(\alpha) - \frac{\sqrt{\zeta^2 + R\zeta}}{\zeta} C J_0(\beta). \tag{32}$$

As the other boundary condition, we have to impose upon φ and ψ the requirement that at $r=a$ (namely $\sigma=1$, $\alpha=(\zeta^2+R\zeta)^{1/2}$, $\beta=\zeta$),

$$\varphi,\ \psi = 0. \tag{33}$$

From (30) and (32), therefore,

$$\frac{\zeta}{R} J_1(\sqrt{\zeta^2 + R\zeta})A + J_1(\zeta)C = 0, \tag{34}$$

and

$$\frac{\zeta^2 + R\zeta}{R\zeta} J_0(\sqrt{\zeta^2 + R\zeta})A + \frac{\sqrt{\zeta^2 + R\zeta}}{\zeta} J_0(\zeta)C = 0. \tag{35}$$

In order that these equations may be satisfied by the constants A and C, either of which does not vanish at least, the determinant

1) Watson, G. N., *ibid.*, p. 18, (5).

$$\begin{vmatrix} \dfrac{\zeta}{R} J_1(\sqrt{\zeta^2+R\zeta}) & J_1(\zeta) \\ \dfrac{\sqrt{\zeta^2+R\zeta}}{R} J_0(\sqrt{\zeta^2+R\zeta}) & J_0(\zeta) \end{vmatrix}$$

should vanish *identically*. $R=\infty$ is one possibility, but then from (7) we know readily that the propagation velocity is large without bounds. Otherwise the determinantal equation

$$\begin{vmatrix} \zeta J_1(\sqrt{\zeta^2+R\zeta}) & J_1(\zeta) \\ \sqrt{\zeta^2+R\zeta}\, J_0(\sqrt{\zeta^2+R\zeta}) & J_0(\zeta) \end{vmatrix} = 0 \tag{36}$$

should be satisfied identically. Expanding the determinant, we have

$$\frac{J_1(\sqrt{\zeta^2+R\zeta})}{\sqrt{\zeta^2+R\zeta}} J_0(\zeta) - \frac{J_1(\zeta)}{\zeta} J_0(\sqrt{\zeta^2+R\zeta}) = 0\,. \tag{37}$$

If this equality should be valid for the special value (or values) of R irrespectively of ζ, then that R will be the *eigenvalue* for this propagation problem. By the formula[1]

$$\frac{J_1(z)}{z} = \frac{J_0(z)+J_2(z)}{2}\,, \tag{38}$$

however, (37) becomes

$$J_2(\sqrt{\zeta_2+R\zeta})\,J_0(\zeta) - J_2(\zeta)\,J_0(\sqrt{\zeta_2+R\zeta}) = 0\,, \tag{39}$$

which cannot be the identity except the case when

$$R = 0\,. \tag{40}$$

Summing up, *the flow pattern has been proved impossible to propagate with a finite velocity.*

Finally, combining (30), (32) and (40), the standing flow pattern is given by

$$\varphi = -(C+\zeta A/R)\,J_0(\zeta\sigma)\,, \tag{41}$$

and

$$\psi = (C+\zeta A/R)\,J_1(\zeta\sigma)\,. \tag{42}$$

In order that both of them may vanish at $\sigma=1$, the quantity $C+\zeta A/R$ needs to be zero identically. But in this case it follows that

$$\varphi,\ \psi \equiv 0\,, \tag{43}$$

so from (14) and (15)

$$\tilde{u},\ \tilde{v} \equiv 0\,, \tag{44}$$

and from (16) and (18)

$$\tilde{\omega} \equiv 0\,. \tag{45}$$

Thus we find that the flow cannot take place at all.

1) Watson, G. N., *ibid.*, p. 17, (1).

APPENDIX III

AN APPROXIMATE TREATMENT FOR THE FLOW PATTERN CAUSED BY THE IMPULSIVE PRESSURE WAVE TRAVELING IN A LONG CIRCULAR PIPE

In APPENDIX II we have proved mathematically that the flow pattern cannot propagate with a finite velocity in a long, rigid, circular pipe filled with a still fluid. Let us imagine next that the flow pattern is traveling in an elastic circular pipe with the velocity that is determined by the elasticity of the pipe wall. The flow pattern propagating sinusoidally with respect to time and space in the form exp $\{ in\,(t-x/V) \}$ was studied in great detail by Wormersley[1] with the aid of some simplifying assumptions.

As to an impulsive pressure, on the other hand, there is no published literature so far as the present writer is aware. However in order to obtain a general idea concerning the flow due to the impulsive pressure propagating in a fluid contained in an elastic tube, the following picture might be of some value as a very rough approximation, provided that the simplification explained at the end of 1 of the foregoing paper should be permissible. Namely we shall calculate the motion of a fluid caused by the impulsive pressure traveling with the velocity determined by the elasticity of the wall, as if the fluid were contained in a rigid pipe and the velocity on the wall vanished. Besides, the problem will become much tractable, if we remember v, the radial component of the velocity, is much smaller than u, the axial component, as long as the radius of the pipe is small enough and $\partial u/\partial x$ is not so large, and further ω, the pressure of the liquid, can be regarded as independent of r, the radial distance from the axis, in the degree of our approximation. For example, let us take the pressure which is expressed in the form

$$\frac{\omega}{\rho} = k\,H\left(\frac{Vt-x}{a}\right), \tag{1}$$

where $H(h)$ is called the unit function and is defined by

$$\begin{aligned} H\,(h\leqq 0) &= 0\,, \\ H\,(h>0) &= 1\,,^{2)} \end{aligned} \tag{2}$$

and k is some constant having the dimension of (velocity)^2. Then obviously ω represents an impulsive pressure of a certain intensity traveling with the velocity V, in the direction of x increasing, and

$$-\frac{\partial}{\partial x}\left(\frac{\omega}{\rho}\right) = \frac{k}{a}\,\delta\left(\frac{Vt-x}{a}\right), \tag{3}$$

1) Womersely, J. R., Oscillatory Motion of a Viscous Liquid in a Thin-Walled Elastic Tube-I: The Linear Approximation for Long Waves, *Phil. Mag.*, vol. 46 (1955), no. 373, pp. 199–221.

2) Carslaw, H. S. and Jaeger, J. C., *Operational Methods in Applied Mathematics* (2nd edition), Oxford University Press (1949), p. 7.

δ being Dirac's delta function.[1] The functions (1) and (3) have the discontinuities at the moving point determined by the equation $x = Vt$, so at this point $\partial u/\partial x$ may become very large, thus violating the above assumptions. Except this one point, however, the pressure distribution being completely uniform, we may safely neglect v and $\partial\omega/\partial r$. There is left one question, which goes "*Can the pressure given by* (1) *travel with a finite velocity* V *without changing its shape in a pipe filled with a still liquid?*" This problem cannot be discussed without taking account of the elasticity of the pipe, and so remains unsolved at present.

Substituting from (3) into the pressure gradient in the equation of motion (2.1), we obtain

$$\frac{\partial u}{\partial t} = \frac{k}{a}\delta\left(\frac{Vt-x}{a}\right) + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2}\right). \tag{4}$$

Now by making use of the relations (3) of APPENDIX II and the variables ϑ, σ and u' defined as before by

$$\frac{x-Vt}{a} \equiv \vartheta\,, \qquad \frac{r}{a} \equiv \sigma\,, \tag{5}$$

and

$$u/V \equiv u'\,, \tag{6}$$

we can arrive at the form

$$\frac{\partial u'}{\partial \vartheta} + \frac{k}{V^2}\delta(-\vartheta) + \frac{1}{R}\left(\frac{\partial^2 u'}{\partial \vartheta^2} + \frac{\partial^2 u'}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial u'}{\partial \sigma}\right) = 0\,, \tag{7}$$

where

$$R \equiv Va/\nu\,. \tag{8}$$

The Fourier transform of this equation with respect to the variable ϑ is found to be

$$\frac{d^2\tilde{u}'}{d\sigma^2} + \frac{1}{\sigma}\frac{d\tilde{u}'}{d\sigma} - p^2\tilde{u}' + i\,Rp\,\tilde{u}' = -R\frac{k}{V^2}\,, \tag{9}$$

or in terms of ζ such that

$$ip = \zeta\,, \tag{10}$$

we have

$$\frac{d^2\tilde{u}'}{d\sigma^2} + \frac{1}{\sigma}\frac{d\tilde{u}'}{d\sigma} + (\zeta^2 + R\zeta)\tilde{u}' = -Rk/V^2\,. \tag{11}$$

The solution compatible with the boundary conditions

$$\tilde{u}' = 0 \quad \text{at} \quad \sigma = 1\,,$$

and

$$\tilde{u}' = \text{finite} \quad \text{at} \quad \sigma = 0\,, \tag{12}$$

is as follows:

$$\tilde{u}' = R\frac{k}{V^2}\frac{J_0(\sqrt{\zeta^2+R\zeta}\cdot\sigma) - J_0(\sqrt{\zeta^2+R\zeta})}{(\zeta^2+R\zeta)\,J_0(\sqrt{\zeta^2+R\zeta})}\,. \tag{13}$$

The inverse transformation can be performed according to the formula

$$u'(\vartheta, \sigma) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \tilde{u}'\, e^{\vartheta\zeta}\, d\zeta\,, \tag{14}$$

2) Carslaw *et al.*, *ibid.*, p. 233.

namely

$$u'(\vartheta, \sigma) = R\frac{k}{V^2}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{J_0(\sqrt{\zeta^2+R\zeta}\cdot\sigma)-J_0(\sqrt{\zeta^2+R\zeta})}{(\zeta^2+R\zeta)\,J_0(\sqrt{\zeta^2+R\zeta})}\,e^{\vartheta\zeta}\,d\zeta. \quad (15)$$

As readily proved, cf. **2**, i), the points given by

$$\zeta^2+R\zeta=0 \quad (16)$$

in the ζ-plane cannot be the poles of the integrand; the only singularities ζ_n arise from

$$\sqrt{\zeta_n^2+R\zeta_n} = j_{0,n} \quad (n=1, 2,\cdots), \quad (17)$$

or

$$\zeta_n = \frac{-R\pm\sqrt{R^2+4\,j_{0,n}^2}}{2}. \quad (18)$$

For shortness we write

$$\zeta_{n,+} \equiv \frac{-R+\sqrt{R^2+4\,j_{0,n}^2}}{2},$$

and

$$\zeta_{n,-} \equiv \frac{-R-\sqrt{R^2+4\,j_{0,n}^2}}{2}, \quad (19)$$

respectively. Then by the well-known formula we obtain: for $\vartheta\geqq 0$

$$u' = 2R\frac{k}{V^2}\sum_{n=1}^{\infty}\left[\frac{\exp(\zeta_{n,-}\cdot\vartheta)\,J_0(j_{0,n}\cdot\sigma)}{j_{0,n}\sqrt{R^2+4\,j_{0,n}^2}\,J_1(j_{0,n})}\right], \quad (20)$$

and for $\vartheta\leqq 0$

$$u' = 2R\frac{k}{V^2}\sum_{n=1}^{\infty}\left[\frac{\exp(\zeta_{n,+}\cdot\vartheta)\,J_0(j_{0,n}\cdot\sigma)}{j_{0,n}\sqrt{R^2+4\,j_{0,n}^2}\,J_1(j_{0,n})}\right], \quad (21)$$

respectively. Fixing x, $\vartheta=0$ corresponds to the instant when the discontinuous pressure front reaches this point, while $\vartheta>0$ and $\vartheta<0$ represent respectively the ranges of time *before* and *after* the arrival of the front. As the numerical examples, for the three cases when $R=1$, 10 and 100, $V^2u'/2k$ are calculated for various values of ϑ and σ. For ϑ not equal to zero, the summations on the right-hand sides of (20) and (21) are cut off at $n=10$. For $\vartheta=0$, however, because the series converges very slowly in the absence of the exponential factors, the summations are advanced to $n=17$. Especially, at the point $\sigma=0$ (the axis of the pipe), the summations are further extended up to $n=39$, and besides the corrections for the remainder are taken into consideration. Namely, for $\vartheta=\sigma=0$, (20) or (21) becomes

$$u' = 2R\frac{k}{V^2}\sum_{n=1}^{\infty}\left[\frac{1}{j_{0,n}\sqrt{R^2+4\,j_{0,n}^2}\,J_1(j_{0,n})}\right], \quad (22)$$

and owing to the approximations of (4.4) and (4.5) the necessary correction is known to be

$$\sum_{n=40}^{\infty}\left[\frac{1}{j_{0,n}\sqrt{R^2+4\,j_{0,n}^2}\,J_1(j_{0,n})}\right]\sim\sum_{n=40}^{\infty}\frac{(-)^{n+1}}{2^{3/2}\,\pi\,n^{3/2}}, \quad (23)$$

provided that

$$2\,j_{0,n} \gg R. \quad (24)$$

For example, when $R=1$, 10 and 100, the approximate (the right-hand side), and the exact (the left-hand side), values of the term $n=40$ in the expression (23) are tabulated below:

exact values			approximate value
$R=1$	$R=10$	$R=100$	$R=1,10,100$
0.000 4490	0.000 4487	0.000 4169	0.000 4491

Making a partial sum s_m for $n=2m$ and $2m+1$ we have approximately

$$s_m = -\frac{3}{32\pi}\frac{1}{m^{5/2}}, \tag{25}$$

and the remainder can be evaluated by

$$R_r \sim -\frac{3}{32\pi}\int_{20}^{\infty}\frac{dm}{m^{5/2}} = -\frac{1}{640\sqrt{5}\,\pi} = -0.000\ 2224\cdots. \tag{26}$$

In FIGURES 20, 21, 22 and TABLE 17 the general features of the function $V^2u'/2k$ are reproduced for ready comparison. The waviness of the velocity profiles observed in the case $R=100$ is due presumably to the cut-off effect of the series.

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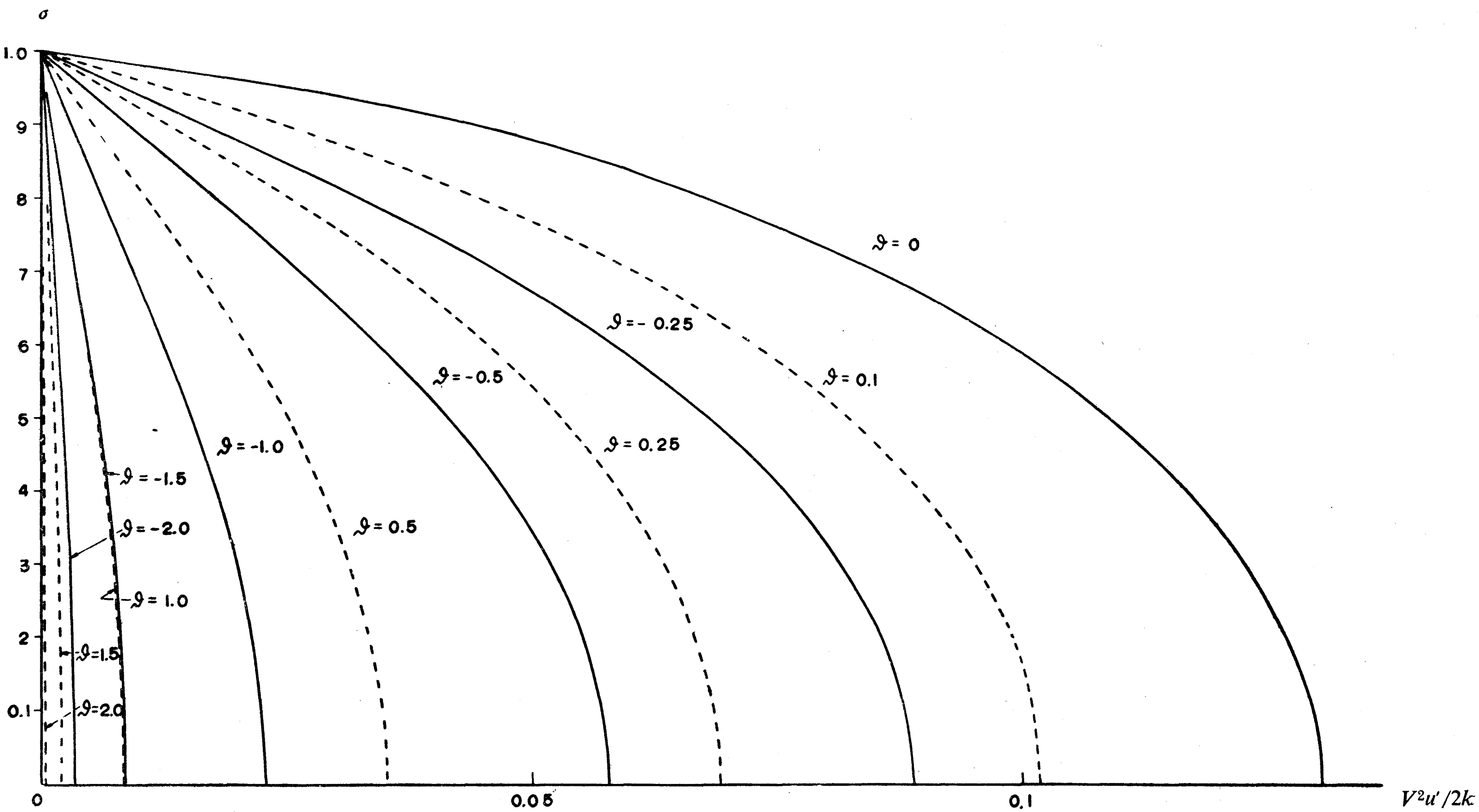

FIG. 20. Velocity profiles when $R=1$.

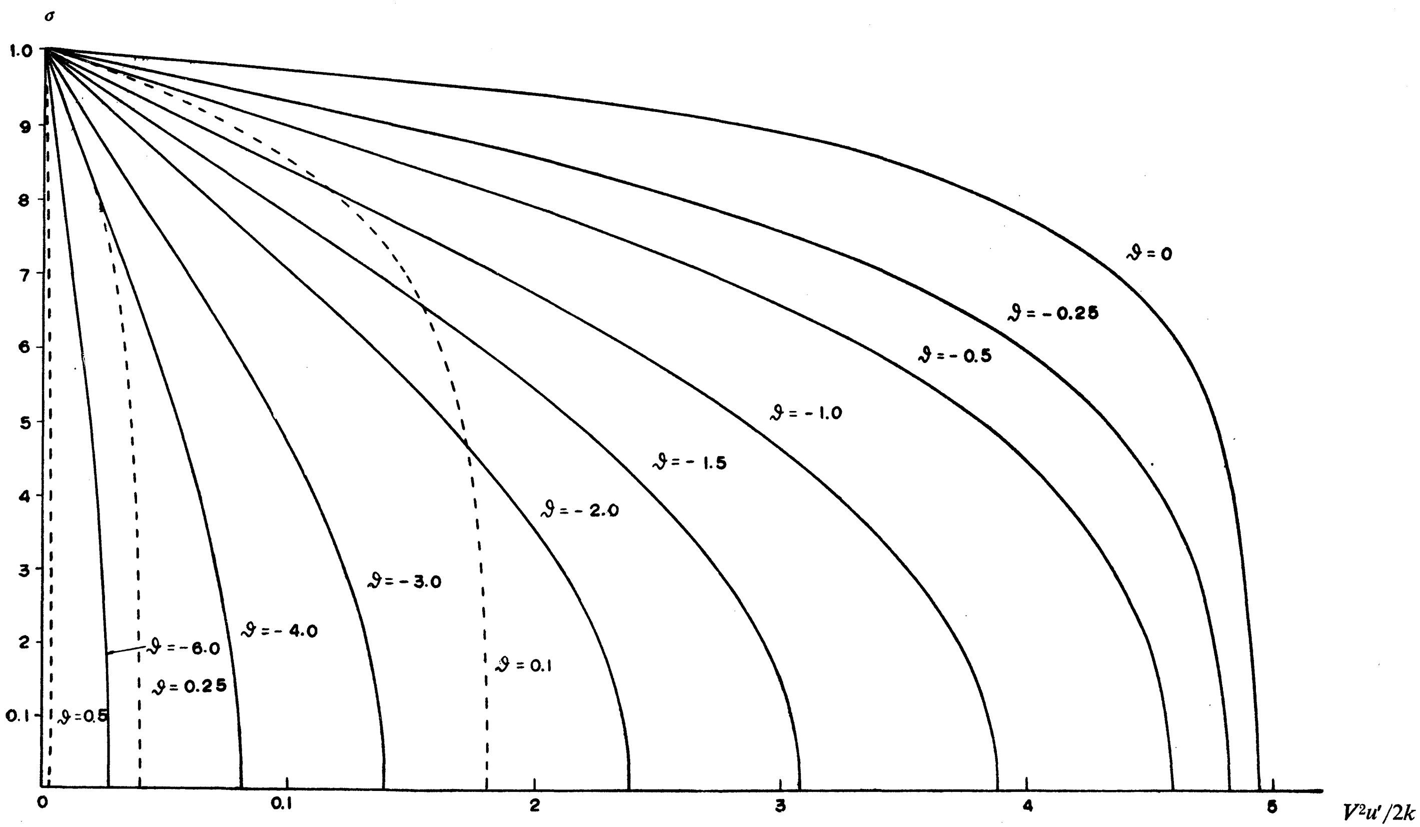

FIG. 21. Velocity profiles when $R=10$.

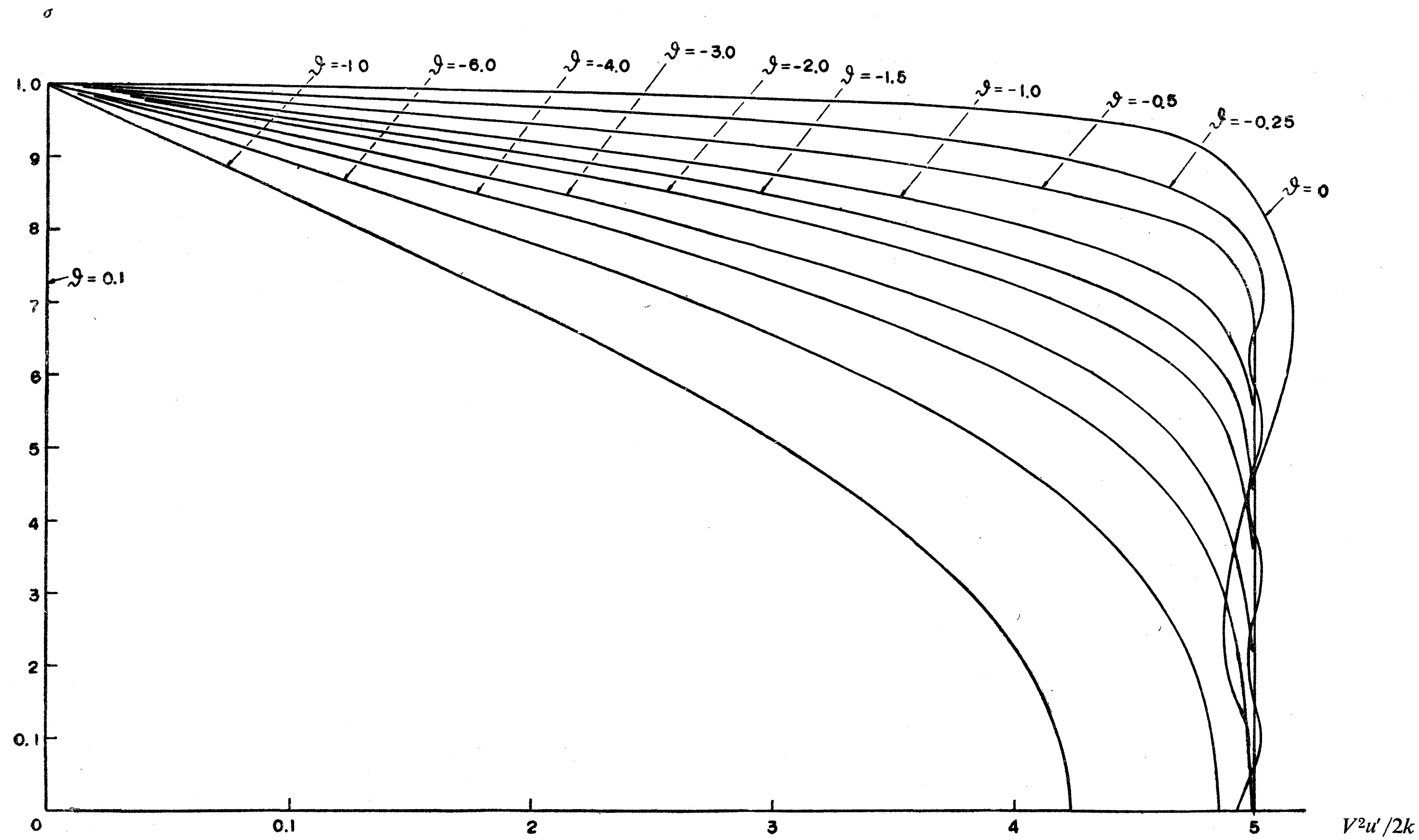

FIG. 22. Velocity profiles when $R=100$.

TABLE 17.

	$\sigma \backslash \vartheta$	−10	−6	−4	−3	−2	−1.5	−1.0	−0.5
	0		0.0000 013	0.0000 65	0.000 461	0.00 326	0.00 864	0.0227	0.0578
	0.1		0. 013	0. 64	0. 454	0. 321	0. 852	0.0224	0.0572
	0.2		0. 012	0. 61	0. 435	0. 307	0. 816	0.0215	0.0552
	0.3		0. 011	0. 57	0. 403	0. 285	0. 757	0.0200	0.0519
$V^2u'/2k$	0.4		0. 010	0. 51	0. 360	0. 255	0. 677	0.0180	0.0474
	0.5		0. 009	0. 44	0. 309	0. 218	0. 581	0.0155	0.0416
($R=1$)	0.6		0. 007	0. 35	0. 250	0. 177	0. 472	0.0126	0.0346
	0.7		0. 005	0. 27	0. 188	0. 133	0. 354	0.0095	0.0266
	0.8		0. 004	0. 17	0. 123	0. 087	0. 233	0.0063	0.0179
	0.9		0. 002	0. 08	0. 060	0. 042	0. 113	0.0031	0.0088
	1.0		0	0	0	0	0	0	0
	0	0.00 300	0.0269	0.0805	0.1391	0.238	0.308	0.388	0.459
	0.1	0. 296	0.0265	0.0794	0.1371	0.235	0.304	0.384	0.456
	0.2	0. 283	0.0254	0.0759	0.1312	0.225	0.293	0.372	0.449
	0.3	0. 262	0.0235	0.0704	0.1217	0.210	0.273	0.351	0.434
$V^2u'/2k$	0.4	0. 235	0.0210	0.0630	0.1089	0.188	0.247	0.322	0.412
	0.5	0. 201	0.0180	0.0540	0.0934	0.162	0.214	0.284	0.379
($R=10$)	0.6	0. 163	0.0146	0.0438	0.0758	0.132	0.175	0.237	0.332
	0.7	0. 122	0.0110	0.0328	0.0569	0.099	0.133	0.183	0.268
	0.8	0. 080	0.0072	0.0216	0.0374	0.066	0.088	0.123	0.188
	0.9	0. 039	0.0035	0.0105	0.0182	0.032	0.043	0.060	0.096
	1.0	0	0	0	0	0	0	0	0
	0	0.424	0.485	0.498	0.500	0.500	0.500	0.500	0.499
	0.1	0.419	0.482	0.497	0.499	0.500	0.500	0.500	0.500
	0.2	0.405	0.474	0.494	0.499	0.500	0.500	0.500	0.499
	0.3	0.382	0.458	0.487	0.496	0.499	0.500	0.500	0.500
$V^2u'/2k$	0.4	0.348	0.431	0.472	0.488	0.498	0.499	0.500	0.500
	0.5	0.305	0.392	0.444	0.470	0.491	0.497	0.500	0.500
($R=100$)	0.6	0.253	0.336	0.397	0.433	0.470	0.486	0.496	0.500
	0.7	0.193	0.266	0.325	0.367	0.419	0.449	0.478	0.498
	0.8	0.129	0.182	0.230	0.267	0.321	0.360	0.410	0.471
	0.9	0.063	0.091	0.117	0.139	0.174	0.202	0.246	0.330
	1.0	0	0	0	0	0	0	0	0

	$\sigma \backslash \vartheta$	−0.25	0	0.1	0.25	0.5	1.0	1.5	2
	0	0.0889	0.1304	0.1018	0.0692	0.0351	0.00 837	0.00 193	0.000 441
	0.1	0.0881	0.1296	0.1012	0.0686	0.0347	0. 826	0. 190	0. 435
$V^2u'/2k$	0.2	0.0857	0.1270	0.0989	0.0667	0.0335	0. 792	0. 182	0. 416
	0.3	0.0816	0.1229	0.0951	0.0635	0.0315	0. 736	0. 169	0. 385
($R=1$)	0.4	0.0758	0.1171	0.0896	0.0590	0.0287	0. 662	0. 151	0. 345
	0.5	0.0681	0.1093	0.0823	0.0530	0.0252	0. 570	0. 130	0. 296
	0.6	0.0584	0.0990	0.0727	0.0455	0.0210	0. 465	0. 105	0. 240
	0.7	0.0466	0.0856	0.0605	0.0363	0.0161	0. 350	0. 079	0. 180
	0.8	0.0325	0.0680	0.0449	0.0253	0.0108	0. 231	0. 052	0. 118
	0.9	0.0166	0.0435	0.0248	0.0129	0.0053	0. 113	0. 025	0. 058
	1.0	0	0	0	0	0	0	0	0
	0	0.482	0.494	0.1799	0.0399	0.00 309	0.0000 18		
	0.1	0.480	0.493	0.1801	0.0397	0. 307	0. 17		
	0.2	0.476	0.490	0.1789	0.0392	0. 302	0. 17		
$V^2u'/2k$	0.3	0.468	0.487	0.1775	0.0384	0. 293	0. 16		
	0.4	0.454	0.483	0.1743	0.0371	0. 278	0. 15		
($R=10$)	0.5	0.432	0.475	0.1693	0.0353	0. 255	0. 13		
	0.6	0.396	0.461	0.1604	0.0325	0. 224	0. 11		
	0.7	0.341	0.434	0.1457	0.0280	0. 181	0. 08		
	0.8	0.256	0.383	0.1192	0.0211	0. 127	0. 06		
	0.9	0.139	0.281	0.0732	0.0115	0. 065	0. 03		
	1.0	0	0	0	0	0	0		

	$\sigma \backslash \vartheta$	—0.25	0	0.1
	0	0.492	0.499	0.0000 21
	0.1	0.503	0.498	0. 23
$V^2u'/2k$	0.2	0.497	0.488	0. 22
	0.3	0.502	0.488	0. 22
$(R=100)$	0.4	0.498	0.494	0. 22
	0.5	0.502	0.504	0. 22
	0.6	0.498	0.513	0. 23
	0.7	0.503	0.515	0. 22
	0.8	0.492	0.506	0. 22
	0.9	0.413	0.482	0. 22
	1.0	0	0	0

ERRATA

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Page 23, line 13, for "perpendicular" read "perpendicularly."

Page 24, line 1, from the bottom, for "$e^{-(1-\eta/)\lambda}$" read "$e^{-(1-\eta)/\lambda}$."

Page 25, line 2, for " $\frac{3}{5}\eta^3$" read " $\frac{3}{5}\eta^5$."

Page 27, line 2, from the bottom, for "calcula-tions" read "calculations."

(J. O.)