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RESPONSE OF THE HIGHER MODES OF THE HULL VIBRATION OF A LARGE TANKER*

By Toyoji KUMAI

The present paper deals with some computations on the higher natural modes and the response of the hull vibration of a large tanker excited by propeller blade exciting force. The calculations are carried out assuming shearing vibration of the beam of variable cross section. The results of the response calculations are checked against those obtained from the generator survey on two tankers. The propeller blade exciting forces are estimated from the accelerations measured on several types of the large tankers. Also, the exciting force permissible to maintain hull vibration within the acceptable limit of an acceleration is estimated in the present calculation.

Introduction. It has already been noted that the chief cause of generation of the higher mode of the hull vibration in turbine-driven ships is the propeller blade frequency exciting force. In accordance with some results of the vibration measurements on large tankers, many resonant peaks for which no cause is apparent have been observed in the higher modes of the hull vibration excited by the propeller blade force. This mode seems to occur once in the range of the frequencies between two adjacent peaks of the natural modes of the free-free bar type in the higher modes of the hull vibrations. The cause of occurrence of this type of peak other than nodal vibration is not discussed in the present paper.

A convenient formula for computing the response of the nodal hull vibration has been given by Lockwood Taylor [1]. Also, the above responses in the higher modes of hull vibration have recently been obtained by means of generator surveys on cargo ships [2], [3] and also on a war ship [4].

Taylor's formula, however, is deduced considering the forced vibrations of a beam of uniform cross section, so that it would be unreasonable to apply the formula to the vibration of a ship-like beam, and it would hardly be effective to explain the response of the higher modes of the hull vibration of a tanker, because of the non-uniform rigidity and the effective mass of the tankers varying over the ship length.

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* Read in Tokyo at the Spring Meeting of the Japan Society of Naval Architects on May 10th. 1958 (in Japanese). Published on "European Shipbuilding" Oslo, No. 5, Vol. VII, 1958.

[1] J. Lockwood Taylor: "Vibration of Ships." Trans. I.N.A., 1930.

[2] A. J. Johnson: "Vibration Tests of All-welded and All-riveted 10,000-ton Dry Cargo Ship." Trans. N.E.C., Vol. 67, 1950-51.

[3] R. T. McGoldrick, V. L. Russo: "Hull Vibration Investigation on S/S Gopher Mariner." Reprint No. 8, S.N.A.M.E., 1955.

[4] J. W. Ramsay: "Aspect of Ship Vibration induced by Twin Propellers." Quart. Trans. I.N.A., Vol. 98, 1956.

On the other hand, it seems that an empirical response factor will easily be obtained from test results of the exciter generating survey on actual ships; the experimental results however will be invalid for the main hull vibration, because the stiffness of the deck where the exciting machine is installed will give a local vibration, because the deck plate below the machine installation would not be sufficiently stiff to excite the main hull vibration of a large tanker. Theoretical investigation into the response is thus necessary in the case of the vibration of the large tanker.

The main difficulties encountered in the calculation of the response are the expressions of the normal function and estimate of the damping factor in the higher mode of the hull vibration of a tanker. The present paper deals with the numerical calculations of the above response of the hull vibration of a tanker by use of the normal function derived from shearing vibration theory under the assumptions of the mathematical representations of the rigidity and the mass distributions of a tanker along her length.

The comparisons of the response factor obtained by the calculations and the measurements during exciter tests on two tankers are shown to verify the theory. The propeller blade exciting forces calculated from the acceleration measured on several tankers are also shown as examples. In the last paragraph, the maximum permissible exciting force to maintain hull vibration within acceptable limits of acceleration is presented.

1. Normal Modes in the Flexural Vibration of a Tanker. In the calculation of the natural frequency of the hull vibration of a ship, the effect of the variable parts of rigidity and mass distributions upon the natural frequency is not conspicuous in the result. With regard to the calculation of the response of the hull vibration, however, this effect is considerable; hence it is of primary importance that the normal mode should approximate as closely as possible that of the actual ship in the response calculation.

An important point in the calculation of the normal function of the flexural shear vibration of ships is the assumption of the appropriate mathematical representations of the rigidity and the mass distributions along the length of the ship. In the present paper, the rigidity and the mass distributions are assumed to be of the power curves of h-th and k-th ($\bar{h}$-th and $\bar{k}$-th) orders of the length co-ordinate respectively from the origin at the extreme after (fore) end of the hull to the after (fore) end of the cargo tanks of a large tanker. The distributions in way of cargo tanks, however, are assumed to be uniform. The normal functions of shearing vibration of the hull accorded to the above assumptions are written in the following equations:

$$\left.\begin{aligned}
&\eta = A_1 \xi^{\mu\nu} J_{-\nu}\left(\frac{\lambda}{\mu}\xi^{\mu}\right), && 0 \leq \xi \leq \xi_1, \\
&\eta = A_2 \cos\alpha\lambda\xi + B_2 \sin\alpha\lambda\xi, && \xi_1 \leq \xi \leq \xi_3, \\
&\eta = A_3 \bar{\xi}^{\bar{\mu}\bar{\nu}} J_{-\bar{\nu}}\left(\frac{\alpha}{\beta}\frac{\lambda}{\bar{\mu}}\bar{\xi}^{\bar{\mu}}\right), && 0 \leq \bar{\xi} \leq \bar{\xi}_1, \\
& && (\bar{\xi}_1 = 1 - \xi_3).
\end{aligned}\right\} \quad (1)$$

The fundamental equation of the shear vibration derived from the above expressions is shown in Appendix, see also [5].

2. Response of Hull Vibration. The expression of the maximum acceleration of forced hull vibration of a ship excited at a certain point is represented by the following equation:

$$a = gF \sum_{n=2}^{\infty} \frac{\eta_n(u) \cdot \eta_n(v)}{L \int_0^1 w\eta_n^2 d\xi \sqrt{\left(1 - \frac{p_n^2}{\omega^2}\right)^2 + 4q_n^2/\omega^2}}, \tag{2}$$

where,

- a acceleration at the position $x = vL$
- F exciting force applied at the position $x = uL$
- g acceleration due to gravity
- η_n normal function of n-node mode of hull vibration
- w effective load per unit length of the ship
- L overall length of the ship
- p_n natural circular frequency of n-node mode of the hull vibration
- q_n damping factor in n-node mode of the hull vibration
- ω forced circular frequency of the hull vibration
- n number of nodes of the hull vibration

The important respect in which to evaluate the above expression is the technique for representation of both the normal function and the damping factor for the n-node mode of the hull vibration. In the present paper, the normal function of the flexural hull vibration of a tanker is assumed to be of the shearing vibration as that previously presented, and the damping factor in the higher mode of the hull vibration is estimated by the existing empirical formula.

In accordance with the results obtained by the author's previous investigations into the above matters [5], [6], the magnitudes of p_n and q_n are replaced by the terms of p_2 and q_2 and the number of nodes of the vibration of a tanker, hence the terms of p_n, q_n and ω are represented as follows;

$$\left.\begin{aligned} p_n &= p_2(n-1), \\ q_n &= q_2(n-1)^{1+3/4} = \frac{p_2\delta_2}{2\pi}(n-1)^{1+3/4}, \end{aligned}\right\} \tag{3}$$

where, δ_2; logarithmic decrement for 2-node vibration,

$$\omega = p_2(m-1+\varepsilon), \quad m = 1, 2, \ldots \ \varepsilon = 0 \sim 1.$$

Substituting from the above expression to (2), the maximum acceleration is given by:

[5] T. Kumai: "Shearing Vibration of Ships." European Shipbuilding, No. 2, Vol. 5, 1956, and Rep. Res. Inst. Appl. Mech., Kyushu Univ. Vol. IV, No. 15, 1956.

[6] T. Kumai: "Damping Factors in the Higher Modes of Ship Vibration." European Shipbuilding, No. 1, Vol. 7, 1958, and Rep. Res. Inst. Appl. Mech, Kyushu University, Vol. VI, No. 21, 1958.

$$a=\frac{gF}{\Delta_1}\sum_{n=2}^{\infty}\frac{\eta_n(u)\cdot\eta_n(v)}{\frac{L}{\Delta_1}\int_0^1 w\eta_n^2 d\xi\sqrt{\left\{1-\frac{(n-1)^2}{(m-1+\varepsilon)^2}\right\}^2+\frac{\delta_2^2}{\pi^2}\frac{(n-1)^{2+3/2}}{(m-1+\varepsilon)^2}}}, \tag{4}$$

where, Δ_1; effective displacement including virtual added mass.

The above equation shows the response of the hull vibration of the tankers in the state of non-resonant vibration. When the vibration attains to the state of resonance, m is taken equal to n and $\varepsilon=0$ in (4), the above equation simplifies as follows:

$$a=\frac{g\pi F}{\Delta_1\delta_2}\phi_n, \tag{5}$$

where,

$$\phi_n=\frac{\eta_n(u)\cdot\eta_n(v)}{\frac{L}{\Delta_1}\int_0^1 w\eta_n^2 d\xi\cdot(n-1)^{3/4}}, \tag{6}$$

also, from (5) we have

$$\phi_n=\frac{\Delta_1\delta_2 a}{g\pi F}. \tag{7}$$

It will be understood that the response factor ϕ_n takes approximately the same value in similar ships with similar loading condition provided that the normal function is determined. On the other hand, ϕ_n is also obtained by the results of the exciting generator survey by use of (7) under the assumption of δ_2. If the acceleration or the exciting force is given for the hull vibration of a tanker, either of the two can be calculated by the use of ϕ_n.

3. Mode Factor and Correction of Acceleration for Normalized Position. In the case of the shearing vibration of the uniform beam, for example, the normal function of the n-node mode of the vibration is written as

$$\eta_n=\cos n\pi\xi. \tag{8}$$

By the use of the above function, the mode factor is simply calculated as follows;

$$1\Big/\frac{L}{\Delta_1}\int_0^1 w\eta_n^2 d\xi=2. \tag{9}$$

As may be seen in (9) the mode factor in the above case is obtained as the same value as that presented by J. Lockwood Taylor [1]. This result is independent of the number of nodes. In the vibrations of the tankers, however, it will be expected that the mode factor obtained from the normal function shown in (1) should show a wide divergence from the uniform section beam shown in Taylor's paper. The calculations of the denominator in the expression of the mode factor is obtained in the following form;

$$\frac{L}{\Delta_1}\int_0^1 w\eta^2 d\xi=\frac{w_2 L}{\Delta_1}\Big[A_1^2\frac{w_1}{w_2}\int_0^{\xi_1}\xi J_{-\nu}^2\Big(\frac{\lambda}{\mu}\xi^{\mu}\Big)d\xi+\int_{\xi_1}^{\xi_3}(A_2\cos\alpha\lambda\xi+B_2\sin\alpha\lambda\xi)^2 d\xi$$
$$+A_3^2\frac{w_3}{w_2}\int_0^{\bar{\xi}_1}\xi J_{-\nu}^2\Big(\frac{\alpha}{\beta}\frac{\lambda}{\mu}\xi^{\mu}\Big)d\xi\Big]. \tag{10}$$

The evaluation of the above integrations of the Bessel functions and trigonometric functions are easily made analytically, and the mode factor of the hull vibration of the tanker according to the foregoing assumptions of the rigidity and the mass distributions may therefore be obtained. The results are applicable to similar ships with similar loading conditions for those of large tankers.

Since the two points $x = uL$ and $x = vL$, the exciting and measuring positions respectively are influenced by the response of the hull vibration as may be seen in the formula, the standard points u and v should be defined in advance when the foregoing calculations and the experiments are compared. In the present investigation, the following positions are defined as standard ones,

$u = 0.05$; near the station at the propeller post in large tankers,

$v = 0$; at the extreme end of the hull.

The correction of the measured acceleration for the above positions is made according to the numerical expression shown later in the present paper.

4. Numerical Examples and Comparison of the Results with Experiments. The numerical calculation presented in this paper is based on the data of a 32,500 ton d.w. tanker recently built in Japan. The distributions of the shear rigidity and the load including hull weight were obtained from construction plans and other design data. The added virtual mass distribution in the vertical hull vibration was approximately determined by the use of the results of the investigations carried out by Prohaska [7]. Fig. 1 shows the rigidity and effective mass distributions obtained from the plans and approximate representation according to the foregoing assumption for mathematical expression in the vertical vibration of the large tanker. As may be seen in the figure, the rigidity and the mass distributions along the length of the entrance and the run may be approximately considered as increasing with the rate of ξ/ξ_1 (assume $h = k = 1$ and $\bar{h} = \bar{k} = 1$) toward the parallel body and the considerable discontinuities are made on the mass distribution at both ends of the cargo tanks in the vertical vibration of a large tanker.

Fig. 2 shows the eigen values of the normal functions of the hull vibration previously mentioned. As will be seen in the figure, the eigen values of the higher modes of the vibration are almost proportional to the number of nodes, neverthless a considerable discontinuity of mass and rigidity distribution appear in the present example.

The vibration profiles of the higher modes from four-node mode to nine-node mode are shown in Fig. 3. Considerable amplitudes appear at the bow and stern.

The profiles of the non-resonant vibration between 6- and 7-node modes are shown in Fig. 4 as an example. As will be seen in the figure, the amplitude near the stern is considerable even in the range of non-resonant vibration, however, the amplitude decreases with increasing distance from the stern.

The numerical values of the mode factors in n-node modes of the vertical vibrations of a tanker in ballast condition are shown in Fig. 5. It will be understood that the numerical results of the mode factors obtained by the foregoing calculations show a wide divergence from those given by Taylor's formula.

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[7] C. W. Prohaska: "Vibrations Verticales du Navire." Assoc. T.M.A., 1947.

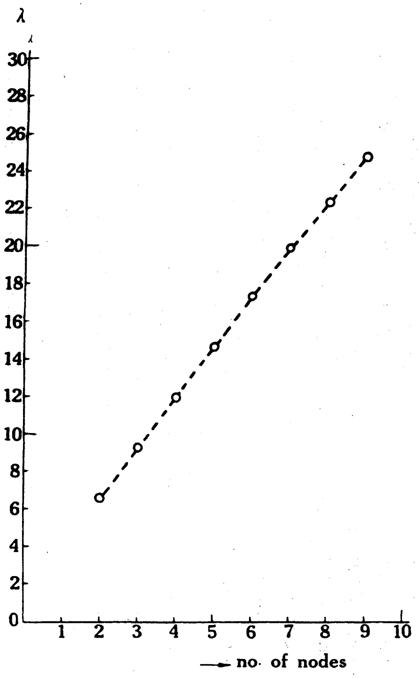

Fig. 2. Eigen values of the vertical vibration of the tanker.

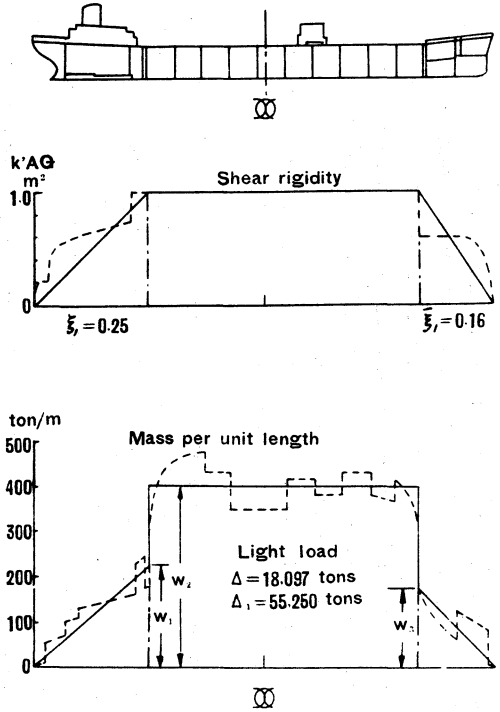

Fig. 1. Distributions of shear rigidity and effective mass including virtual added mass.

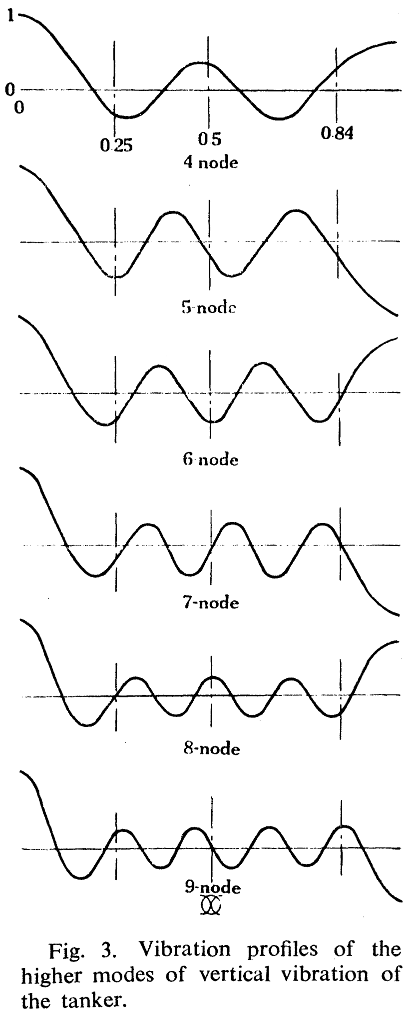

Fig. 3. Vibration profiles of the higher modes of vertical vibration of the tanker.

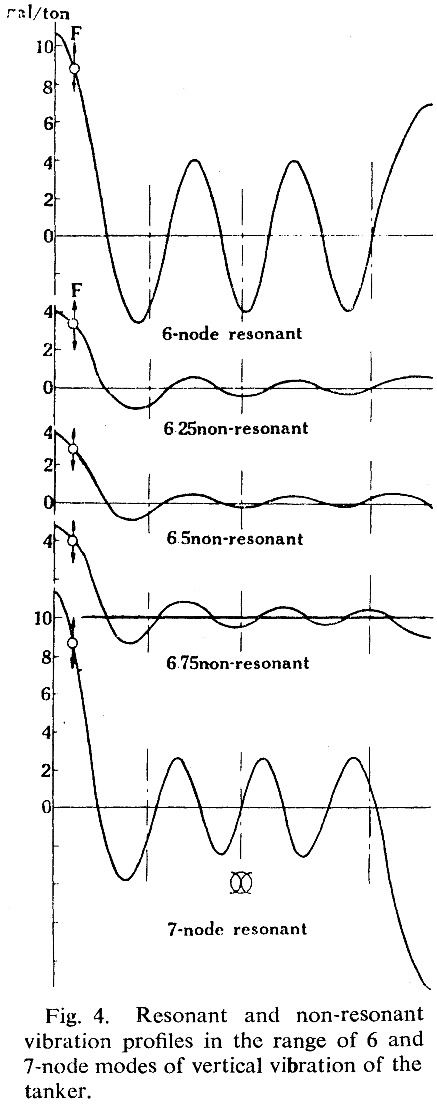

Fig. 4. Resonant and non-resonant vibration profiles in the range of 6 and 7-node modes of vertical vibration of the tanker.

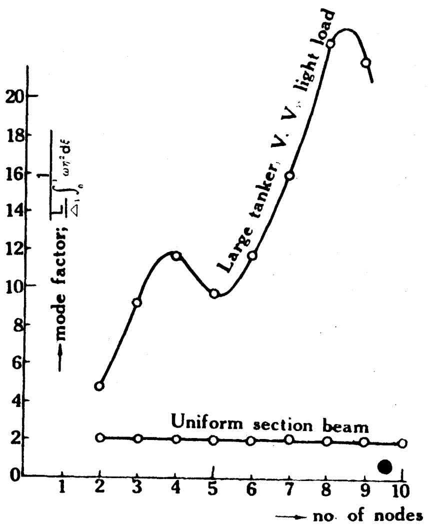

Fig. 5. Mode factors in vibrations of uniform beam and large tanker.

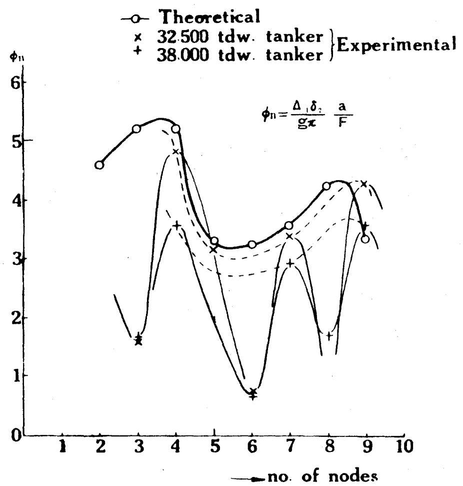

Fig. 6. Comparison of response factors obtained by calculation and measured in generator tests in two large tankers.

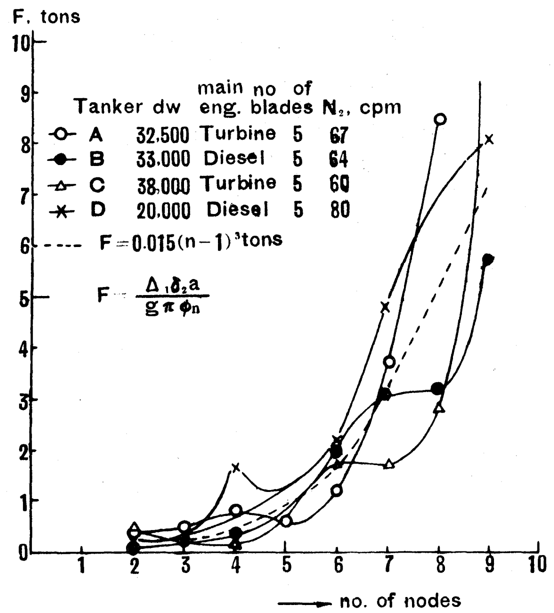

Fig. 7. Excited forces computed from the measuring acceleration on several large tankers.

The response factor ϕ_n computed by the present calculations was compared with those obtained from the results of the exciter tests on two tankers carried out at the Harima Shipbuilding and Engineering Works. Considerable discrepancies appear between results of the theory and the experiments as will be seen in Fig. 6. However, the results show fairly good agreement in part. It may be assumed that the above discrepancies between theory and experiment are due to the local vibration of the deck plating on which the exciting machine was installed.

Since the magnitude of ϕ_n calculated under the assumptions mentioned above may be applicable to similar tankers with similar loading condition, the propeller blade exciting forces of several tankers were calculated by use of ϕ_n from the measured results of the accelerations for the number of nodes of the hull vibrations as shown in Fig. 7. In the results, considerable blade exciting force appears over the 7-node mode of the vertical hull vibration, and the upper limit of the frequencies corresponding to the $8 \sim 9$-node modes of the vertical vibrations are about 550 cpm in the service speed of tankers with 5-bladed propeller and about 450 cpm in tankers with 4-bladed propeller, both in the light condition. It is supposed from the appearance of the curves in Fig. 7 that the four-bladed propeller will be better than the five-bladed one for vibration prevention if we wish to avoid the acute vibration reaction in such a high frequency range as from 450 cpm to 550 cpm provided that the resonance of the frequency 450 cpm can be avoided at service speed.

As may be seen in Fig. 7, the exciting forces in several tankers are widely different from each other in the higher mode of the hull vibration. It is supposed that the cause of these wide differences among the tankers are due to the different design of the screw aperture and the form of the run in each hull.

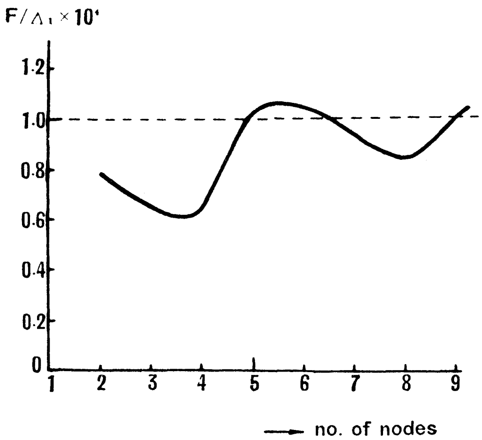

Fig. 8. Permissible exciting force with acceptable limit of acceleration 60 gal for vertical vibration.

[8] T. Kumai: "Some Measurement of Acceleration of Hull Vibration and Human Sensitivity to Vibration." European Shipbuilding, No. 3, Vol. 6, 1957, and Rep. Res. Inst. Appl. Mech., Kushu Univ., Vol. V, No. 17, 1957.

5. Permissible Exciting Force within Acceptable Limit of Acceleration. The threshold of unpleasantness of the vibration with an acceleration is estimated to be about 15 gal in a cabin for horizontal vibration[8]. The threshold of unpleasantness with an acceleration becomes approximately twice the above limit in vertical vibration, and the acceleration at the extreme end of the ship is also assumed to be twice the above value. The allowable acceleration at the normalized position defined by the foregoing calculation may therefore be estimated at about 60 gal. By use of the present formula, permissible exciting force corresponding to permissible acceleration is easily obtained as shown in Fig. 8. The rapid estimate of the permissible exciting force due to a five-bladed propeller to maintain hull vibration within acceptable limits is quickly made from the effective vibration of a tanker in ballast condition, namely, $F = 10^{-4}\Delta_1$ tons, as may be seen in Fig. 8.

Conclusion. The response factors ϕ_n including the term of acceleration, exciting force and effective displacement and logarithmic decrement of the large tankers were obtained up to nine-node modes of the vertical vibration in ballast condition. Comparison of the magnitude of ϕ_n in the theory and the experiments showed fairly good agreement. By use of the ϕ_n, the propeller exciting forces in several types of tankers were obtained from the measured results of the acceleration aboard ship.

The permissible exciting force to maintain the hull vibration within the acceptable limit of an acceleration was estimated by the simple formula.

APPENDIX

Normal Function of Shear Vibration of the Hull of a Tanker.

The fundamental equation of the shear vibration of a beam of variable cross section with variable load such as a ship's hull is generally expressed in the following form;

$$\frac{\partial}{\partial x} k' AG \frac{\partial y}{\partial x} = \frac{w}{g} \frac{\partial^2 y}{\partial t^2}, \tag{a}$$

where,

$k'AG$	variable shear rigidity
y	amplitude of the hull vibration
x	length coordinate with origin at the extreme end of the hull
t	time coordinate
w	load per unit length
g	acceleration due to gravity

putting $y/L = \eta \cos pt$,
$x/L = \xi$,

where L extreme length of the hull
p natural circular frequency

and also assuming $k'AG = (k'A)_1 G(\xi/\xi_1)^h$ and $w = W_1(\xi/\xi_1)^k$ in (a), we get the following Bessel equation:

$$\frac{d^2\eta}{d\xi^2} + \frac{h}{\xi}\frac{d\eta}{d\xi} + \lambda^2 \xi^{k-h} \eta = 0, \tag{b}$$

with the solutions

$$\eta = A_1 \xi^{\mu\nu} J_{-\nu}(\lambda/\mu \cdot \xi^\mu) + B_1 \xi^{\mu\nu} J_\nu(\lambda/\mu \cdot \xi^\mu), \tag{c}$$

where

$$\lambda^2 = \frac{p^2 w_1 L^2}{g(k'A)_1 G \cdot \xi_1^{k-h}}, \quad \mu = \frac{2+k-h}{2}, \quad \nu = \frac{1-h}{2+k-h}.$$

The second term in the above solutions is inapplicable to the present problems because it makes a singular point at $\xi = 0$ provided that the μ and ν take appropriate values ($\mu\nu < 1/2$) in assumption of the distributions of the rigidity and the mass in tankers. Accordingly, the integration constant B_1 may be taken as zero, and we have the following normal function in the variable part of the stern of a ship:

$$\eta = A_1 \xi^{\mu\nu} J_{-\nu}(\lambda/\mu \cdot \xi^\mu) \qquad 0 \leq \xi \leq \xi_1, \tag{d}$$

with the slope

$$\frac{d\eta}{d\xi} = -A \cdot \lambda \xi^{\mu(1+\nu)-1} J_{-\nu+1}(\lambda/\mu \cdot \xi^\mu). \tag{e}$$

Since the slope is not to be infinite at $\xi = 0$, μ should be taken as larger than 1/2 in the above solution. On the other hand, the normal function of the hull vibration in way of the parallel part of the ship is easily obtained as follows:

$$\eta = A_2 \cos \alpha \lambda \xi + B_2 \sin \alpha \lambda \xi \qquad \xi_1 \leq \xi \leq \xi_3. \tag{f}$$

The normal function of the vibration of the fore part of the hull of a tanker is assumed to be the same as (d) provided that the coordinate ξ is taken as zero at the extreme fore end of the hull, as follows;

$$\eta = A_3 \bar{\xi}^{\bar{\mu}\bar{\nu}} J_{-\bar{\nu}}\left(\frac{\alpha}{\beta} \cdot \frac{\lambda}{\bar{\mu}} \bar{\xi}^{\bar{\mu}}\right), \quad 0 \leq \bar{\xi} \leq \bar{\xi}_1, \tag{g}$$

where,

$$\alpha = \frac{\lambda_2}{\lambda}, \quad \beta = \frac{\lambda_2}{\lambda_3}, \quad \lambda_3^2 = \frac{p^2 w_3 L^2}{g(k'A)_3 G \bar{\xi}^{\bar{k}-\bar{h}}}, \quad \bar{\mu} = \frac{2+\bar{k}-\bar{h}}{2}, \quad \bar{\nu} = \frac{1-\bar{h}}{2+\bar{k}-\bar{h}}.$$

The following boundary conditions should be considered:

$$\left.\begin{aligned}
&\xi = 0; \quad \eta = 1, \\
&\xi = \xi_1; \quad \eta_1 = \eta_2 \quad \text{and} \quad (k'A)_1 G \frac{d\eta_1}{d\xi} = (k'A)_2 G \frac{d\eta_2}{d\xi}, \\
&\xi = \xi_3 \quad \text{or} \quad \bar{\xi} = \bar{\xi}_1; \quad \eta_2 = \eta_3 \quad \text{and} \\
&\qquad (k'A)_2 G \frac{d\eta_2}{d\xi} = -(k'A)_3 G \frac{d\eta_3}{d\xi}.
\end{aligned}\right\} \tag{h}$$

Four integration constants and the eigen values are determined by the above conditions respectively as follows;

$$\left.\begin{aligned}
&A_1 = \Gamma(1-\nu) \cdot (\lambda/2\mu)^{\nu}, \quad (\Gamma(\ ); \text{ Gamma function}), \\
&A_2 = A_1 \frac{\xi_1^{\mu\nu}}{\alpha'} \left\{ J_{-\nu}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \alpha' \cos \alpha \lambda \xi_1 + \xi_1^{\mu-1} J_{-\nu+1}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \sin \alpha \lambda \xi_1 \right\}, \\
&B_2 = A_1 \frac{\xi_1^{\mu\nu}}{\alpha'} \left\{ J_{-\nu}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \alpha' \sin \alpha \lambda \xi_1 - \xi_1^{\mu-1} J_{-\nu+1}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \cos \alpha \lambda \xi_1 \right\}, \\
&A_3 = \frac{A_2 \cos \alpha \lambda \xi_3 + B_2 \sin \alpha \lambda \xi_3}{\bar{\xi}_1^{\bar{\mu}\bar{\nu}} J_{-\bar{\nu}}\left(\frac{\alpha}{\beta} \frac{\lambda}{\bar{\mu}} \bar{\xi}_1^{\bar{\mu}}\right)}.
\end{aligned}\right\} \tag{i}$$

$$\begin{aligned}
&\frac{J_{-\nu}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \alpha' \cos \alpha \lambda \xi_1 + \xi_1^{\mu-1} J_{-\nu+1}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \sin \alpha \lambda \xi_1}{J_{-\nu}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \alpha' \sin \alpha \lambda \xi_1 - \xi_1^{\mu-1} J_{-\nu+1}\left(\frac{\lambda}{\mu}\xi_1^{\mu}\right) \cos \alpha \lambda \xi_1} \\
&= \frac{J_{-\bar{\nu}}\left(\frac{\alpha}{\beta}\frac{\lambda}{\bar{\mu}}\bar{\xi}_1^{\bar{\mu}}\right) \beta' \cos \alpha \lambda \xi_3 - \bar{\xi}_1^{\bar{\mu}-1} J_{-\bar{\nu}+1}\left(\frac{\alpha}{\beta}\frac{\lambda}{\bar{\mu}}\bar{\xi}_1^{\bar{\mu}}\right) \sin \alpha \lambda \xi_3}{J_{-\bar{\nu}}\left(\frac{\alpha}{\beta}\frac{\lambda}{\bar{\mu}}\bar{\xi}_1^{\bar{\mu}}\right) \beta' \sin \alpha \lambda \xi_3 + \bar{\xi}_1^{\bar{\mu}-1} J_{-\bar{\nu}+1}\left(\frac{\alpha}{\beta}\frac{\lambda}{\bar{\mu}}\bar{\xi}_1^{\bar{\mu}}\right) \cos \alpha \lambda \xi_3}.
\end{aligned} \tag{j}$$

where, $\alpha' = \lambda_2(k'A)_2/\lambda(k'A)_1$, $\beta' = \lambda_2(k'A)_2/\lambda(k'A)_3$.

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