ON THE THEORY OF HOMOGENEOUS AXISYMMETRIC TURBULENCE (II)

OHJI, Michio
Research Institute for Applied Mechanics, Kyushu University

https://doi.org/10.5109/7162484

出版情報：Reports of Research Institute for Applied Mechanics. 6 (24), pp.153-171, 1958. 九州大学応用力学研究所
バージョン：
権利関係：

Reports of Research Institute for Applied Mechanics
Vol. VI, No. 24, 1958

ON THE THEORY OF HOMOGENEOUS AXISYMMETRIC TURBULENCE (II)

By Michio OHJI

This paper shows explicit relations between the defining scalars of the second-order axisymmetric $\boldsymbol{r}$-tensor $R_{ij}(\boldsymbol{s}; \boldsymbol{r})$ and those of the $\boldsymbol{\kappa}$-tensor $\Phi_{ij}(\boldsymbol{s}; \boldsymbol{\kappa})$ in homogeneous incompressible turbulence. Since R_{ij} and Φ_{ij} are the three-dimensional scalar Fourier transforms each other, we are in general lead to some integral relations involving the Bessel functions of the first kind, J_0 and J_1 (or J_2). It is also seen that in a certain special case they degenerate into expressions with elementary functions, the results then being largely simplified. Further, the Chandrasekhar's representation of $R_{ij}(\boldsymbol{s}; \boldsymbol{r})$ is referred in these connexions. And the last section contains an illustrative example in which the double velocity correlation coefficients appropriate for the spectrum of turbulence behind fine wire gauze are computed after our present formulas.

1. Introduction. The preceding paper of the author [3] (referred as I, hereafter)[1] set out the way of representing a field of homogeneous and axisymmetric turbulence especially in $\boldsymbol{\kappa}$-space, but there the explicit relations between the defining scalars of the double velocity correlation tensor $R_{ij}(\boldsymbol{s}; \boldsymbol{r})$ and those of the energy spectrum tensor $\Phi_{ij}(\boldsymbol{s}; \boldsymbol{\kappa})$ have been left untouched. For isotropic turbulence, on the other hand, we know that the Fourier transform relation between two isotropic tensors $R_{ij}(\boldsymbol{r})$ and $\Phi_{ij}(\boldsymbol{\kappa})$ yields the scalar relations:

$$\left.\begin{aligned} R(r) &= 2\int_0^\infty \mathcal{E}(\kappa)\frac{\sin\kappa r}{\kappa r}\,\mathrm{d}\kappa\,, \\ \mathcal{E}(\kappa) &= \frac{1}{\pi}\int_0^\infty R(r)\,\kappa r\sin\kappa r\,\mathrm{d}r\,, \end{aligned}\right\} \tag{1.1}$$

where

$$R(r) = \overline{u^2}(f+2g)\,, \quad \text{and} \quad \int_0^\infty \mathcal{E}(\kappa)\,\mathrm{d}\kappa = \frac{\overline{u^2}}{2}\,, \tag{1.2}$$

$f(r)$ and $g(r)$ being the Kármán's longitudinal and lateral velocity correlation coefficients, respectively.

Our present purpose is thus to supplement the foregoing analysis by finding the scalar relations analogous to (1.1) on the basis of

$$R_{ij}(\boldsymbol{s}; \boldsymbol{r}) = \int \Phi_i\,(\boldsymbol{s}; \boldsymbol{\kappa})\,\mathrm{e}^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{\kappa}\,, \tag{1.3}$$

[1] Unless particularly noticed, the notations and terminologies in I will be followed throughout the present paper as well.

and its inversion. In mathematics, no special difficulties will arise to do so; referring to some familiar outcomes from the theory of Bessel functions, we can proceed straightforwardly for the most part. Nevertheless, the results obtained will be of occasional use, some of which are found in the later sections.

2. General expressions of R's in terms of Φ's. To begin with, let us introduce an abbreviation:

$$[\ldots]^{\mathfrak{F}} = \int(\ldots)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}d\boldsymbol{\kappa}, \tag{2.1}$$

where $(\ldots)$ stands for any quantity, scalar or tensor, considered as a function of $\boldsymbol{\kappa}$. Then (1.3) can be written as

$$R_{ij}(\boldsymbol{s};\boldsymbol{r}) = [\Phi_{ij}(\boldsymbol{s};\boldsymbol{\kappa})]^{\mathfrak{F}}$$

or, in view of I; (3.5) and I; (4.5), as

$$\begin{aligned} & R_1 r_i r_j + R_2\delta_{ij} + R_3 s_i s_j + R_4(s_i r_j + r_i s_j) \\ & = [\Phi_1\kappa_i\kappa_j + \Phi_2\delta_{ij} + \Phi_3 s_i s_j + \Phi_4(s_i\kappa_j + \kappa_i s_j)]^{\mathfrak{F}}. \end{aligned} \tag{2.2}$$

We shall now derive four separate scalar relations from this. They are most readily obtained by making R_{ii}, $R_{ij}s_i s_j$, $R_{ij}r_i r_j$ and $R_{ij}h_i h_j$, successively, where $\boldsymbol{h}$ is a unit vector which lies in the plane containing the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ and is perpendicular to $\boldsymbol{s}$. Namely, putting

$$\left.\begin{aligned} &\boldsymbol{r}\cdot\boldsymbol{r} = r^2, \quad \boldsymbol{r}\cdot\boldsymbol{s} = rm, \quad \boldsymbol{r}\cdot\boldsymbol{h} = rm', \\ &\boldsymbol{\kappa}\cdot\boldsymbol{\kappa} = \kappa^2, \quad \boldsymbol{\kappa}\cdot\boldsymbol{s} = \kappa\mu, \quad \boldsymbol{\kappa}\cdot\boldsymbol{h} = \kappa\varpi, \end{aligned}\right\} \tag{2.3}$$

and

$$\boldsymbol{\kappa}\cdot\boldsymbol{r} = \kappa r\sigma,$$

there come the equations:

$$\left.\begin{aligned} & r^2R_1 + 3R_2 + R_3 + 2rmR_4 \\ & \quad = [\kappa^2\Phi_1 + 3\Phi_2 + \Phi_3 + 2\kappa\mu\Phi_4]^{\mathfrak{F}}, \\ & r^2m^2R_1 + R_2 + R_3 + 2rmR_4 \\ & \quad = [\kappa^2\mu^2\Phi_1 + \Phi_2 + \Phi_3 + 2\kappa\mu\Phi_4]^{\mathfrak{F}}, \\ & r^2R_1 + R_2 + m^2R_3 + 2rmR_4 \\ & \quad = [\kappa^2\sigma^2\Phi_1 + \Phi_2 + m^2\Phi_3 + 2m\kappa\sigma\Phi_4]^{\mathfrak{F}}, \\ & r^2m'^2R_1 + R_2 \\ & \quad = [\kappa^2\varpi^2\Phi_1 + \Phi_2]^{\mathfrak{F}}, \end{aligned}\right\} \tag{2.4}$$

in turn. Note that from the definition of $\boldsymbol{h}$, we have

$$\boldsymbol{s}\cdot\boldsymbol{h} = 0 \quad \text{and} \quad m^2 + m'^2 = 1. \tag{2.5}$$

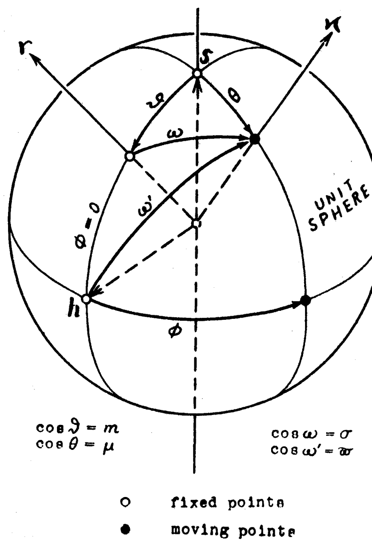

Fig. 1. Coordinates diagram.

In order to evaluate the quantities on

the right-hand sides of (2.4), it is convenient to use the spherical polar coordinates shown in Fig. 1.[2] Let denote

$$\boldsymbol{\kappa}=(\kappa,\theta,\phi)\,,\qquad \boldsymbol{r}=(r,\vartheta,0)\,,\qquad \boldsymbol{h}=(1,\pi/2,0)\,,$$

in these coordinates; then together with the obvious relations:

$$\left.\begin{aligned} &\mu=\cos\theta\,,\qquad m=\cos\vartheta\,,\qquad m'=\sin\vartheta=\sqrt{1-m^2}\,,\\ &\sigma=m\cos\theta+m'\sin\theta\cos\phi\,,\\ &\varpi=\sin\theta\cos\phi\,,\end{aligned}\right\}\tag{2.6}$$

we see

$$\begin{aligned}[\text{scalar}]^{\mathfrak{F}}&=\int_0^\infty\int_0^\pi\int_0^{2\pi}(\text{scalar})\,\mathrm{e}^{\iota\kappa r\sigma}\kappa^2\sin\theta\,\mathrm{d}\kappa\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &=\int_0^\infty\kappa^2\mathrm{d}\kappa\int_0^\pi\mathrm{e}^{\iota\kappa rm\cos\theta}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}(\text{scalar})\,\mathrm{e}^{\iota\kappa rm'\sin\theta\cos\phi}\mathrm{d}\phi\,,\end{aligned}\tag{2.7}$$

so long as the definite integrals in question are assumed to exist.[3] The way of finding the desired expressions has been hereby prepared. First solve the equation (2.4) with respect to R_1, R_2, R_3 and R_4. Next, rewrite the solutions after (2.7), and refer to the theorem (Hansen-Bessel's formula, see e.g. [4] 2.2):

$$\int_0^{2\pi}\mathrm{e}^{\iota z\cos\phi}\cos n\phi\,\mathrm{d}\phi=2\pi\,\iota^n J_n(z)\,,\tag{2.8}$$

for arbitrary z, where n is a positive integer and $J_n(z)$ is the Bessel function of the first kind of order n.

The final answer can be summarized as follows:

$$\left.\begin{aligned} r^2R_1&=-\frac{C_1}{1-m^2}\,,\\ R_2&=A_2+D_1\,,\\ R_3&=A_1+A_3+2A_4-(B_1+D_1)-\frac{m^2C_1-2m\sqrt{1-m^2}(a_1+b_4)}{1-m^2}\,,\\ rmR_4&=\frac{m^2C_1-m\sqrt{1-m^2}(a_1+b_4)}{1-m^2}\,,\end{aligned}\right\}\tag{2.9}$$

provided that

---

2) Strictly speaking, the vectors $\boldsymbol{r}$ and $\boldsymbol{\kappa}$ cannot be drawn in the same space. In Fig. 1, therefore, we had better suppose that $\boldsymbol{r}$ and $\boldsymbol{\kappa}$ are both reduced to dimensionless vectors by introducing some common length factor at our disposal.

3) In the following analysis we shall always assume this premise whose justification will be done aposteriori in another future report.

$$\left.\begin{aligned}
A_1 &= 4\pi \int_0^\infty \kappa^4 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_1 \times J_0(\kappa\beta \sin\theta) \cos(\kappa\alpha \cos\theta) \sin\theta \, \mathrm{d}\theta = [\kappa^2 \Phi_1]^{\mathfrak{F}}, \\
A_2 &= 4\pi \int_0^\infty \kappa^2 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_2 \times J_0(\kappa\beta \sin\theta) \cos(\kappa\alpha \cos\theta) \sin\theta \, \mathrm{d}\theta = [\Phi_2]^{\mathfrak{F}}, \\
A_3 &= 4\pi \int_0^\infty \kappa^2 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_3 \times J_0(\kappa\beta \sin\theta) \cos(\kappa\alpha \cos\theta) \sin\theta \, \mathrm{d}\theta = [\Phi_3]^{\mathfrak{F}}, \\
A_4 &= 4\pi \int_0^\infty \kappa^3 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_4 \cos\theta \times J_0(\kappa\beta \sin\theta) \cos(\kappa\alpha \cos\theta) \sin\theta \, \mathrm{d}\theta = [\kappa\Phi_4 \cos\theta]^{\mathfrak{F}}, \\
B_1 &= 4\pi \int_0^\infty \kappa^4 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_1 \times J_0(\kappa\beta \sin\theta) \cos(\kappa\alpha \cos\theta) \sin^3\theta \, \mathrm{d}\theta, \\
C_1 &= 4\pi \int_0^\infty \kappa^4 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_1 \times J_2(\kappa\beta \sin\theta) \cos(\kappa\alpha \cos\theta) \sin^3\theta \, \mathrm{d}\theta, \\
D_1 &= 4\pi \int_0^\infty \kappa^4 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_1 \times \frac{J_1(\kappa\beta \sin\theta)}{\kappa\beta} \cos(\kappa\alpha \cos\theta) \sin^2\theta \, \mathrm{d}\theta \ \left(= \frac{B_1 + C_1}{2}\right), \\
a_1 &= 4\pi \int_0^\infty \kappa^4 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_1 \times J_1(\kappa\beta \sin\theta) \sin(\kappa\alpha \cos\theta) \cos\theta \sin^2\theta \, \mathrm{d}\theta, \\
b_4 &= 4\pi \int_0^\infty \kappa^3 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_4 \times J_1(\kappa\beta \sin\theta) \sin(\kappa\alpha \cos\theta) \sin^2\theta \, \mathrm{d}\theta,
\end{aligned}\right\} \tag{2.10}$$

where

$$\alpha = rm, \quad \beta = r\sqrt{1-m^2},$$

and use is made of the fact that Φ_1, Φ_2, Φ_3 are even, while Φ_4 is an odd, functions of $\cos\theta$ (cf. p. 170, Addendum).

These exhaust the subject in all generality, but, in particular, the conditions of solenoidality I; (4.6) or

$$\kappa^2\Phi_1 + \Phi_2 + \kappa\Phi_4 \cos\theta = 0, \quad \text{and} \quad \Phi_3 \cos\theta + \kappa\Phi_4 = 0, \tag{2.11}$$

should be taken into account here. Thus, it is found immediately that

$$A_1 + A_2 + A_4 = 0, \tag{2.12}$$

as well as

$$\left.\begin{aligned} [\Phi_3 \cos^2\theta]^{\mathfrak{F}} + A_4 = 0, \quad \text{or} \quad A_3 + A_4 = B_3, \\ a_3 + b_4 = 0 \end{aligned}\right\}, \tag{2.13}$$

and [4]

4) B_3 and a_3 are analogues of B_1 and a_1, where $\kappa^2\Phi_1$ is replaced by Φ_3 in their integrands: viz.

$$B_3 = 4\pi \int_0^\infty \kappa^2 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_3 \times J_0(\kappa\beta \sin\theta) \cos(\kappa\alpha \cos\theta) \sin^3\theta \, \mathrm{d}\theta,$$

$$a_3 = 4\pi \int_0^\infty \kappa^2 \mathrm{d}\kappa \int_0^{\pi/2} \Phi_3 \times J_1(\kappa\beta \sin\theta) \sin(\kappa\alpha \cos\theta) \cos\theta \sin^2\theta \, \mathrm{d}\theta.$$

by means of which we are able to eliminate the contributions from Φ_2 and Φ_4 in (2.9).

Of course, the same result will be attained by direct termwise integration of the tensor form (2.2), but the present derivation is much less confusing, for we have now only to do with the integrations of the scalar quantities (cf. I; Appendix).

3. Special case where Φ_1 and Φ_3 are independent of $\mu = \cos\theta$. In the case under the title, we can carry out the integrations with respect to θ at once, and the foregoing expressions are considerably reduced. Namely, in virtue of Sonine's second finite integral as it is called,[5] we have

$$\int_0^{\pi/2} J_\nu(\kappa\beta\sin\theta)\cos(\kappa\alpha\cos\theta)\sin^{\nu+1}\theta\,\mathrm{d}\theta = (1-m^2)^{\nu/2}\psi_\nu(\kappa r)\,, \tag{3.1}$$

$$\int_0^{\pi/2} J_\nu(\kappa\beta\sin\theta)\sin(\kappa\alpha\cos\theta)\cos\theta\sin^{\nu+1}\theta\,\mathrm{d}\theta = m(1-m^2)^{\nu/2}\psi_{\nu+1}(\kappa r)\,, \tag{3.2}$$

where $\Re(\nu) > -1$ and

$$\psi_\nu(\kappa r) = \sqrt{\frac{\pi}{2\kappa r}}J_{\nu+\frac{1}{2}}(\kappa r)\,. \tag{3.3}$$

If ν is an integer, $\psi_\nu(z)$ is sometimes called the spherical Bessel function of order ν,[6] which can be expressed as a certain combination of elementary functions $1/z$, $\sin z$ and $\cos z$. Note that

$$z[\psi_{\nu-1}(z) + \psi_{\nu+1}(z)] = (2\nu+1)\psi_\nu(z)\,, \tag{3.4}$$

$$\frac{\mathrm{d}\psi_\nu(z)}{\mathrm{d}z} + \frac{\nu+1}{z}\psi_\nu(z) = \psi_{\nu-1}(z)\,, \tag{3.5}$$

and specifically

$$\left.\begin{aligned}
\psi_{-1}(z) &= \frac{\cos z}{z}\,,\\
\psi_0(z) &= \frac{\sin z}{z}\,,\\
\psi_1(z) &= \frac{1}{z}\left(\frac{\sin z}{z} - \cos z\right),\\
\psi_2(z) &= \frac{1}{z}\left\{\frac{3}{z}\left(\frac{\sin z}{z} - \cos z\right) - \sin z\right\},\\
&\dots
\end{aligned}\right\} \tag{3.6}$$

---

5) Sonine's second finite integral is in general:
$$\int_0^{\pi/2} J_\mu(z\cos\theta)\,J_\nu(Z\sin\theta)\cos^{\mu+1}\theta\sin^{\nu+1}\theta\,\mathrm{d}\theta = \frac{z^\mu Z^\nu}{(z^2+Z^2)^{(\mu+\nu+1)/2}}J_{\mu+\nu+1}(\sqrt{z^2+Z^2})$$
for $\Re(\mu) > -1$, $\Re(\nu) > -1$ (see [4] p. 376). In our case $z = \kappa\alpha = \kappa rm$, $Z = \kappa\beta = \kappa r\sqrt{1-m^2}$ and so $z^2 + Z^2 = \kappa^2 r^2$. Then, (3.1) and (8.2) are obtained by putting $-1/2$ and $+1/2$ in turn because of the relations $J_{-1/2}(z) = \sqrt{2/\pi z}\cos z$ and $J_{1/2}(z) = \sqrt{2/\pi z}\sin z$.

6) G. N. Watson avoids explicit use of the term 'spherical Bessel function' and quotes a number of related functions introduced by various authors. The present definition of ψ_ν comes from A. Sommerfeld and his school. [4] **3.41.**

Assuming now Φ_1 and Φ_3 are independent of μ (or θ), and making use of these theorems in (2. 10) with reference to (2. 12) and (2. 13), we find the expressions (2. 9) reduce to:

$$\left.\begin{aligned} R_1(r) &= -\frac{G_1(r)}{r^2}, \\ R_2(r, m) &= \frac{1}{3}\left[-2F_1(r) + G_1(r) + F_3(r) + (1-3m^2)G_3(r)\right], \\ R_3(r) &= \frac{1}{3}\left[F_3(r) - 2G_3(r)\right], \\ R_4(r, m) &= \frac{mG_3(r)}{r}, \end{aligned}\right\} \quad (3.7)$$

where we put

$$\left.\begin{aligned} 4\pi\int_0^\infty \kappa^4 \Phi_1(\kappa)\psi_0(\kappa r)\,d\kappa = F_1(r), \quad 4\pi\int_0^\infty \kappa^2 \Phi_3(\kappa)\psi_0(\kappa r)\,d\kappa = F_3(r), \\ 4\pi\int_0^\infty \kappa^4 \Phi_1(\kappa)\psi_2(\kappa r)\,d\kappa = G_1(r), \quad 4\pi\int_0^\infty \kappa^2 \Phi_3(\kappa)\psi_2(\kappa r)\,d\kappa = G_3(r). \end{aligned}\right\} \quad (3.8)$$

It may be worth-while to note here that owing to (3. 4) and (3. 5) there exist two intrinsic relations

$$G_1 + \frac{r}{3}\frac{\partial}{\partial r}(F_1 + G_1) = 0 \quad \text{and} \quad G_3 + \frac{r}{3}\frac{\partial}{\partial r}(F_3 + G_3) = 0, \quad (3.9)$$

between these four scalar functions.

Further, a little inspection of (3. 7) tells us that R_1 and R_3 are independent of m and that R_3 and R_4 are determined by Φ_3 alone. Consequently, when $\Phi_3 \equiv 0$, then $R_3 = R_4 = 0$ with vanishing F_3 and G_3, and we have

$$R_1(r) = -\frac{G_1(r)}{r^2}, \quad R_2(r) = -\frac{2}{3}F_1(r) + \frac{1}{3}G_1(r), \quad (3.10)$$

which are nothing but the isotropic relations. In fact, if we write

$$-\frac{2}{3}F_1(r) - \frac{2}{3}G_1(r) = u^2 f(r), \quad -\frac{2}{3}F_1(r) + \frac{1}{3}G_1(r) = \overline{u^2}g(r), \quad (3.11)$$

and

$$-4\pi\kappa^4\Phi_1(\kappa) = \mathcal{E}(\kappa), \quad (3.12)$$

respectively, we readily get the first of (1. 2) from (3. 8) and (3. 10), whereas (3. 9) furnishes the well-known relation

$$g = f + \frac{1}{2}r\frac{\partial f}{\partial r}. \quad (3.13)$$

4. Inverse relations. We shall now invert the problem, seeking the expressions of Φ's in terms of R's. Again with the aid of another abbreviation:

$$[\ldots]^{-\mathfrak{F}} = \int (\ldots) e^{-\iota \boldsymbol{\kappa} \cdot \boldsymbol{r}} d\boldsymbol{r}, \tag{4.1}$$

the relation inverse to (2.2) is

$$\begin{aligned} &8\pi^3 [\Phi_1 \kappa_i \kappa_j + \Phi_2 \delta_{ij} + \Phi_3 s_i s_j + \Phi_4 (s_i \kappa_j + \kappa_i s_j)] \\ &\quad = [R_1 r_i r_j + R_2 \delta_{ij} + R_3 s_i s_j + R_4 (s_i r_j + r_i s_j)]^{-\mathfrak{F}}, \end{aligned} \tag{4.2}$$

but when (2.6) is referred we have

$$e^{-\iota \boldsymbol{\kappa} \cdot \boldsymbol{r}} = e^{-\iota \kappa r \mu \cos \vartheta} \times e^{-\iota \kappa r \mu' \sin \vartheta \cos \phi},$$

where $\mu' = \sin\theta = \sqrt{1-\mu^2}$, and hence

$$[\text{scalar}]^{-\mathfrak{F}} = \int_0^\infty r^2 dr \int_0^\pi e^{-\iota \kappa r \mu \cos \vartheta} \sin \vartheta \, d\vartheta \int_0^{2\pi} (\text{scalar}) e^{-\iota \kappa r \mu' \sin \vartheta \cos \phi} d\phi, \tag{4.3}$$

in contrast with (2.7). It is, therefore, evident that the analysis in section 2 is still valid here, so long as the appropriate exchanges are made between integration variables and fixed variables. Thus, without repeating the similar manipulations in detail, we find that the general expressions are:

$$\left.\begin{aligned} 8\pi^3 (\kappa^2 \Phi_1) &= -\frac{\tilde{C}_1}{1-\mu^2}, \\ 8\pi^3 \Phi_2 &= A_2 + \tilde{D}_1, \\ 8\pi^3 \Phi_3 &= \tilde{A}_1 + \tilde{A}_3 + 2\tilde{A}_4 - (\tilde{B}_1 + \tilde{D}_1) - \frac{\mu^2 \tilde{C}_1 - 2\mu\sqrt{1-\mu^2}(\tilde{a}_1 + \tilde{b}_4)}{1-\mu^2}, \\ 8\pi^3 (\kappa \mu \Phi_4) &= \frac{\mu^2 \tilde{C}_1 - \mu\sqrt{1-\mu^2}(\tilde{a}_1 + \tilde{b}_4)}{1-\mu^2}, \end{aligned}\right\} \tag{4.4}$$

(cf. (2.9)), on the right-hand sides of which we have only to reinterprete each of the defining equations (2.10) in such a manner as

$$\tilde{A}_1 = 4\pi \int_0^\infty r^4 dr \int_0^{\pi/2} R_1 \times J_0(r\tilde{\beta} \sin\vartheta) \cos(r\tilde{\alpha} \cos\vartheta) \sin\vartheta \, d\vartheta, \tag{4.5}$$

and the like, where

$$\tilde{\alpha} = \kappa\mu, \quad \tilde{\beta} = \kappa\mu' = \kappa\sqrt{1-\mu^2},$$

are to be understood. (From comparison between (4.3) and (2.7) we are suggested that $\alpha = rm$ and $\beta = rm'$ must be replaced by $-\kappa\mu$ and $-\kappa\mu'$ respectively, not by $\kappa\mu$ and $\kappa\mu'$. It is true, but since all members of (2.10) are invariant on the simultaneous changes of the signs of α and β, we can assume $\tilde{\alpha}$ and $\tilde{\beta}$ as above.)

In contrast with the foregoing case, however, the accessory relations similar to (2.12) and (2.13) are no longer valid here, because the solenoidal conditions for $R_{ij}(\boldsymbol{s}; \boldsymbol{r})$ make their appearances quite differently (see I; (3.6)). We had rather substitute from (4.4) into (2.11) to get the correct relations between $\tilde{A}_1$, $\tilde{A}_2$, etc. In this way, we have

$$\tilde{A}_2+\tilde{D}_1-\tilde{C}_1=(\tilde{A}_1+2\tilde{A}_2+\tilde{A}_3+2\tilde{A}_4-\tilde{D}_1)\mu^2=\frac{\mu}{\sqrt{1-\mu^2}}(\tilde{a}_1+\tilde{b}_4), \quad (4.6)$$

and it should be noted that the first and the second equations of (4.4) (and hence $\tilde{C}_1$, $\tilde{D}_1$, $\tilde{A}_2$) suffice to determine $\Phi_{ij}(s;\boldsymbol{\kappa})$ completely.

Now let us proceed to the inverse problem of section **3**, considering the special case where R_1 and R_3 are independent of $m=\cos\vartheta$. Though it is also possible to go along *via* (4.4), the following way seems to be the shortest for our present purpose. Namely, in view of linear characters of the Fourier transform operation and of the solenoidal conditions, we can suppose the former circumstances are reversible in the sense that the inversion of (3.8) would give the adequate answer to this problem. But we have, from the Hankel inversion theorem,[7] a lemma saying:

$$\left.\begin{aligned} &\text{if} && \tilde{\varphi}(r)=\int_0^\infty \varphi(\kappa)\psi_\nu(\kappa r)\,\mathrm{d}\kappa, \qquad \Re(\nu)\geqq -1,\\ &\text{then} && \varphi(\kappa)=\frac{2}{\pi}\int_0^\infty \kappa^2 r^2\,\varphi(r)\,\psi_\nu(\kappa r)\,\mathrm{d}r, \end{aligned}\right\} \quad (4.7)$$

provided that $\varphi(\kappa)$ is piecewise continuous in the range $(0,\infty)$ of κ and that

$$\int_0^\infty \left|\frac{\varphi(\kappa)}{\kappa}\right|\mathrm{d}\kappa<\infty \quad (4.8)$$

is ensured, for $\sqrt{2r/\pi}\,\tilde{\varphi}(r)$ and $\varphi(\kappa)/\kappa\sqrt{\kappa}$ are Hankel transform of order $\nu+1/2$. So assuming here for the moment that the condition (4.8) is fulfilled for our functions (cf. footnote 3) on p. 155), an application of (4.7) to (3.8) in combination with (3.7) and (3.9) yields:

$$\left.\begin{aligned} \kappa^2\Phi_1(\kappa)&=-\frac{1}{2\pi^2}\int_0^\infty r^4 R_1(r)\psi_2(\kappa r)\,\mathrm{d}r,\\ \Phi_3(\kappa)&=-\frac{1}{2\pi^2}\int_0^\infty r^2\Big[R_3(r)+\frac{1}{r}\int_0^r R_3(r')\mathrm{d}r'\Big]\psi_2(\kappa r)\,\mathrm{d}r, \end{aligned}\right\} \quad (4.9)$$

from which $\Phi_2(\kappa,\mu)$ and $\Phi_4(\kappa,\mu)$ are found through (2.11). When furthermore $R_3(r)\equiv 0$, these reduce to the second of the isotropic relations (1.2), as is readily seen with reference to $(3.9)\sim(3.12)$.

5. Spectrum functions and Chandrasekhar's correlation functions. According to S. Chandrasekhar [2] (or I; section 3), the defining scalars of $R_{ij}(s;\boldsymbol{r})$ are expressed as:

$$\begin{aligned} R_1&=\Big(\frac{1}{r}\frac{\partial q_1}{\partial r}-\frac{m}{r^2}\frac{\partial q_1}{\partial m}-\frac{1}{r^2}\frac{\partial^2 q_1}{\partial m^2}\Big)+\Big(\frac{1}{r}\frac{\partial q_2}{\partial r}-\frac{m}{r^2}\frac{\partial q_2}{\partial m}\Big),\\ R_2&=-\Big[r\frac{\partial q_1}{\partial r}+2q_1+m\frac{\partial q_1}{\partial m}-(1-m^2)\frac{\partial^2 q_1}{\partial m^2}\Big] \end{aligned}$$

[7] For instance, I. N. Sneddon: *Fourier transforms.* McGraw-Hill (1951) Chap. 2.

$$\left.\begin{aligned} &-\left[r(1-m^2)\frac{\partial q_2}{\partial r}+q_2-m(1-m^2)\frac{\partial q_2}{\partial m}\right], \\ R_3 &= -\frac{\partial^2 q_1}{\partial m^2}+\left(r\frac{\partial q_2}{\partial r}+q_2-m\frac{\partial q_2}{\partial m}\right), \\ R_4 &= \left(\frac{1}{r}\frac{\partial q_1}{\partial m}+\frac{m}{r}\frac{\partial^2 q_1}{\partial m^2}\right)-\left(m\frac{\partial q_2}{\partial r}-\frac{m^2}{r}\frac{\partial q_2}{\partial m}\right), \end{aligned}\right\} \quad (5.1)$$

where q_1 and q_2, which may be called provisionally Chandrasekhar's correlation functions, are the defining scalars of the second-order skew axisymmetric tensor $q_{ij}(\boldsymbol{s}\,;\boldsymbol{r})$. Previously, the present author pointed out that these expressions in $\boldsymbol{r}$-space, representing the decomposition of turbulence into quasi-isotropic and transverse terms, correspond to those in $\boldsymbol{\kappa}$-space obtained by putting

$$\Phi_1 = -(Y_1+Z_1) \quad \text{and} \quad \Phi_3 = -\kappa^2 Y_1 . \qquad (5.2)$$

See I; sections **8** and **9**. Then what are the concrete relations between q_1, q_2 and Y_1, Z_1? The mathematical arguments presented above will be now applied to this question.

In principle, we may substitute from (5.2) into (2.10) and then compare (2.9) with (5.1). Or as a more accessible alternative, we have

$$r^2 R_1+2R_2+R_3+2rmR_4=[\kappa^2(\mu^2+\varpi^2)\Phi_1+2\Phi_2+\Phi_3+2\kappa\mu\Phi_4]^{\mathfrak{F}},$$

on adding the second and the fourth of (2.4), together with

$$r^2 R_1+R_2+m^2R_3+2rmR_4=[\kappa^2\sigma^2\Phi_1+\Phi_2+m^2\Phi_3+2m\kappa\sigma\Phi_4]^{\mathfrak{F}},$$

which is nothing but the third of (2.4). If we use (5.1) on the left-hand sides, while (5.2) on the right-hand sides, of these two expressions, there come a pair of equations:

$$\left.\begin{aligned} &\left(r\frac{\partial q_1}{\partial r}+4q_1+m\frac{\partial q_1}{\partial m}\right)+q_2=-4\pi\int_0^\infty \kappa^4 d\kappa\int_0^{\pi/2} Z_1\times P\sin\theta\, d\theta-[\kappa^4 Y_1]^{\mathfrak{D}}, \\ &2q_1+(1-m^2)q^2=-4\pi\int_0^\infty \kappa^4 d\kappa\int_0^{\pi/2} Z_1\times Q\sin\theta\, d\theta-(1-m^2)[\kappa^4 Y_1]^{\mathfrak{D}}, \end{aligned}\right\} \quad (5.3)$$

where

$$\left.\begin{aligned} P &= J_0(\kappa r\sqrt{1-m^2}\sin\theta)\cos(\kappa rm\cos\theta) \\ &\quad+\frac{J_1(\kappa r\sqrt{1-m^2}\sin\theta)}{\kappa r\sqrt{1-m^2}\sin\theta}\cos(\kappa rm\cos\theta)\sin^2\theta, \\ Q &= (1-m^2)P-(1-2m^2)J_0(\kappa r\sqrt{1-m^2}\sin\theta)\cos(\kappa rm\cos\theta)\sin^2\theta \\ &\quad+2m\sqrt{1-m^2}J_1(\kappa r\sqrt{1-m^2}\sin\theta)\sin(\kappa rm\cos\theta)\sin\theta\cos\theta, \end{aligned}\right\} \quad (5.4)$$

and

$$[\kappa^4 Y_{\mathrm{I}}]^{\mathfrak{D}}$$

$$= 4\pi \int_0^\infty \kappa^4 d\kappa \int_0^{\pi/2} Y_{\mathrm{I}} \times \frac{J_1(\kappa r\sqrt{1-m^2}\sin\theta)}{\kappa r\sqrt{1-m^2}\sin\theta} \cos(\kappa r m\cos\theta)\sin^3\theta\, d\theta\,. \quad (5.5)$$

The equations (5. 3) seem to represent the general relations between q_1, q_2 and Y_{I}, Z_{I} in the simplest context, but still it is a formidable task to solve them with respect to q_1 and q_2. What we can say at this stage is only that the solutions must be written symbolically as

$$q_1 = q_1[Z_{\mathrm{I}}] \qquad \text{and} \qquad q_2 = q_2[Y_{\mathrm{I}}, Z_{\mathrm{I}}] \qquad (5.6)$$

in the formal sense. To prove this, we shall assume

$$\left.\begin{aligned} q_1 &= -2\pi \int_0^\infty \kappa^4 d\kappa \int_0^{\pi/2} Z_{\mathrm{I}} \times Q \sin\theta\, d\theta + q_1'\,, \\ q_2 &= -[\kappa^4 Y_{\mathrm{I}}]^{\mathfrak{D}} + q_2'\,, \end{aligned}\right\} \qquad (5.7)$$

for q_1 and q_2, and insert these into (5. 3); the results are

$$\left.\begin{aligned} &\left(r\frac{\partial q_1'}{\partial r} + 4q_1' + m\frac{\partial q_1'}{\partial m}\right) + q_2' \\ &\quad = -2\pi \int_0^\infty \kappa^4 d\kappa \int_0^{\pi/2} Z_{\mathrm{I}}\left(2P - r\frac{\partial Q}{\partial r} - 4Q - m\frac{\partial Q}{\partial m}\right)\sin\theta\, d\theta\,, \\ &\text{and} \quad 2q_1' + (1-m^2)q_2' = 0\,, \end{aligned}\right\} \qquad (5.8)$$

showing that both q_1' and q_2' are related to Z_{I} alone.

In view of (5. 6) thus proved, which in turn suggests us certain inverse relations

$$Z_{\mathrm{I}} = Z_{\mathrm{I}}[q_1] \qquad \text{and} \qquad Y_{\mathrm{I}} = Y_{\mathrm{I}}[q_1, q_2]\,, \qquad (5.9)$$

we arrive at a conclusion like this: that is, the decomposition represented by (5. 1) and that represented by (5. 2) are in general not entirely equivalent, but we have to suppose the quasi-isotropic term in $\boldsymbol{\kappa}$-space (i.e. Z_{I}) contributes to the transverse term in $\boldsymbol{r}$-space (i.e. q_2) by the amount of q_2' and conversely the quasi-isotropic term in $\boldsymbol{r}$-space (i.e. q_1) has some components of the transverse term in $\boldsymbol{\kappa}$-space (i.e. Y_{I}). As a consequence, the words 'isolated quasi-isotropic turbulence' *say* would be nonsense without adding the phrase either 'in $\boldsymbol{\kappa}$-space' or 'in $\boldsymbol{r}$-space' for definiteness.[8] In $\boldsymbol{\kappa}$-space, for instance, $Y_{\mathrm{I}} \equiv 0$ is to be assumed, and we see from (5. 7), (5. 8)

$$q_1 = -2\pi \int_0^\infty \kappa^4 d\kappa \int_0^{\pi/2} Z_{\mathrm{I}} \times Q \sin\theta\, d\theta + q_1' \qquad \text{and} \qquad q_2 = -\frac{2}{1-m^2}q_1'\,,$$

where in general q_1' (and hence q_2) $\neq 0$, the field being not quasi-isotropic in $\boldsymbol{r}$-space. Similarly, even if $q_2 \equiv 0$ in $\boldsymbol{r}$-space, we cannot expect $Y_{\mathrm{I}} = 0$ always. Nevertheless,

8) This is not surprising, since *quasi-isotropy* is not a proper type of symmetry defined by the transformation group of coordinate axes.

so far as the following *special* cases are concerned, it is not difficult to prove how we are certainly free from such a inconsistency of representations.

(1) Isolated transverse turbulence:

In this case $Z_{\mathrm{I}} = 0$, and so the equations (5.8) become simply

$$r\frac{\partial q_1'}{\partial r} + 4q_1' + m\frac{\partial q_1'}{\partial m} + q_2' = 2q_1' + (1-m^2)q_2' = 0\,, \tag{5.8'}$$

for which we have by definition

$$q_1'(=q_1) = 0\,(=E_{//}) \qquad \text{at} \qquad r=0\,,$$

(cf. I; (9.2)). The relevant solution of (5.8') subject to this condition is

$$q_1' = q_2' = 0\,,$$

so that we find

$$q_1 = 0 \qquad \text{and} \qquad q_2 = -[\kappa^4 Y_{\mathrm{I}}]^{\mathfrak{D}}\,, \tag{5.10}$$

and conversely $Z_{\mathrm{I}} = 0$, $Y_{\mathrm{I}} = Y_{\mathrm{I}}[q_2]$ will follow when $q_1 \quad 0$.

(2) Z_{I} and Y_{I} are independent of μ:

In this case it turns out identically that

$$\int_0^{\pi/2} Z_{\mathrm{I}} \times \left(2P - r\frac{\partial Q}{\partial r} - 4Q - m\frac{\partial Q}{\partial m}\right)\sin\theta\,\mathrm{d}\theta = Z_{\mathrm{I}}\int_0^{\pi/2}(\quad {''} \quad)\sin\theta\,\mathrm{d}\theta = Z_{\mathrm{I}} \times 0 = 0$$

because of (3.1) and (3.2) together with (3.4), (3.5), and hence (5.8') is obtained again. Furthermore, since we have from (5.7)

$$(q_1)_{r=0} = (q_1')_{r=0} - \frac{4}{3}\pi\int_0^\infty \kappa^4 Z_{\mathrm{I}}\,\mathrm{d}\kappa$$

on the one hand and

$$-(q_1)_{r=0} = E_{//} = 2\pi\int_0^\infty \kappa^4 Z_{\mathrm{I}}\,\mathrm{d}\kappa\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta = \frac{4}{3}\pi\int_0^\infty \kappa^4 Z_{\mathrm{I}}\,\mathrm{d}\kappa$$

on the other (see I; (9.2), (5.6), (5.7) and (8.4)), the condition at $r=0$ must be $(q_1')_{r=0} = 0$ as before. Accordingly, $q_1' = q_2' = 0$ is warranted here too, and (5.7) results in

$$\left.\begin{aligned} q_1(r) &= -4\pi\int_0^\infty \kappa^4 Z_{\mathrm{I}}(\kappa)\frac{\psi_1(\kappa r)}{\kappa r}\,\mathrm{d}\kappa\,, \\ q_2(r) &= -4\pi\int_0^\infty \kappa^4 Y_{\mathrm{I}}(\kappa)\frac{\psi_1(\kappa r)}{\kappa r}\,\mathrm{d}\kappa\,, \end{aligned}\right\} \tag{5.11}$$

through carrying out the integrations with respect to θ. To be noted, these q_1 and q_2 do not depend on m, and in virtue of (4.7) the inverse relations are found to be:

$$\left.\begin{aligned} Z_{\mathrm{I}}(\kappa) &= -\frac{1}{2\pi^2}\int_0^\infty r^4 q_1(r)\frac{\psi_1(\kappa r)}{\kappa r}\,\mathrm{d}\kappa\,, \\ Y_{\mathrm{I}}(\kappa) &= -\frac{1}{2\pi^2}\int_0^\infty r^4 q_2(r)\frac{\psi_1(\kappa r)}{\kappa r}\,\mathrm{d}\kappa\,, \end{aligned}\right\} \tag{5.12}$$

provided that we assume $\int_0^\infty |\kappa^2 Z_{\mathrm{I}}| d\kappa < \infty$ and $\int_0^\infty |\kappa^2 Y_{\mathrm{I}}| d\kappa < \infty$ as well. Thus, the field of turbulence in this case may be pictured as is composed of two parts; one is *truly* isotropic in particular and the other is transverse.

In this way, the exact correspondence $q_1 \leftrightarrows Z_{\mathrm{I}}$, $q_2 \leftrightarrows Y_{\mathrm{I}}$ is just established for the cases (1) and (2). Now a question will arise: what are the physical implications of these special models? The answer will be given in future from another point of view.

6. Correlation coefficients of turbulence behind a damping screen. One of the practical methods of producing homogeneous turbulence which is distinctly axisymmetric is perhaps to distort a stream of grid turbulence by passage through a plane sheet of fine wire gauze or a damping screen, although such a device, as its name indicates, aims primarily at a reduction of undesirable residual fluctuations in wind-tunnel flow. The theory developed by G. I. Taylor and G. K. Batchelor in the present connexion (see [**1**] **4. 2**) seems to give a first-order explanation of this 'gauze-effect,' predicting the relation between the energy spectrum tensors of turbulence before and after distortion. Then it may be of some interest to calculate the velocity correlation functions with the Taylor-Batchelor's theoretical result as a typical example of axisymmetric case.

(1) Project in general. The starting point of our calculation is found also in I; section **8** (pp. 79, 80). Specifically, let the turbulence be isotropic far upstream; we have then from I; (8'. 5)

$$\left.\begin{aligned} \Phi_1''(\kappa, \mu) &= \frac{\mathcal{E}'(\kappa)}{4\pi\kappa^4(1-\mu^2)}\,[\mu^2 \Omega(\mu) - a^2]\,, \\ \Phi_3''(\kappa, \mu) &= \frac{\mathcal{E}'(\kappa)}{4\pi\kappa^2(1-\mu^2)}\,[\Omega(\mu) - a^2]\,, \end{aligned}\right\} \qquad (6.1)$$

for the turbulence far downstream, where $\mathcal{E}'(\kappa)$ is the prescribed energy spectrum function of undistorted isotropic field, a a dynamical constant, and $\Omega(\mu)$ stands for a certain known function formerly written as JJ^*. For simplicity's sake we shall omit $'$ and $''$ hereafter.

It is now a feasible, though somewhat tedious, procedure to evaluate the corresponding R-functions in every details after the formulas of section **2**, but for the time being we have to confine ourselves to some representative correlation functions or coefficients. In Fig. 2, nine variations of them are shown schematically, which however degenerate into the following five because of axisymmetry. That is, they are:

$$\left.\begin{aligned} f_{//}(r) &= \frac{R_{11}(r,0,0)}{R_{11}(0,0,0)} = \frac{(r^2R_1 + R_2 + R_3 + 2rR_4)_{m=1}}{u_{//}^2}\,, \\ g_{//}(r) &= \frac{R_{11}(0,r,0)}{R_{11}(0,0,0)} = \frac{R_{11}(0,0,r)}{R_{11}(0,0,0)} = \frac{(R_2+R_3)_{m=0}}{u_{//}^2}\,, \\ f_{\perp}(r) &= \frac{R_{22}(0,r,0)}{R_{22}(0,0,0)} = \frac{R_{33}(0,0,r)}{R_{33}(0,0,0)} = \frac{(r^2R_1+R_2)_{m=0}}{u_{\perp}^2}\,, \\ g_{\perp}^{(1)}(r) &= \frac{R_{22}(r,0,0)}{R_{22}(0,0,0)} = \frac{R_{33}(r,0,0)}{R_{33}(0,0,0)} = \frac{(R_2)_{m=1}}{u_{\perp}^2}\,, \end{aligned}\right| \qquad (6.2)$$

$$g_{\perp}^{(2)}(r) = \frac{R_{22}(0,0,r)}{R_{22}(0,0,0)} = \frac{R_{33}(0,r,0)}{R_{33}(0,0,0)} = \frac{(R_2)_{m=0}}{u_{\perp}{}^2}, \quad \Bigg\}$$

referring to the s-system (i.e. $s_1 = 1$, $s_2 = s_3 = 0$, I; p. 70). Besides putting

$$u_{//}{}^2 = \eta_{//}\overline{u^2}, \quad u_{\perp}{}^2 = \eta_{\perp}\overline{u^2}, \tag{6.3}$$

and estimating each of (6.2) in terms of (6.1), we have, for instance,

$$f_{//}(r) = \frac{1}{u^2\eta_{//}}\int_0^\infty \mathcal{E}(\kappa)\cos(r\kappa\mu)\,d\kappa \int_0^1 (1-\mu^2)\,\Omega(\mu)\,d\mu,$$

and the like *via* (2.9) and (2.10) (note that $d\mu = \sin\theta\, d\theta$), but in virtue of the isotropic relation:

$$2\int_0^\infty \mathcal{E}(\kappa)\frac{\sin\kappa r}{\kappa r}\,d\kappa = R(r) = \overline{u^2}(f+2g) = \overline{u^2}\Big(3f + r\frac{\partial f}{\partial r}\Big) \tag{6.4}$$

(see (1.1), (1.2) and (3.13)), the integral with respect to κ can be expressed by $R(r)$ and hence by Kármán's f, g. In this and similar ways, we get finally

$$\left.\begin{aligned}
f_{//} &= \frac{1}{\eta_{//}}\int_0^1 (1-\mu^2)\,\Omega\,\mathcal{H}\,d\mu, \quad g_{//} = \frac{1}{\eta_{//}}\int_0^1 (1-\mu^2)\,\Omega\,\mathcal{K}_0\,d\mu,\\
f_{\perp} &= \frac{1}{2\eta_{\perp}}\int_0^1 [(a^2+\mu^2\Omega)\mathcal{K}_0 - (a^2-\mu^2\Omega)\mathcal{K}_2]\,d\mu,\\
g_{\perp}^{(1)} &= \frac{1}{2\eta_{\perp}}\int_0^1 (a^2+\mu^2\Omega)\,\mathcal{H}\,d\mu,\\
g_{\perp}^{(2)} &= \frac{1}{2\eta_{\perp}}\int_0^1 [(a^2+\mu^2\Omega)\mathcal{K}_0 + (a^2-\mu^2\Omega)\mathcal{K}_2]\,d\mu,
\end{aligned}\right\} \tag{6.5}$$

where

$$\eta_{//} = \frac{3}{2}\int_0^1 (1-\mu^2)\,\Omega\,d\mu, \quad \eta_{\perp} = \frac{3}{4}\int_0^1 (a^2+\mu^2\Omega)\,d\mu, \tag{6.6}$$

the analytical expressions of which are found in I; (8'.9) and (8'.10), and

$$\left.\begin{aligned}
\mathcal{H}(r\mu) &= \int_0^\infty \mathcal{E}(\kappa)\cos(r\kappa\mu)\,d\kappa = \frac{1}{2}\frac{d}{d(r\mu)}[r\mu R(r\mu)],\\
\mathcal{K}_0(r\sqrt{1-\mu^2}) &= \int_0^\infty \mathcal{E}(\kappa)J_0(r\kappa\sqrt{1-\mu^2})\,d\kappa = \frac{2}{\pi}\int_0^1 \frac{\mathcal{H}(\chi r\sqrt{1-\mu^2})}{\sqrt{1-\chi^2}}\,d\chi,\\
\mathcal{K}_2(r\sqrt{1-\mu^2}) &= \int_0^\infty \mathcal{E}(\kappa)J_2(r\kappa\sqrt{1-\mu^2})\,d\kappa = \frac{2}{\pi}\int_0^1 \frac{\mathcal{H}(\chi r\sqrt{1-\mu^2})}{\sqrt{1-\chi^2}}(2\chi^2-1)\,d\chi.
\end{aligned}\right\} \tag{6.7}$$

Further, the Taylor-Batchelor theory tells us that

$$\Omega(\mu) = \frac{4a^2\mu^2 + (1-\mu^2)(ak-1-a)^2}{(k+1+a)^2 + [4-(k+1+a)^2]\mu^2} \tag{6.8}$$

k being connected with a through

$$a^2(1+k)=1.21\,. \tag{6.9}$$

Thus, we are able to calculate the five coefficients $f_{//}$, $g_{//}$, ... for any given a (or k), so long as the profile of $f(r)$ is known.[9]

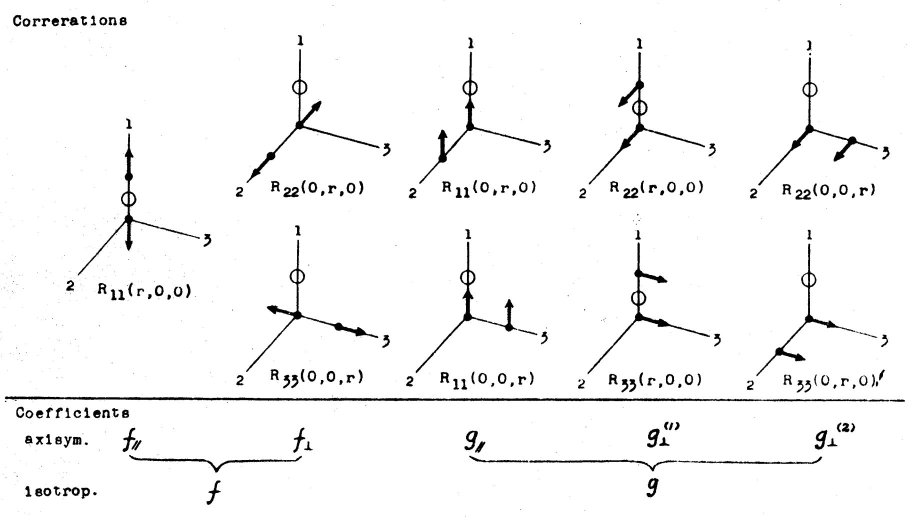

Fig. 2. Correlation Diagrams. (Small circles indicate the axis of symmetry.)

(2) Assumed profile of $f(r)$. We shall now take an expedient to approximate the observed profile of $f(r)$ in the form

$$f(r)\equiv f\left(\frac{r}{M}\right)=\exp\left[b-\sqrt{c^2(r/M)^2+b^2}\right], \tag{6.10}$$

containing two parameters b and c, where M is the mesh-length of the turbulence producing grid. The curve represented by (6.10) falls off monotonically from the value 1 at $r=0$ to 0 as $r\to\infty$, with the parabolic asymptote

$$f(r)\approx 1-\frac{r^2}{2\lambda^2}; \qquad \frac{\lambda}{M}=\frac{\sqrt{b}}{c} \tag{6.11}$$

for vanishing r, and so it resembles the actual profile of $f(r)$ at least in these general features. By the way, if (6.10) is assumed, we obtain

$$R(r)=\left(3-\frac{c^2r^2}{\sqrt{c^2r^2+b^2}}\right)\times f(r),$$

(from (6.4)), and

$$\mathcal{H}(\xi)=\frac{1}{2}\left[3-\frac{6c^2\xi^2}{\zeta}+\frac{c^4\xi^4}{\zeta^2}\left(1+\frac{1}{\zeta}\right)\right]\times f(\xi); \qquad \zeta=\sqrt{c^2\xi^2+b^2},$$

9) It should be remembered that for the purpose of finding the turbulent energy attenuations only the Taylor-Batchelor theory does not require the knowledge of $f(r)$ at all.

from which $\mathcal{K}_0$ and $\mathcal{K}_2$ can be computed after (6. 7). Note that here M has been used as a unit of length.

For the following calculation, let us assume

$$b = 0.1\,, \quad c = 2.5 \quad (\text{and hence } \lambda = 0.126\,M)\,, \tag{6.12}$$

in particular. The assumed profile then, as is seen from Fig. 3, roughly corresponds to the experimental correlation curve observed for a moderate Reynolds number ($UM/\nu \approx 10^3 \sim 10^5$, say) and in some earlier stage of decay. Of course, it may be conceivable to improve the approximation any further, but perhaps our present assumption is the mathematically simplest one, even though not the best.

(3) Numerical results. In this paper, the calculation has been carried out only when $ak = 1 + a$, giving in combination with (6. 9)

$$a = 0.567 \quad \text{and} \quad k = 2.76\,. \tag{6.13}$$

Correspondingly, (6. 8) becomes

$$\Omega(\mu) = \frac{0.06859\,\mu^2}{1 - 0.7867\,\mu^2}\,, \tag{6.14}$$

and the theoretical attenuation factors of the turbulent energies are found to be

$$\eta_{//} = 0.0228 \quad \text{and} \quad \eta_{\perp} = 0.268 \tag{6.15}$$

from (6. 6), showing $u_{\perp}^2/u_{//}^2 = 0.268/0.0228 = 11.7$ which is near the maximum degree of anisotropy predicted by the theory. (The fields in question is, therefore, an axisymmetric one of oblate symmetry, i.e. $\overline{u_{\perp}^2} > \overline{u_{//}^2}$.)

Now, since $\mathcal{H}$, $\mathcal{K}_0$ and $\mathcal{K}_2$ have been already given, we are able to perform the (numerical) integrations in (6. 5). The final results of this numerical work are shown in the table as well as in Figs. 4, 5 and 6.

TABLE

r/M	Assumed profiles		Calculated profiles				
	f	g	$f_{//}$	$f_{\perp}$	$g_{//}$	$g_{\perp}^{(1)}$	$g_{\perp}^{(2)}$
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	0.844	0.746	0.682	0.862	0.813	0.759	0.717
0.2	0.664	0.501	0.386	0.702	0.611	0.524	0.453
0.3	0.519	0.326	0.182	0.568	0.451	0.354	0.268
0.4	0.405	0.203	0.046	0.461	0.329	0.235	0.143
0.6	0.246	0.062	−0.097	0.304	0.166	0.093	0.005
0.8	0.149	0.000	−0.141	0.205	0.075	0.028	−0.045
1.2	0.055	−0.027	−0.122	0.100	0.000	−0.009	−0.048
1.6	0.020	−0.020	−0.077	0.057	−0.019	−0.023	−0.028
2.4	0.003	−0.005	−0.023	0.031	−0.018	−0.002	+0.005
$-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}\right)_{r=0}$	31.25	62.50	79.01	25.93	38.66	59.21	71.09

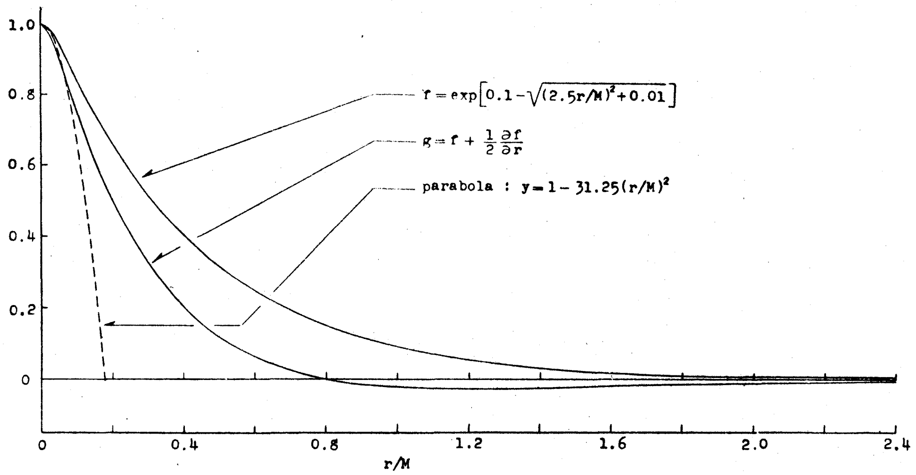

Fig. 3. Assumed correlation profiles.

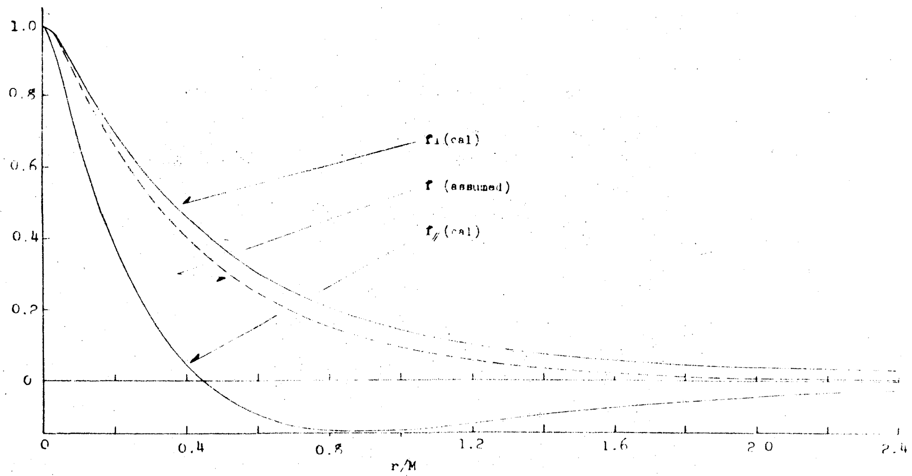

Fig. 4. f-type correlation profiles.

To the author's knowledge, no correlation measurements behind a damping screen have been reported so far. Therefore, we cannot compare directly the present results with the experiments, but in view of the fact that the theory tends to overestimate the values of $u_\perp^2/u_{//}^2$ (see [1] Fig. 4. 1), the degree of anisotropy is likely to be exaggerated as regards the correlation profiles too. Furthermore, the use of the provisional function (6. 10) will introduce another uncertainty. Nevertheless, apart from the quantitative refinement in these respects, we may suppose the above calculation suggests us some general aspects of axisymmetric turbulence. That is, for example, certain correlation coefficients can have wavy profiles (like $g_\perp^{(2)}$), or can assume fairly large negative values (like $f_{//}$). These features which have not

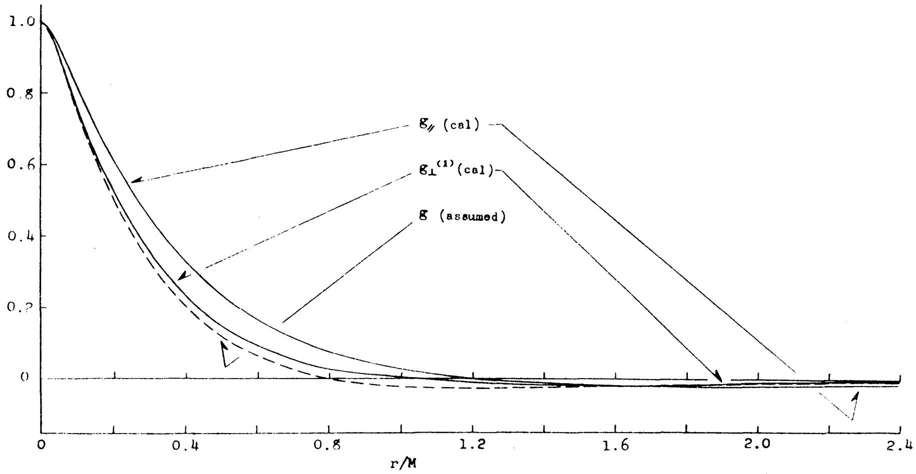

Fig. 5. g-type correlation profiles (1).

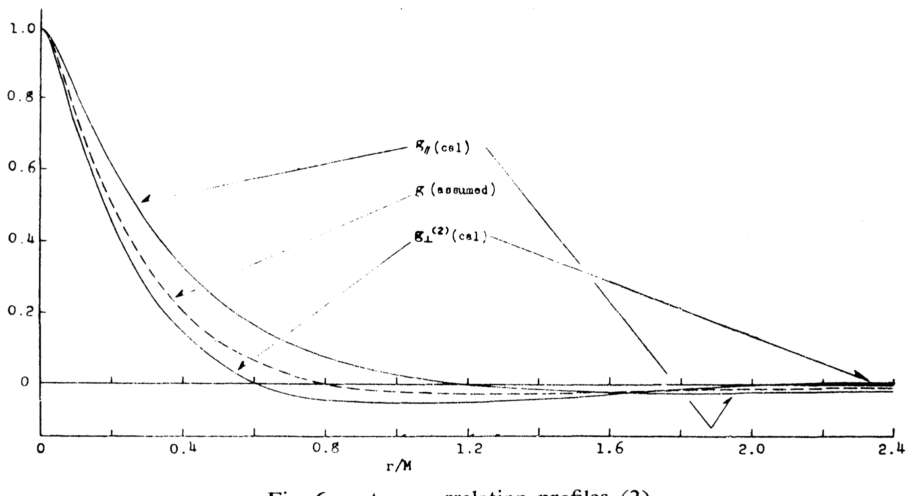

Fig. 6. g-type correlation profiles (2).

been observed in isotropic turbulence seem to be rather peculiar to anisotropic turbulence and the turbulent shear flow (the wake, the boundary layer, etc.).[10)]

In conclusion, the author wishes to express his hearty thanks to Professor Hikoji Yamada for his interest in this work and for his instructive suggestions.

---

10) In this connexion, the recent paper of H. L. Grant (*Journ. Fluid Mech.* Vol. 4, Part 2 (1958) 149) is consulted as an interesting instance.

References

[1] Batchelor, G. K. *The theory of homogeneous turbulence.* Cambridge University Press (1953).
[2] Chandrasekhar, S. The theory of axisymmetric turbulence. *Philos. Trans.* A, **242** (1950) 557.
[3] Ohji, M. On the theory of homogeneous axisymmetric turbulence. *Rep. Res. Inst. App. Mech.* (this journal) Vol. 4, No. 22 (1958) 63. (referred as I in the present paper).
[4] Watson, G. N. *The theory of Bessel functions.* Cambridge University Press (1922).

(Received December 9, 1958; Communicated by H. Yamada)

Addendum to the previous paper I

Previously it has been stated that Φ_1 and Φ_3 defining $\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ are *even* functions of $\kappa\mu$ (I; p. 68). In effect, this statement is equivalent to asserting R_1, R_2, R_3 of $R_{ij}(\boldsymbol{s}\,;\boldsymbol{r})$ are even, while R_4 is an odd, functions of rm, for which Batchelor has provided the proof (I; reference [1]). Of course, the similar arguments are possible in $\boldsymbol{\kappa}$-space as follows.

First, in virtue of the homogeneity condition I; (4.2):

$$\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = \Phi_{ji}(\boldsymbol{s}\,;-\boldsymbol{\kappa})\,, \tag{A. 1}$$

we have

$$\begin{aligned}&\Phi_1\kappa_i\kappa_j+\Phi_2\delta_{ij}+\Phi_3 s_i s_j+\Phi_4 s_i\kappa_j+\Phi_5\kappa_i s_j\\&=\Phi_1^-\kappa_i\kappa_j+\Phi_2^-\delta_{ij}+\Phi_3^- s_i s_j-\Phi_5^- s_i\kappa_j-\Phi_4^-\kappa_i s_j\,,\end{aligned} \tag{A. 2}$$

where $\Phi_1^-(\kappa,\kappa\mu)=\Phi_1(\kappa,-\kappa\mu)$ and so on, as in I; p. 74. Thus, it is found that

$$\Phi_1=\Phi_1^-\,,\quad \Phi_2=\Phi_2^-\,,\quad \Phi_3=\Phi_3^-\,,\quad \Phi_4=-\Phi_5^-\,,\quad \Phi_5=-\Phi_4^-, \tag{A. 3}$$

showing Φ_1, Φ_2 and Φ_3 are even in $\kappa\mu$. But this does not complete the proof, since (A. 2) cannot be identified with I; (4.7) until $\Phi_4=\Phi_5$ is confirmed.

Considering however, the solenoidal condition (with respect to the index i, say): $\kappa_i\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})=0$, we obtain

$$\kappa^2\Phi_1+\Phi_2+\kappa\mu\Phi_4=0 \quad\text{and}\quad \kappa\mu\Phi_3+\kappa^2\Phi_5=0\,, \tag{A. 4}$$

respectively, from which it follows that Φ_4 and Φ_5 are odd in $\kappa\mu$ (because Φ_1, Φ_2 and Φ_3 are even). So in combination with (A. 3), there come

$$\Phi_4=-\Phi_4^-=-\Phi_5^-\,,\quad \Phi_5=-\Phi_5^-=-\Phi_4^-\,,\quad\text{and hence}\quad \Phi_4=\Phi_5\,.\quad \text{Q.E.D.}$$

As is seen above, the property (A. 1) is essential for this proof which are obviously valid for any second-order axisymmetric true tensor:

$$\chi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})=\chi_1\kappa_i\kappa_j+\chi_2\delta_{ij}+\chi_3 s_i s_j+\chi_4 s_i\kappa_j+\chi_5\kappa_i s_j\,,$$

and the followings have been established.

(1) When $\chi_{ij}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})=\chi_{ji}(\boldsymbol{s}\,;\,-\boldsymbol{\kappa})$ is satisfied:

(a) If χ_{ij} is solenoidal in *either* of its indices, it must be symmetric in the indices, being soienoidal in *another* index as well; χ_1, χ_2 and χ_3 are even functions of $\kappa\mu$, but $\chi_4=\chi_5$ are odd ones. $\Phi_{ij}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})$ belongs to this class.

(b) If χ_{ij} is not solenoidal in either of its indices, it is *not necessarily* symmetric in the indices, being not solenoidal in another index as well; χ_1, χ_2 and χ_3 are still even in $\kappa\mu$, but $\chi_4=-\chi_5^-$ (and so $\chi_5=-\chi_4^-$) remain arbitrary. $\Gamma_{ij}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})$ and $\Pi_{ij}(\boldsymbol{\kappa}\,;\,\boldsymbol{s})$ are of this class (cf. I; section **7**).

(2) When $\chi_{ij}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})\neq\chi_{ji}(\boldsymbol{s}\,;\,-\boldsymbol{\kappa})$:

The foregoings do not hold; it is also possible that χ_{ij} is solenoidal in *one* index only. $\kappa_k\Upsilon_{(ik)j}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})$ or $\kappa_k\Upsilon_{i(k_\cdot)}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})$ is an instant (cf. *loc. cit.*).

Incidentally, a note on the terminologies will be added. Instead of the terms 'axial turbulence' and 'transverse turbulence' introduced in I; section **8**, it may be more familiar to call them 'longitudinal...' and 'lateral...,' respectively. The reason why these latter words were avoided is that they had been already used to specify the orientation of the separation vector $\boldsymbol{r}$ in connexion with the terms 'longitudinal correlation function' etc. (I; section **5** (1)).

M. O.

ERRATA

On the Theory of Homogeneous Axisymmetric Turbulence

By Michio OHJI

(Reports of Research Institute for Applied Mechanics vol. VI, No. 22, 1958)

Page 75, equation (7.9c), for $\Gamma_3=2\kappa^2(\mu\Upsilon_4^o+\Upsilon_6^e)$,
read $\Gamma_3=2\kappa^2(\mu\Upsilon_4^o-\Upsilon_6^e)$.
equation (7.10), for $\cdots+\kappa(\Pi s_i\kappa_j-\Pi^-\kappa_i s_j)$,
read $\cdots+\kappa(\Pi^- s_i\kappa_j-\Pi\kappa_i s_j)$.
equation (7.12), for $\cdots=\iota\Pi(\mu\kappa_j-\kappa s_j)$,
read $\cdots=-\iota\Pi(\mu\kappa_j-\kappa s_j)$.
equation (7.13), for $\cdots=\iota\Pi^-(\mu\kappa_i-\kappa s_i)$,
read $\cdots=-\iota\Pi^-(\mu\kappa_i-\kappa s_i)$.
Page 79, line 16, for $\beta=\kappa_1(\kappa_2^2+\kappa_2^3)^{-1/2}$,
read $\beta=\kappa_1(\kappa_2^2+\kappa_3^2)^{-1/2}$.