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ON A CURVILINEAR STRAIN GAGE FOR STRESS ANALYSIS

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On the basis of the well-known fact that the strain states at any point in the elastic body under loads are expressed by the quadratic surface as shown by Eq. (1) in the text, we discuss fully the relations between stress and strain components in the present problems, from which we find always the stress and strain states at a given point on the surface of the elastic body, provided that we have the strain components [e_{xx}, e_{yy}, e_{xy}, shown in the text] at the point. Now, since we use gages for finding stress states at a given point, we recognize the assumption that the strain states on the basic domain, where the strain gage is pasted, are the same as those at the given point, that is, the centre of the gage. Here, we propose curvilinear strain gages with such types as their baisc domains are small ring ones surrounding the given point and as we find the strain states on the ring domain as truthfully as we can. On the other hand, we induce the formulae, from which we find the strain components [e_{xx}, e_{yy}, e_{xy}], if only we find linear strains in the curvilinear directions situated by the gages on the ring domain. From the above descriptions, only measuring the curvilear strains of the gages on the ring domain after having got deformations, we find the strain components [e_{xx}, e_{yy}, e_{xy}] at the given point and the problem is solved. As an actual example of the curvilinear strain gages, we take the foil strain gage of circular type for our experiments. The diameter of the smallest of the gages being in use of our experiments is 3 mm. Then, its basic area is fairly smaller than that of usual Rosette gage and we have generally the results with higher accuracy than those obtained by using these. Therefore, the present gages come fully in the useful for the experiments of finding accurately the strain states being necessary to the stress analysis.

I. Introduction.

A strain-gage used commonly for measuring skin stress of an elastic body under the loads is one of straight-type or a combination of 3 straight gages which are not situated in parallel positions with one another, that is, "Rosette Type [Equiangular, Rectangular, T—Delta etc.]". Moreover, when we know directions of principal strains at a given point on the surface of the body, we use a combination of two straight gages which are perpendicular to each other. We generally need to use a "Rosette-Type" gage for measuring skin stress of a structure under loads and the gage has fairly wide area for its base. Therefore, to measure the strain states at a given point on the surface by using the gage is to find the respective mean values of the linear strains in the 3 directions occupied by the 3 straight gages. Namely, assuming the basic area of the gage to be a point and pasting the gage upon a given point on the surface of the structure, we measure the linear strains of the 3 straight gages after having got deformations. Then, we find the stress and strain states at the given point from formulae between stress and strain components in use of these linear strains. Therefore, for finding measurements with high accuracies, it is to be desired that its area is as small as we can. On the other hand, the maximum skin stress on the structure which is the most important value for designing it, generally, occurs at a small space which is situated in special position on its surface. Then for measuring these strain states, we can not use frequently the strain gage, because the basic area of the gage is too large for the small space, where-on the maximum skin stress occurs. Even if we use the gage for measuring the states, the value found by the gage is considerably different from the true value by reason of the descreptions stated above and, therefore, for researches requiring high accuracy on its results the gage is not suitable, because we are not content with its acuracy of the results obtained by using the gage.

For the purpose of removing defects in many ways described above, we propose a new type of a strain gage consisting of 3 curvilinear strain gages, forms of which are quite similar with one another and their centres of which are in common. Putting the centre of the gage upon a given point exactly, we measure the curvilinear strains of the gage on the ring domain surrounding the given point. From these strains obtained above and the relations between stress and strain components, we find the stress and strain states at the given point.

II. Preliminary Items.

As well known, stress and strain states at any point in an elastic body under external forces are generally expressed by following equation of quadratic surface.

$$ax^2 + by^2 + cz^2 + \mathrm{d}\,xy + fyz + h\,zx = k \qquad (1) ,$$

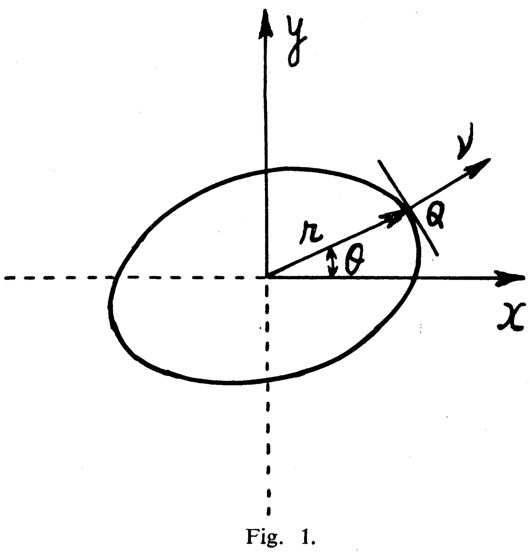


Fig. 1.

in which the coefficients represent the stress or strain components respectively as

follows $a=(\widehat{xx}, e_{xx})$, $b=(\widehat{yy}, e_{yy})$, $c=(\widehat{zz}, e_{zz})$, $d=(2\widehat{xy}, e_{xy})$,

$f=(2\widehat{yz}, e_{yz})$, $h=(2\widehat{zx}, e_{zx})$, $k=const.$

and $(\widehat{xx}, \widehat{yy}, \widehat{zz}, \widehat{xy}, \widehat{yz}, \widehat{zx}) =$ stress components

$(e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{yz}, e_{zx}) =$ strain components .

And we know also the following relation

$$al^2 + bm^2 + cn^2 + d\,lm + f\,mn + h\,nl = k/r^2 = k' \qquad (2),$$

in which r denotes the radius vector of the quadratic surface, l, m, n direction cosines of the radius vector, k' $\widehat{rr}$ stress component or e_r strain, $\widehat{rr}$ normal stress component to r-plane passing through the origin, e_r linear strain in the r-direction at the origin and r-plane the plane having the normal in r-direction. Further, by signs L, M, N representing the direction cosines of the out-ward drawn normal ν to the quadratic surface at the point Q, on which the radius vector intersects with the surface, we have the following relations from Eq. (1) and the formulae of analytical geometry

$$(L,\ M,\ N) = \frac{1}{H}\left\{\left(al+\frac{1}{2}dm+\frac{1}{2}hn\right),\ \left(\frac{1}{2}dl+bm+\frac{1}{2}fn\right),\ \left(\frac{1}{2}hl+\frac{1}{2}fm+cn\right)\right\} \qquad (3),$$

in which

$$H=\sqrt{\left(al+\frac{1}{2}dm+\frac{1}{2}hn\right)^2+\left(\frac{1}{2}dl+bm+\frac{1}{2}fn\right)^2+\left(\frac{1}{2}hl+\frac{1}{2}fm+cn\right)^2}\ .$$

Here, in the case of strain discussion, the direction ν shows that of relative displacement of the point Q to the origin after having got the deformation of the elastic body under external forces and, in the case of stress discussion, the direction ν shows that of the stress intensity F_r acting on the r-plane passing through the origin and magnitude of the stress intensity F_r is given by the equations

$$F_r = \widehat{rr}/\cos\varphi\,, \qquad \cos\varphi = lL + mM + nN = \widehat{rr}/H$$
$$\therefore\ \ F_r = H \qquad (4),$$

in which H is already shown by Eqs. (3) in stress discussion and φ denotes the angle between the radius vector r and the normal ν.

Next, we describe relations, from which we find all the stress components and the other strain ones and, moreover, all principal stresses and strains at the origin, if only we find the strain components $[e_{xx}, e_{yy}, e_{xy}]$ by experiments or some methods. Now, let us assume that the stress states acting on an infinitesimal small imaginary plane passing through a given point are known and we call the plane z-plane for the convenience of the following descriptions. The case treated in the present problems, that is, such cases as we find skin stress states at a given point on the surface of a structure under external forces by using a strain gage, corresponds to that assumed above. In this case, first, from the assumptions we know 3 stress components $[\widehat{zx}, \widehat{yz}, \widehat{zz}]$. Then, if only we find 3 strain components $[e_{xx}\ e_{yy}\ e_{xy}]$ by means of experiments or some methods, we find all the other stress and strain components as follows:

First, we know the components of $[\widehat{xy}, e_{yz}, e_{zx}]$ directly from the following relations.

$$(\widehat{xy}, \widehat{yz}, \widehat{zx}) = G(e_{xy}, e_{yz}, e_{zx}) \qquad (5_1)$$

where G denotes torsional rigidity and the normal strain component of e_{zz} from the following relations

$$\widehat{zz} = 2G\, e_{zz} + \lambda\Delta = (2G+\lambda)\, e_{zz} + \lambda\Delta_1$$

$$\therefore\ e_{zz} = (\widehat{zz} - \lambda\Delta_1)/(2G+\lambda) \qquad (5_2),$$

in which

$$\Delta = e_{xx} + e_{yy} + e_{zz}, \qquad \Delta_1 = e_{xx} + e_{yy},$$

$$\lambda = \textit{Lamé's constant.}$$

And, moreover, all the other stress components are found from the relations

$$(\widehat{xx}, \widehat{yy}) = 2G(e_{xx}, e_{yy}) + \lambda\Delta \qquad (5_3).$$

Next, as to principal stress and strain, from Eqs. (2), (3) and their definitions we have the following relations

$$\frac{1}{l}\left(al+\frac{1}{2}dm+\frac{1}{2}hn\right) = \frac{1}{m}\left(\frac{1}{2}dl+bm+\frac{1}{2}fn\right)$$

$$= \frac{1}{n}\left(\frac{1}{2}hl+\frac{1}{2}fm+cn\right) = K'' \qquad (6),$$

in which from their definitions we have the relations

$$(L, M, N) = (l, m, n)$$

and K'' denotes principal stress intensity [T] or principal strain [ε].

From the above equations, eliminating direction *cosines* [l, m, n], we have the equation

$$\begin{vmatrix} a-K'' & \frac{1}{2}d & \frac{1}{2}h \\ \frac{1}{2}d & b-K'' & \frac{1}{2}f \\ \frac{1}{2}h & \frac{1}{2}f & c-K'' \end{vmatrix} = 0$$

and, moreover, expanding the above determinant, we have

$$K''^3 - \alpha K''^2 + \beta K'' - \gamma = 0 \qquad (7).$$

Now, representing 3 roots of Eq. (7) by K_1'', K_2'', K_3'', we have, as well known, the following relations

$$\alpha = a + b + c = K_1'' + K_2'' + K_3''$$

$$\beta = bc + ca + ab - \frac{1}{4}\left(d^2 + f^2 + h^2\right) = K_1''K_2'' + K_2''K_3'' + K_3''K_1''$$

$$\gamma = abc - \frac{1}{4}\left(af^2 + bh^2 + cd^2 - dfh\right) = K_1''K_2''K_3''\ .$$

These roots K_1'', K_2'', K_3'' are nothing but 3 principal stress intensities or 3 principal strains. And from Eqs. (6) and formula of direction cosines, their directions are given by the equations

$$(l,\ m,\ n)_i = \frac{\pm 1}{D_i}\Big[\ \{2(K_i'' - b)h + df\},\ \{2(K_i'' - a)f + dh\},\ \{4(K_i''-a)(K_i''-b) - d^2\}\Big] \qquad (8)\ ,$$

in which

$$D_i = \sqrt{\{2(K_i''-b)h+df\}^2 + \{2(K_i''-a)f+dh\}^2 + \{4(K_i''-a)(K_i''-b)-d^2\}^2} \quad (8)\ .$$

Here, if only we find 3 strain components $[\,e_{xx},\ e_{yy},\ e_{xy}\,]$ by experiments or some method, we know all the coefficients in Eqs. (7) and (8) as stated previously and, therefore, from Eqs. (7) and (8) we find magnitudes of principal stresses, principal strains and their directions. After all, if only we find strain components $[\,e_{xx},\ e_{yy},\ e_{xy}\,]$, the present problems are solved as stated above.

By the bye, for the convenience of actually finding the stress and strain states at a given point in use of the strain components $[\,e_{xx},\ e_{yy},\ e_{xy}\,]$ found by utilizing strain gages, we consider overagain the above descriptions for such the case as we assume z-plane to be a principal plane. The case corresponds to that treated in the present problems, namely, such cases as we find skin stress of a stracture, surface of which is under liquid pressure [p] or in no load. First from the assumption that z-plane is a principal plane, we know

$$\widehat{zz} = p\ (= T_3)\ , \qquad \widehat{zx} = \widehat{yz} = 0\ , \qquad e_{zx} = e_{yz} = 0\ .$$

Utilizing these values for the previous results, needless to say, we find all the other unknown quantities. But it suits us better to use the following relations, namely, from the relations between stress and stain components

$$(\,e_{xx},\ e_{yy},\ e_{zz}\,) = \frac{1}{E}\left\{(1+\sigma)\,(\,\widehat{xx},\ \widehat{yy},\ p\,) - \sigma\,\Theta\right\}$$

$$\varepsilon_i = \frac{1}{E}\left\{(\,1 + \sigma\,)\,T_i - \sigma\,\Theta\right\}$$

where

$$\Theta = \widehat{xx} + \widehat{yy} + p = T_1 + T_2 + p$$

E = Young's Modulus and σ=Poisson's ratio , we have

$$\left.\begin{aligned}(\widehat{xx},\ \widehat{yy}) &= \frac{E}{1-\sigma^2}\left\{(e_{xx},\ e_{yy}) + \sigma(e_{yy},\ e_{xx})\right\} + \frac{\sigma}{1-\sigma}p \\ (T_1,\ T_2) &= \frac{E}{1-\sigma^2}\left\{(\varepsilon_1,\ \varepsilon_2) + \sigma(\varepsilon_2,\ \varepsilon_1)\right\} + \frac{\sigma}{1-\sigma}p \\ \widehat{zz} &= T_3 = p,\quad \widehat{xy} = G\,e_{xy}\end{aligned}\right\} \quad (9_1),$$

$$\left.\begin{aligned}e_{zz} &= \frac{\sigma}{1-\sigma}\left\{\frac{p}{\lambda} - (e_{xx} + e_{yy})\right\} \\ \varepsilon_3 &= \frac{\sigma}{1-\sigma}\left\{\frac{p}{\lambda} - (\varepsilon_1 + \varepsilon_2)\right\}\end{aligned}\right\} \quad (9_2).$$

From the above descriptions, if only we find 3 strain components $[e_{xx},\ e_{yy},\ e_{xy}]$, we know all the other stress and strain components from Eqs. (9).

Further, in this case, Eqs. [6] are rewritten as follows

$$\left.\begin{aligned}(a-K'')\,l + \frac{1}{2}dm &= 0 \\ \frac{1}{2}dl + (b-K'')\,m &= 0\end{aligned}\right\}$$

Therefore, from the above equations K'' and its direction are found as follows

$$\left.\begin{aligned}K_i'' &= \frac{1}{2}\left\{(a+b) \pm \sqrt{(a-b)^2 + d^2}\right\} \\ \tan\varphi_i &= \frac{d}{2(K''_i - b)} = \frac{d}{(a-b) \pm \sqrt{(a-b)^2 + d^2}}\end{aligned}\right\} \quad (10),$$

in which φ_i denotes the angle between x-axis and K_i'' direction.

After all, as to principal strains, from 2nd of Eqs. $[9_2]$ and Eqs. [10] we have the equations

$$\left.\begin{aligned}\varepsilon_{i=1\,2} &= \frac{1}{2}\left(A \pm \sqrt{B^2 + C^2}\right), \qquad \varepsilon_3 = \frac{\sigma}{1-\sigma}\left(\frac{p}{\lambda} - A\right), \\ \tan\varphi_{i=1.2} &= \frac{C}{B \pm \sqrt{B^2 + C^2}},\end{aligned}\right\} \quad (11),$$

in which

$$A = e_{xx} + e_{yy},\quad B = e_{xx} - e_{yy},\quad C = e_{xy}.$$

On the other hand, as to principal stresses, needless to say, we find them from 3rd of Eqs. $[9_1]$ and Eqs. [10] in the same manner as stated above. But for fiinding principal stresses actually, it suits us better to use the equations

$$\left.\begin{aligned}T_{i=1.2} &= \frac{E}{2}\left\{\frac{A}{1-\sigma} \pm \frac{\sqrt{B^2+C^2}}{1+\sigma}\right\} + \frac{\sigma}{1-\sigma}p \\ T_3 &= p\end{aligned}\right\} \quad (12)$$

which are reduced from Eqs. (9) and (II). And their directions coincide with the principal strain directions shown by Eqs. (11).

Last, from Eqs. (4) we find the stress intensity $[F_r]$ acting on the place passing through a given point, normal to which is in r-direction, as follows

$$F_r = \sqrt{\widehat{xx}^2\, l^2 + \widehat{yy}^2\, m^2 + 2\, \widehat{xy}\,(\widehat{xx} + \widehat{yy})\, lm + \widehat{xy}^2\,(1-n^2) + p^2\, n^2} \qquad (13)\,,$$

in which $[l,\ m,\ n]$ denote the direction cosines of the normal r and the direction of the stress F_r is given by Eqs. (3) in the stress discussions.

III. Theoritical Principle.

We take z-axis in the direction of the outward-drown normal to the surface of the elastic body at a given point which is the object of our measurements and x- and y-axes perpendicular to each other on the surface, namely, these are in orthogonal axes of right hand system. Then, first, on the surface the direction cosine n is always zero and, accordingly, Eq. (2) is rewritten as follows

$$e_{xx}\, l^2 + e_{yy}\, m^2 + e_{xy}\, lm = e_r \qquad (14)\,.$$

Therefore, in such cases as we find strain states at a given point by utilizing usual straight type strain gages, measuring linear strains $[e_r]_i$ occupied by 3 straight gages which are not situated in parallel positions with one another, we find 3 strain components $[e_{xx},\ e_{yy},\ e_{xy}]$ from symultaneous equations consisting of Eqs. (14) in connection with these strains $[e_r]_i$ respectively. Then, now we know these components, as fully described in previous part, we find all other strain and stress components at the given point from the relations between stress and strain components and the present problems are solved.

Here, let us consider the curvilinear strain gages proposed by author. First, we treat curvilinear strain gages of volute type and circular one.

[i] Strain gages of volute type.

As well known, a volute curve is shown by the eqution

$$\rho = \rho_0\, e^{p\theta} \qquad (15)\,,$$

in which ρ_0, ρ_1 denote radial vectors at the starting point and the finishing one, θ_1 amplitude of radius vector ρ_1 and $p = \frac{1}{\theta_1} \log \rho_1 / \rho_0$.

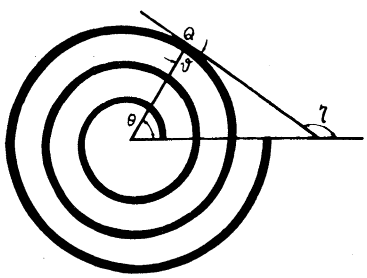


Fig. 2.

Fig. 3.

Now, representing the angle between the fundamental line and the direction of the tangent to the curve at some point Q ($\rho_1\ \theta$) by sign η, from the formulae of analytical geometry we have the relations

$$\eta = \theta + \vartheta, \quad \vartheta = \cot^{-1} p \quad (= \text{const.}) \qquad (16),$$

in which ϑ denotes the angle between the radius vector and the tangent.

Then, from Eq. (14) and Eq. (16) linear strain [e_t] of the gage in the direction of the tangent to the volute curve at the point Q is given by the equation

$$e_t = e_{xx} - (e_{xx} - e_{yy}) \sin^2 \eta + \frac{1}{2} e_{xy} \sin 2\eta .$$

Now, adopting the volute curve satisfing the equation $\vartheta \doteqdot \frac{\pi}{2}$ as the form of the strain gage, the above equation is rewritten as follows

$$e_t = \frac{1}{2}(A - B\cos 2\theta - C\sin 2\theta) \qquad (17),$$

in which

$$A = e_{xx} + e_{yy}, \qquad B = e_{xx} - e_{yy}, \qquad C = e_{xy} .$$

Therefore, utilizing Eq. (15) and (17), the total elongation ϕ of the strain gage after having got deformation is expressed by the equations

$$\phi = \int_{\theta_0}^{\theta} e_t\, ds, \qquad ds = \rho\, d\theta$$

$$= \frac{1}{2}\rho_0 \left| A\frac{e^{p\theta}}{p} - \frac{p\, e^{p\theta}}{p^2+4}\left\{\left(\frac{2}{p}B + C\right)\sin 2\theta + \left(B - \frac{2}{p}C\right)\cos 2\theta\right\}\right|_{\theta_0}^{\theta}$$

where ds denotes some elemental length of the strain gage and θ_0 the amplitude of radius vector ρ_0.

Then, the total strain e of the strain gage in a given section is expressed by the equation

$$e = \phi \Big/ \int_{\theta_0}^{\theta} \rho\, d\theta = \frac{1}{2}A + B\,\Theta_1 + C\,\Theta_2 \qquad (18),$$

in which

$$\Theta_1 = \frac{-p^2}{2(p^2+4)} \left| e^{p\theta}\left(\frac{2}{p}\sin 2\theta + \cos 2\theta\right)\right|_{\theta_0}^{\theta} \Big/ \left| e^{p\theta} \right|_{\theta_0}^{\theta},$$

$$\Theta_2 = \frac{-p^2}{2(p^2+4)} \left| e^{p\theta}\left(\sin 2\theta - \frac{2}{p}\cos 2\theta\right)\right|_{\theta_0}^{\theta} \Big/ \left| e^{p\theta} \right|_{\theta_0}^{\theta} .$$

And the above equation is nothing but linear one with variables [A, B, C], that is, strain components [e_{xx}, e_{yy}, e_{xy}]. Therefore, measuring linear strains [e_1, e_2, e_3] of the three strain gages, after having got deformation, which have a common centre at a given point and are not in the same positions with one another, we have always simultaneous equations consisting of 3 equations which are nothing but Eqs. (18) for these strains respectively and always find unknown constants

"note":-θ in Eq. (15) is the angle which add $2n\pi$ ($n=0, 1, 2\cdots$) to θ in Eq. (16). But, for the convenience of the following descriptions we adopt the same sign θ in both equations, because we have no interference in the following descriptions owing to such expressions as stated above.

$[e_{xx}, e_{yy}, e_{xy}]$ from them. Therefore, we know the stress and strain states at the given point as fully discussed in the previous part and, accordingly, the present problems are solved.

[ii] Strain gages of circular type.

In this case, first, from Eqs. (14) and (17) the strains e_r and e_t in the radial and tangential directions of the circular strain gage at a given point are found as follows

$$e_r = \frac{1}{2}(A + B\cos 2\theta + C\sin 2\theta),$$

$$e_t = \frac{1}{2}(A - B\cos 2\theta - C\sin 2\theta),$$

in which $A = e_{xx} + e_{yy}, \qquad B = e_{xx} - e_{yy}, \qquad C = e_{xy}$.

Here, let us consider a circular strain gage consisting of n circular curves of n concentric circles which are situated in the section between two radii in the directions $[\theta_0, \theta_1]$ and these radii as shown in Fig. (4).

For the convenience of the following treatments, we take the concentric circles with equal intervals of

$$\Delta\rho = \frac{\rho_1 - \rho_0}{n-1}, \qquad \delta = \frac{\Delta\rho}{\rho_1},$$

in which ρ_1, ρ_0 denote the outer and inner radii.

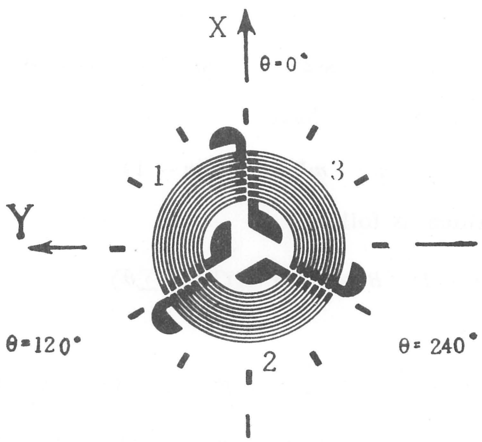


Fig. 4.

Then, the total elongation ϕ_t in the circular direction of the strain gage stated above is given by the equations

$$\phi_t = \sum_{i=1}^{n} {}_i\rho \int_{\theta_0}^{\theta_1} e_t \, d\theta = N\cdot P\cdot F \qquad (19_1),$$

in which

$$N = n\left\{1 - \frac{1}{2}(n-1)\delta\right\}, \qquad N\cdot\rho_1 = \frac{n}{2}(\rho_1 + \rho_0),$$

$$F = \frac{1}{2}\left\{A\,\Delta\theta - \sin\Delta\theta\,(B\cos\Sigma\theta + C\sin\Sigma\theta)\right\}$$

$_i\rho$ = radius of their each circle, $\quad {}_1\rho = \rho_0\,, \quad {}_n\rho = \rho_1$

$\Delta\,\theta = \theta_1 - \theta_0\,, \qquad \Sigma\,\theta = \theta_1 + \theta_0\,.$

And the total elongations $[\phi_{r0},\ \phi_{r1}]$ in the two radial directions at both ends of the strain gage are given by the equations

$$\phi_{r0} = n_i^*\ \delta\ \rho_1\ e_{r0}\,, \qquad \phi_{r1} = n_i\ \delta\ \rho_1\ e_{r1} \qquad (19_2)\,,$$

in which e_{r0}, e_{r1} denote linear strains in the radial directions. Now taking the radial parts of the gage in such connections as shown in Fig (4), we have

$$n = \text{odd number} : \qquad n_i = n_i^* = \frac{n-1}{2}$$

$$n = \text{even number} : \qquad n_i = \frac{n}{2}\,, \qquad n_i^* = \frac{n}{2} - 1\,.$$

On the other hand, the total length [L] of the strain gage is given by the equation

$$L = N\rho_1\,(\theta_1 - \theta_0) + (n-1)\,\delta\rho_1 \qquad (19_3)\,.$$

From the above results, the total strain of the strain gage is given by equation

$$e = \frac{N\cdot F + \delta\,(n_i^*\ e_{r0} + n_i\ e_{r1})}{N\,(\theta_1 - \theta_0) + (n-1)\,\delta} \qquad (20_1)\,,$$

in which

$$e_{ri} = \frac{1}{2}\,(A + B\cos 2\theta_i + C\sin 2\theta_i) \quad (i = 0,\ 1)\,.$$

Here, taking n of odd number, we have

$$n_i = n_i^* = \frac{1}{2}\,(n-1)$$

and Eq. (20_1) is rewritten as follows

$$e = \frac{1}{2}A + \Gamma_1\,(B\cos\Sigma\theta + C\sin\Sigma\theta) \qquad (20_2)\,,$$

in which

$$\Gamma_1 = (\delta\ n_i\cos\Delta\theta - \frac{N}{2}\sin\Delta\theta)\ /\ \{N\Delta\theta + (n-1)\,\delta\}$$

Moreover, taking the total length of the gage in the radial directions as small, or taking such devices in technique for radial parts of the gage, as we have almost no charge in the electric resistance of the gage by the strains which the gage receives in the two radial parts at both ends after having got deformation under loads, Eq. (20_1) is also rewritten as follows

$$e \doteqdot \frac{1}{2}A + \Gamma_2\,(B\cos\Sigma\theta + C\sin\Sigma\,\theta) \qquad (20_3)\,,$$

in which

$$\Gamma_2 = -\sin\Delta\theta\ /\ 2\,\Delta\theta\,.$$

In the above discussions, we know clearly that the equations (20_1), (20_2) and (20_3) are all linear equations with variables $[A, B, C]$, that is, strain components $[e_{xx}, e_{yy}, e_{xy}]$. Therefore, if only we find linear strains $[e_1, e_2, e_3]$ of these gages, after having got deformation under loads, which have a common centre and are not in the same position with one another, we have always simultaneous equations consisting of three equations which are nothing but Eqs. (20) for these strains and always find three unknown values $[e_{xx}, e_{yy}, e_{xy}]$ from them as stated previously, and, accordingly, find the stress and strain states at the given point as fully discussed in previous part. After all, we find all the requirements at the given point and the present problems are solved.

Here, as an actual example of circular strain gages stated above, we adopt a set consisting of three circular gages, each of which has angle $[0^\circ \sim 120^\circ]$, $[120^\circ \sim 240^\circ]$, $[240^\circ \sim 360^\circ]$ respectively for the included section between the two radii at both ends of each gage and n curves of the concentric circles in each section stated above and, moreover, take such intervals of the circles as $\delta = 4.1728 \times 10^{-2}$. First putting the centres of these gages on a given point exactly, we paste these ones on the surface of an elastic body perfectly for finding the strain states at the given point. Then measuring the linear strains $[e_1, e_2, e_3]$ of these gages after having got deformations under loads, from the simultaneous equations consisting of Eqs. (20_2) concerned with these strains, we find $[A, B, C]$ and, accordingly, find $[e_{xx}, e_{yy}, e_{xy}]$ as follows:-

$$\left.\begin{aligned} A &= \frac{2}{3}(e_1 + e_2 + e_3) \\ B &= -1.604231\,\{2\,e_2 - (e_3 + e_1)\} \\ C &= 2.778609\,(e_3 - e_1) \end{aligned}\right\} \qquad (20_1),$$

$$\left.\begin{aligned} e_{xx} &= 1.13544876\,(e_1 + e_3) - 1.2708975\,e_2 \\ e_{yy} &= -0.4687821\,(e_1 + e_3) + 1.9375642\,e_2 \\ e_{xy} &= 2.778609\,(e_3 - e_1) \end{aligned}\right\} \qquad (21_2)$$

and from the simultaneous equations consisting of Eqs. (20_3) concerned with these strains, we have

$$\left.\begin{aligned} A &= \frac{2}{3}(e_1 + e_2 + e_3) \\ B &= -1.61232\,\{2\,e_2 - (e_3 + e_1)\} \\ C &= 2.792547\,(e_3 - e_1) \end{aligned}\right\} \qquad (21_1'),$$

$$\left.\begin{aligned} e_{xx} &= 1.139468\,(e_1 + e_3) - 1.278935\,e_2 \\ e_{yy} &= -0.472801\,(e_1 + e_3) + 1.945602\,e_2 \\ e_{xy} &= 2.792547\,(e_3 - e_1) \end{aligned}\right\} \qquad (21_2').$$

These descriptions tell us that both results shown by Eqs. (21_2) and $(21_2')$ are regarded to be equal within allowable errors for using these strain gages. Therefore, for such intervals between the concentric circles as we take in the gages, we see that the variation of electric resistance caused by the strains of the gages in the radial directions at both ends is negligible small.

In conclusion, if only we find the linear strains $[e_1,\ e_2,\ e_3]$ of these gages stated above after having got deformations under loads, that is, if only we use the strain gages, we always find strain components $[e_{xx},\ e_{yy},\ e_{xy}]$ from Eqs. (21) and, accordingly, find stress states at the given point, that is, the centre of the gages and the problems are solved.

[iv] On an experimental result measured by using the foil strain gage with circular type.

For the purpose of finding degree of error contained in the experimental result measured by using the foil strain gage with circular type, we find experimentally normal stress component $[\widehat{xx}]$ acting on the x-axis direction at a given point on a strip under tension by using the gage and, on the other hand, find also the stress component at the given point from well-known stress formulae and then compare these two results. First, the formula used for calculating the stress component is as follows* : –

$$\begin{aligned}\frac{\widehat{xx}}{T} = 1.00492 + \frac{2.02}{\rho^2}\times 10^{-2} + 2 \Bigg| & \left\{\frac{1.275}{\rho^4}\times 10^{-3} + 1.62\times 10^{-2} - 2.61\times 10^{-2}\rho^2\right\} \\ & + \left\{\frac{3.05}{\rho^6}\times 10^{-7} - \frac{3.06}{\rho^4}\times 10^{-6} + 2.46\times 10^{-2}\rho^2 \right. \\ & \left. \qquad - 3.435\times 10^{-2}\rho^4\right\} \\ & + \left\{\frac{4.473}{\rho^8}\times 10^{-10} - \frac{6.41}{\rho^6}\times 10^{-9} + 1.62\times 10^{-2}\rho^4 \right. \\ & \left. \qquad - 2.436\times 10^{-2}\times\rho^6\right\} \\ & + \left\{\frac{4.384}{\rho^{10}}\times 10^{-13} - \frac{7.161}{\rho^8}\times 10^{-12} + 7.672\times \right. \\ & \left. \qquad 10^{-3}\rho^6 - 1.305\times 10^{-2}\rho^8\right\} \Bigg|\end{aligned}$$

in which $\rho = y/b$, $T = P/bh$, $P =$ Tension .

Next, we show dimensions of the test pieces and the foil strain gages employed in our experiments.

Test pieces

breadth (b)	length of parallel part in x-axis direction	thickness (h)	radius of hole (r)	$\lambda = r/b$
20 mm	140 mm	4 mm	4 mm	0.2

Young's Modulus E = 20400 kg/mm^2
material of strip = mild steel

* R. C. J. Howland: Phil. Trans. Roy. Soci. Lond. Series A. Vol. 229, 1930, p. 49∼86.

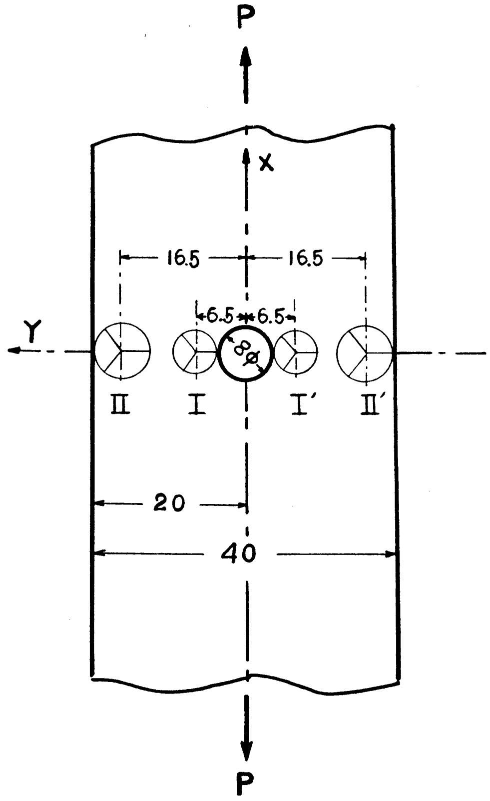
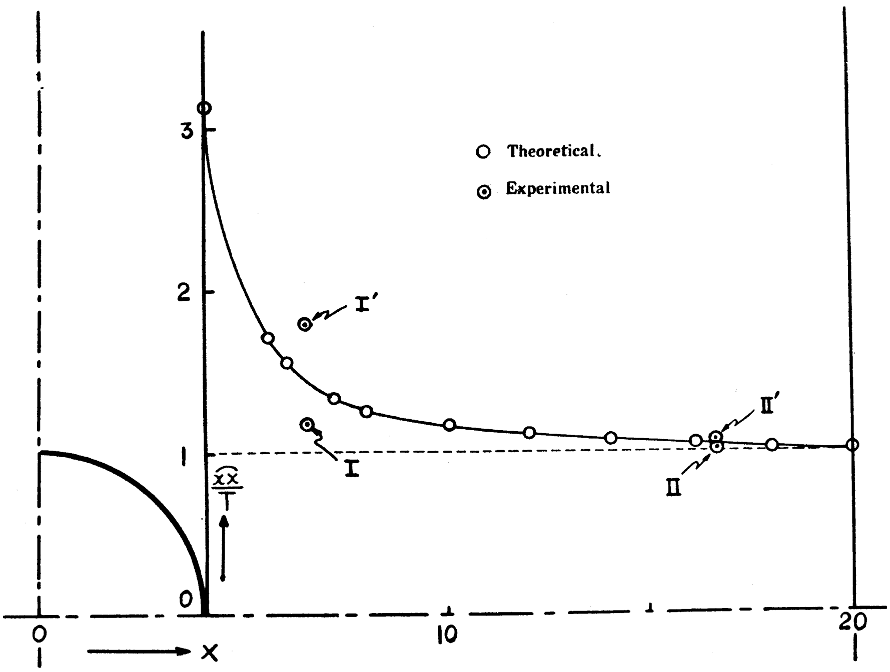

P
X
16.5
16.5
6.5
6.5
8ϕ
Y
II
I
I′
II′
20
40
P

Fig. 5.

Fig. 6.


3
2
1
0
O Theoretical.
⊙ Experimental
I′
I
II′
II
$\frac{\widehat{xx}}{T}$
0
10
20
X

Foil strain gages

diameter of gage	electric resistance
5 mm∼7 mm	60 Ω

In our experiments, we paste these gages at the points 6.5 mm and 16.5 mm distant from the origin on the y-axis in such manners as we take always the basic lines of the gages in the same direction on the y-axis.

It means the fact that the two gages being at the symmetrical points for x-axis are pasted in inverse directions for each other. Now, needless to say, since we make use of these gages, we recognize the assumption that the strain states in the basic domain where the strain gage is pasted are the same as those at the given point, that is, the centre of the gage. Therefore, it is to be desired that its basic area is as small as we can and, if only we use the gage under the condition satisfying the assumption stated above, we find experimentally accurate stress states at the given point independently of the direction of the basic line of the pasted gage. But, when we use the gage in such domains, where-on the inclinations of strains states are sharp, the assumption descrived above is not satisfied truthfully and some error is generally contained in the result found experimentally by using the gage and, further-more, the result is always depended on the direction of the gage pasted at the given point. In our experimental results shown by Fig. [6], these facts are clearly recognized. Namely, the experimental results found by using the two gages pasted at the points 16.5 mm distant from the origin in the both sides of the x-axis nearly coincide with each other. But there are some difference between these results found by using the two gages pasted at the points 6.5 mm distant from the origin in the both sides of the x-axis. This difference is caused by the facts that the gage pasted on the right side domain is largely affected by the large stresses occuring in the neighbor-hood of the hole boundary and, on the contrary, the gage pasted on the left side domain is largely affected by the small stresses occuring in the domain on the far side from the origin. In these cases, it is clear that mean value of these both results is more accurate than both of them. From the above descriptions in such domains, where-on the inclinations of strain states are sharp, it is necessary for finding the result with high accuracy that we take mean value of both results found by using the two gages pasted in such directions as the basic lines of the gages are in reverse for each other at the given point.

Last, it is to be regretted that the above results are unable to be compared with ones found by using the usual Rosette type gage. Because its basic area is too large to the dimension of the breadth of the test piece and, even if we find stress states at a given point on the strip by using the gage, the results are too meaning-less for the true values.

[v] Advantages claimed for the gages.

[a] As mentioned in the previous part, since we make use of the gage, we re-

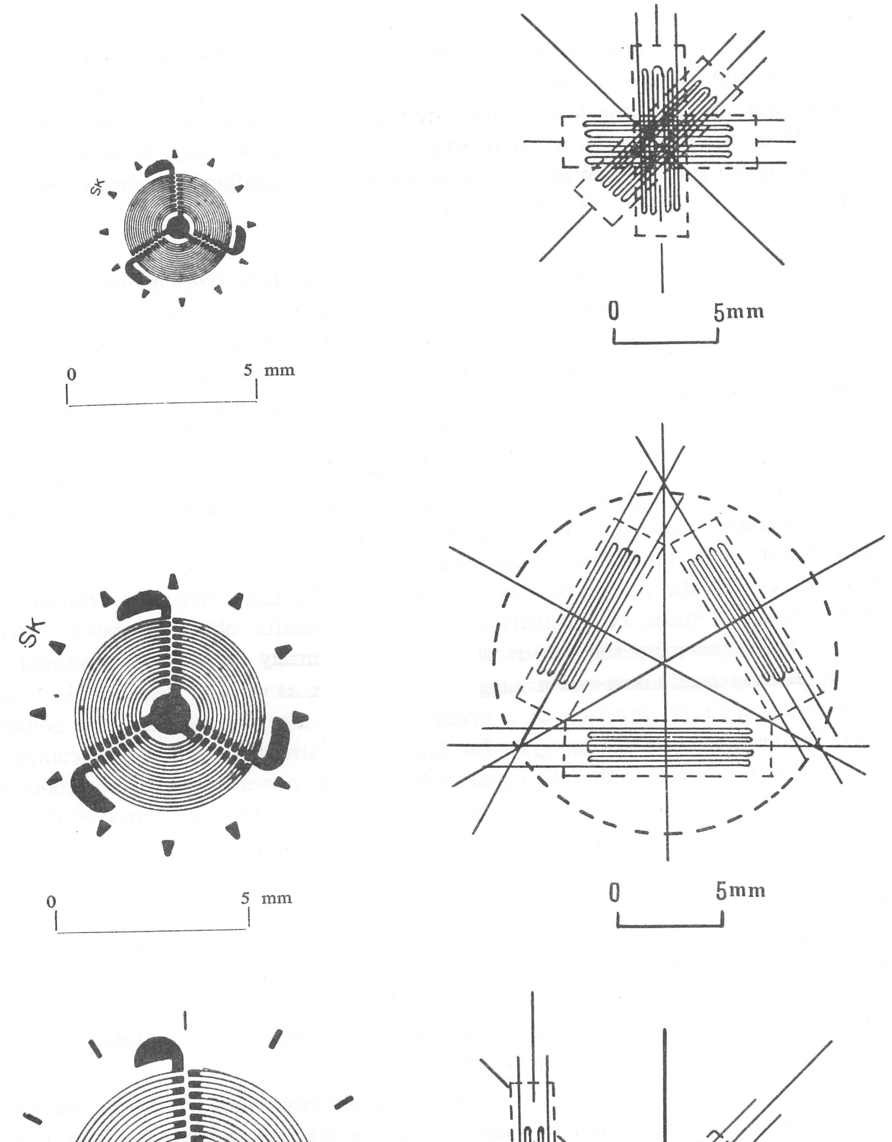


Fig. 7

cognize the assumption that the strain states on the basic domain, where the strain gage is pasted, are the same as those at the given point, that is, the centre of the gage. Therefore, it is to be desired that its basic area is as small as we can. Now, the diameter of the smallest of the circular gages in use for our experiments is 3 mm. Then, its basic area is fairly smaller than that of usual Rosette gage and the gages come fully in useful for the experiments of finding accurately the strain states being necessary to the strain analysis.

" note "

For the purpose of introducing the gages and their sizes in use for our experiments, we show photographs of these gages and usual Rosette ones in Fig. 7 with the proviso that those are quintuple enlarged photos and these three and half times as large as the sizes of the originals on our negative for expressing clearly those patterns :— namely, diameters of these curvilinear gages are 3 mm, 5 mm and 7 mm in sizes of the originals.

[b] Measuring the linear strains in the peripheral directions on the total circumference and ones together in the radial directions of the circle, the centre of which coincides with that of the gage, on the small ring domain surrounding a given point, and taking all these linear strains into considerations, we find the stress states at the point, that is, the centre of the gage, from the relations between stress and strain components in use of the results obtained above. Therefore, we have generally the results with higher accuracy than those obtained by utilizing usual Rosette-type strain gages. Furthermore, as we are able to take linear strain in an arbitary direction at a given point on the ring domain into consideration in our experimental analysis for finding the stress states at the centre of the gage on the basis of the theory described in the present report, it is more desirable for raising the accuracy to design the gages with such types of the patterns as we take the same degree of quantities of the linear strains in the radial directions of the circle as those in the peripheral directions in our stress calculation.

[c] In the circular gage consisting of 3 elementary gages, the starting points of these gages, at which these ones are joined to the lead wires respectively, may be used in putting together. It suits us pretty better to decrease two joining points in the gage for saving labor, because we generally need to find stress states at many points on the surface of a structure.

[d] For finding the strain states on the ring domain more truthfully, we think that it is appropriate to design in such ways as we take the curves, such as volute one etc, into the patterns of the strain gages, because some quantities of both the linear strains in radial and tangentical directions at a given point on the circle stated in the part [b] are generally included in the linear strain in the curvilinear direction at the point.

Last, I express my gratitude to Mr. S. Sumi for his valuable advice on designing the pattern of the gage and his helps in our experiments.

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