# PERMANENT GRAVITY WAVES ON WATER OF UNIFORM DEPTH

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# PERMANENT GRAVITY WAVES ON WATER OF UNIFORM DEPTH

By Hikoji YAMADA

**Summary.** *Lévi-Civita*'s formulation (as an eigenvalue problem) of permanent water wave problem, which deals with a field function, holomorphic in the unit circle and satisfying a certain boundary condition on the circumference of it, are extended to the periodic waves in a canal of uniform depth, deep water waves and solitary wave being two special cases of it Application of an iterative procedure on this formulation supplies sufficiently accurate results, example being given. Somc discussions on mass transport and eigenvalue are also included.

§ 1. **Transformation onto unit circle.** We have already treated the periodic waves of permanent type on deep water,(1) which is the *Lévi-Civita*'s case,(2) and the solitary wave in a canal of uniform depth.(3) In these cases the leading principles of calculation are the transformation $z=z(\zeta)$ of the water region $z$ (complex) onto a unit circle $|\zeta|\leqq 1$, and the determination of the field quantity $\Omega(\zeta)=\theta+i\tau$, where $\theta$ is the direction angle of the flowing velocity and $\tau=\log q$, $q$ being the magnitude (measured by means of the velocity $U$ at infinity as unit) of the velocity, by an iterative procedure as accurate as we are required. In this paper we deal with the periodic irrotational waves in a canal of uniform depth 4) by the same principles, such that the two cases mentioned above are the special cases of our present one.

We observe the waves from the coordinates system $O$-$xy$ which follows after the permanent waves as fast as the waves, so that the wave form stands fixed relative to the axes, and water flows steadily from left to right (say). The origin $O$ is at one wave crest, $x$-axis is horizontal and directed to right, $y$-axis vertical and upward (Fig. 1 $a$).

Let the wave-length be denoted by $L$ and the mean depth by $D$. We define the wave velocity $U$ by the formula

$$UL=\int_{-L/2}^{L/2} u(x,y)\,dx=\varphi\left(\frac{L}{2}\right)-\varphi\left(-\frac{L}{2}\right), \tag{1}$$

where $\varphi$ is the potential function and $u(x,y)$ is the horizontal component of the

(1) H. Yamada, Reports of Res. Inst. Appl. Mech. Kyushu Univ. (this journal), vol. 5 (1957), pp. 37–52 and 143–155.
(2) T. Lévi-Civita, Math. Annalen, vol. 93 (1925), p. 264.
(3) H. Yamada, Reports of Res. Inst. Appl. Mech. Kyushu Univ. (this journal), vol. 5 (1957), pp. 53–67 and vol. 6 (1958), pp. 35-47.
(4) This is the case treated by K. J. Struik, Math. Annalen, vol. 95 (1926), pp. 595-634; our calculation in this paper is, we think, much more simple and perspective.

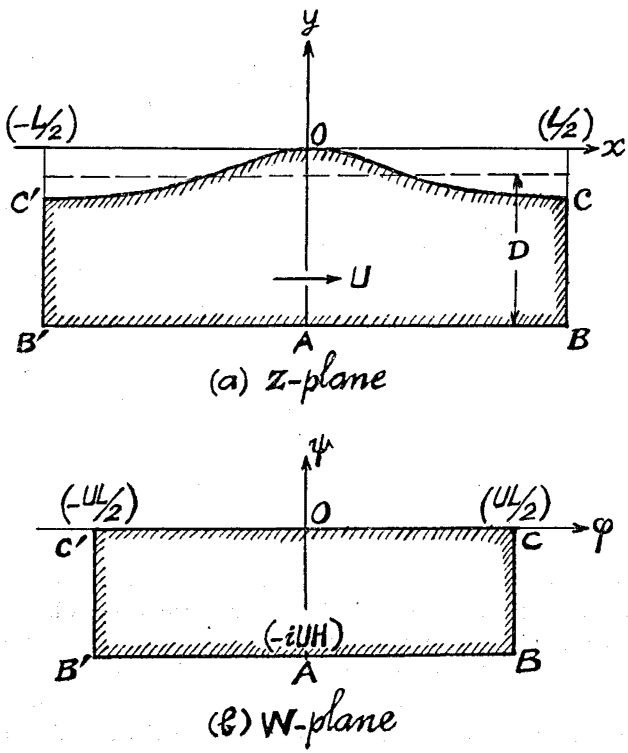


Fig. 1.

flow velocity. The wave velocity thus defined is coincident with that of *Stokes*' waves when $D$ tends to infinity, and with that of solitary wave when $L$ tends to infinity.

The complex potential function

$$W(z)=\varphi+i\psi, \quad (2)$$

in which $\psi$ is the stream function and arbitrary constant is fixed so that $W(0)=0$, maps the physical $z$-plane onto the $W$-plane as shown in Fig. 1*b*, the same alphabetical letters indicating the same points of special interest. The distance $OA$ of the $W$ plane is the flux through any sectional plane of the flow, and when we denote it by $UH$, by use of above defined $U$, $H$ is the length nearly equal to $D$, but differs a little; the difference has an important physical meaning which will be referred to later.

Now we introduce the complete elliptic integrals, $K(k)$ and $K'=K(k')$, of the first kind with the modulus $k$ and its complementary modulus $k'=\sqrt{1-k^2}$, and define the numerical value of $k$ by the relation

$$\frac{K'}{K}=2\frac{H}{L}. \tag{3}$$

With $k$, $k'$, $K$ and $K'$ thus defined we do successive transformations:

$$W_1=\frac{2K}{UL}W, \tag{4}$$

$$Z=sn(W_1+iK',k)=\frac{1}{k\cdot sn(W_1,k)}, \tag{5}$$

$$Z_1=\log\frac{\frac{1}{k}+Z}{\frac{1}{k}-Z}=\log\frac{sn(W_1,k)+1}{sn(W_1,k)-1}, \tag{6}$$

and

$$\zeta = -\tanh^2\left(\frac{Z_1}{4}\right), \qquad (7)$$

or

$$i\sqrt{\zeta} = \frac{e^{Z_1/2}-1}{e^{Z_1/2}+1}. \qquad (7')$$

In these transformations *sn* is the *Jacobian* elliptic function with modulus $k$, and the mappings of each steps are as shown in the figures 2*a*–2*d* which will be understood easily. One point of notice is the cut (barrier) from $\zeta=0$ to $\zeta=-1$, the points $B$, $C$ lying on one side and the points $B'$, $C'$ on the other side of the cut. The position of $B$ (or $B'$) is

$$\zeta_B = 1-\frac{2}{k^2}(1-\sqrt{1-k^2}). \quad (8)$$

By this series of transformations one wave-length region of the $W$-plane (from one trough to next trough) is mapped onto the unit circle (with cut) in the $\zeta$-plane, the transformations being combined into one equation:

$$sn\left(\frac{2K}{UL}W, k\right) = -i\frac{1-\zeta}{2\sqrt{\zeta}}. \quad (9)$$

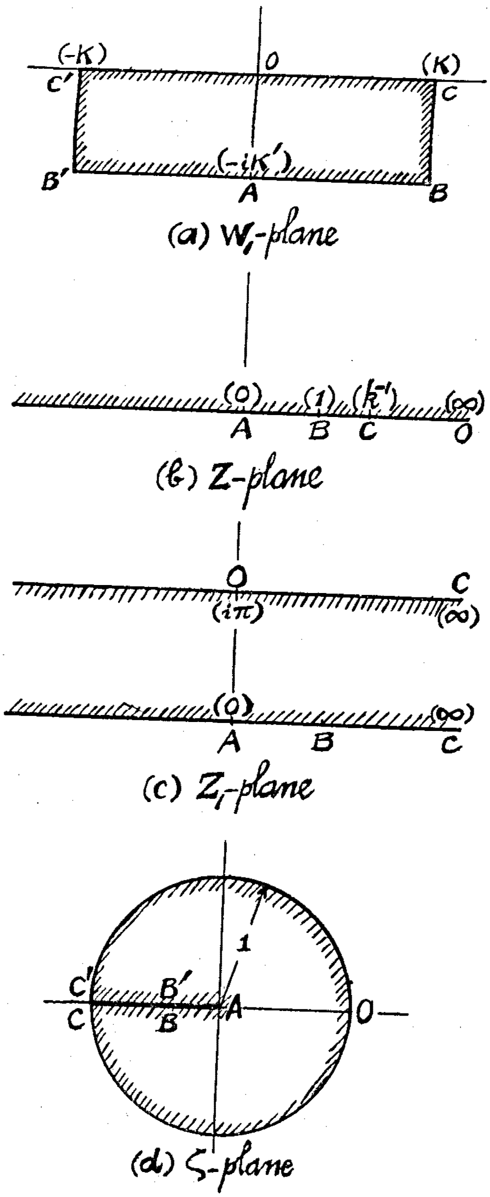


Fig. 2.

**§ 2. Correspondence between $z$ and $\zeta$.** (9) is the first relation we aimed at, but as our object is the relation between the physical plane $z$ and the unit circle $\zeta$, we must have anew one relation between $z$ and $W$. Such a relation is the complex velocity:

$$\frac{dW}{dz} = |v|e^{-i\theta}, \qquad (10)$$

where $|v|$ and $\theta$ is the speed and direction of flowing water, $\theta$ being the angle measured upwards from the horizontal direction (to right). We denote $|v|/U$ by $q$ and $\log q$ by $\tau$, in accordance with the special cases stated above, and then (10) becomes to

$$\frac{1}{U}\frac{dW}{dz} = q\,e^{-i\theta} = e^{-i\Omega(z)}, \qquad (10')$$

the field quantity $\Omega(z)$ being defined by the relation:

$$\Omega(z) = \theta + i\tau = i \log\left(\frac{1}{U}\frac{dW}{dz}\right). \tag{11}$$

Evidently $\Omega(z)$ is holomorphic at every interior point of the water region, and may have singular points at the boundary of the region.

As $z$ has to be a certain function of $\zeta$, holomorphic at every interior point of the cut unit circle, we have from (10')

$$\frac{1}{U}\cdot\frac{dW}{d\zeta}\cdot\frac{d\zeta}{dz} = e^{-i\Omega(\zeta)} \qquad i.e. \qquad dz = \left(\frac{1}{U}\frac{dW}{d\zeta}\right)e^{i\Omega(\zeta)}d\zeta\,, \tag{12}$$

where $\Omega(\zeta)$ is the transformed of $\Omega(z)$ on the $\zeta$-plane. On the other hand, differentiating (9) with regard to $\zeta$, we have

$$\frac{1}{U}\frac{dW}{d\zeta} = \frac{L}{2K}\frac{1}{cn\left(\frac{2K}{UL}W\right)dn\left(\frac{2K}{UL}W\right)}\cdot\frac{i}{4}\frac{1+\zeta}{\sqrt{\zeta^3}}.$$

Also from (9) it results

$$cn\left(\frac{2K}{UL}W\right) = \frac{1+\zeta}{2\sqrt{\zeta}}, \qquad dn\left(\frac{2K}{UL}W\right) = \sqrt{1+\frac{k^2(1-\zeta)^2}{4\zeta}}\,,$$

and making use of these relarions in above one, we have

$$\frac{1}{U}\frac{dW}{d\zeta} = i\frac{L}{4K}\frac{1}{\sqrt{\zeta^2+\frac{k^2}{4}\zeta(1-\zeta)^2}}. \tag{13}$$

Finally combining (12) and (13), it results

$$dz = i\frac{L}{4K}\frac{1}{\sqrt{\zeta^2+\frac{k^2}{4}\zeta(1-\zeta)^2}}e^{i\Omega(\zeta)}d\zeta\,, \tag{14}$$

which is our required relation between $z$ and $\zeta$.

In (14), however, $\Omega(\zeta)$ is unknown presently, and the determination of it is the essential point of our wave problem. When it is determined (14) can be integrated, giving $z = z(\zeta)$ which is our final object.

As our wave form is symmetrical about the vertical line through the crest *i.e.* $Oy$-axis, it can easily be verified that the function $-i\Omega(z) = \tau - i\theta$ has conjugate complex values at every pair of symmetrical points $z = x + iy$ and $z = -x + iy$. To this pair of points, on the other hand, corresponds a pair of two points of $W$-plane, situated symmetrically about $\psi$ axis, by the due choice of origin of $W$ above stated. But now these two points becoming, by the transfarmation (9), to a pair of conjugate points in the $\zeta$-plane, we know that $-i\Omega(\zeta)$ has a pair of complex conjugate values at every pair of conjugate points in the unit circle $|\zeta| \leq 1$. Above all it takes real value on the real axis because of the horizontal velocity at points under crest or trough, or at the bottom, and all along the cut $-1 \leq \zeta < 0$ values of the two sides coincide by reason of the periodicity of waves. Thus the function $-i\Omega(\zeta)$, and consequently $\Omega(\zeta)$ itself, is not only holomorphic in the cut unit circle, but also in the unit circle without cut.

The precise functional from of $\Omega(\zeta)$ has to be determined by the condition prescribed on the boundary $|\zeta|=1$, which corresponds to the free surface of the physical plane. If desired, we can expand $\Omega(\zeta)$ into a power series of $\zeta$:

$$\Omega(\zeta)=i\sum_{n=0}^{\infty}a_n\zeta^n \tag{15}$$

about the origin $\zeta=0$ and the expansion coefficients $a_n$'s which are all real, are to be determined by the boundary condition.

§ 3. **Surface condition.** Along the streamline which constitute the free surface of water prevails a constant (atmospheric) pressure and by the *Bernoulli*'s equation we have

$$q^2+\frac{2g}{U^2}y=\text{const}, \tag{16}$$

or differentiating this along the arc $s$ of this streamline

$$q\frac{dq}{ds}+\frac{g}{U^2}\sin\theta=0, \tag{17}$$

where $\theta$ is the inclination of free surface velocity.

When we take $dz$ along the free surface this is equal to $ds\,e^{i\theta}$ and has to correspond to $d\zeta=ie^{i\sigma}d\sigma$ on the unit circle $\zeta=e^{i\sigma}$, $\sigma$ being the arc length of the unit circle. The correspondence is given by (14) and by use of the expression (11) we arrive easily at the relation:

$$ds=-\frac{L}{4K}\frac{d\sigma}{q\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}. \tag{18}$$

When we take this relation into (17), which is the surface consition in the physical plane, it can now be trasformed into a condition in the $\zeta$-plane:

$$q^2\frac{dq}{d\sigma}=p\frac{\sin\theta}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}, \tag{19}$$

where

$$p=\frac{gL}{4KU^2}. \tag{20}$$

(19) is the boundary condition at $|\zeta|=1$, which is, as cited above, neccessary for the determination of $\Omega(\zeta)$, and $p$ is the eigenvalue which has to be determined simultaneously with $\Omega(\zeta)$. From $p$ the wave velocity $U$ follows at once.

Now our mathematical formulation is thus settled, and we can see that the two cases treated already (*Stokes*' waves and solitary wave) are the special cases ($k=0$ and $k=1$) of our present one:

For $k=0$ we have $K=\pi/2$, $K'=\infty$ and then $H=\infty$ by use of (3), *i.e.* the case of deep water. In this case (14), (19) and (20) reduce to

$$\left.\begin{aligned} dz &= i\frac{L}{2\pi}\frac{1}{\zeta}e^{i\Omega(\zeta)}d\zeta\,, \\ q^2\frac{dq}{d\sigma} &= p\sin\theta\,, \\ p &= \frac{gL}{2\pi U^2}\,, \end{aligned}\right\} \tag{21}$$

and this is the very formulation of *Lévi-Civita.*[6]

For $k=1$ we have $K=\infty$, $K'=\pi/2$ and then $L=\infty$ by (3), *i.e.* the case of solitary wave. Then making use of (3) again we know that

$$\frac{L}{2K}=\frac{H}{K'}=\frac{2H}{\pi}\,,$$

and by means of this (14), (19) and (20) reduce to

$$\left.\begin{aligned} dz &= i\frac{2H}{\pi}\frac{1}{\sqrt{\zeta}\,(1+\zeta)}e^{i\Omega(\zeta)}d\zeta\,, \\ q^2\frac{dq}{d\sigma} &= p\sin\theta\,\sec\left(\frac{\sigma}{2}\right), \\ p &= \frac{gH}{\pi U^2}\,, \end{aligned}\right\} \tag{22}$$

which is our formulation of the solitary wave recently reported.[7]

§ 4. **Iterative determination of $\Omega$.** As the direct determination of $\Omega(\zeta)$, which satisfies the condition (19) is difficult generally, we give here an iterative procedure which enables us the determination of any desired order of accuracy. For the sake of simple calculation[8] we use the variable $Q(\sigma)$ defined by

$$\frac{q}{(3p)^{1/3}}=Q \qquad i.e. \qquad \tau=\log Q+\frac{1}{3}\log(3p) \tag{23}$$

instead of $q$. Then (19) is changed into

$$\frac{dQ^3}{d\sigma}=\frac{\sin\theta}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}\,,$$

and when integrated it becomes to

$$Q^3(\sigma)-Q^3(0)=\int_0^\sigma\frac{\sin\theta(\sigma')}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma'\,, \tag{24}$$

which we use in place of (19).

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(6) See the paper of the footnote (1) or (2).

(7) *c.f.* footnote (3).

(8) The procedure here described is the simplified version of the method used in our preceding reports.

To determine the wave precisely one more conatant, it is well known, is required and we take as this constant the ratio of the surface velocities at crest and trough:

$$\rho=\frac{Q(0)}{Q(\pi)}=\frac{q(0)}{q(\pi)}, \tag{25}$$

and making use of this in (24), with $\sigma=\pi$, we have the relation which connects $Q(0)$ to the given $\rho$:

$$Q^3(0)=\frac{\rho^3}{1-\rho^3}\int_0^\pi \frac{\sin\theta(\sigma')}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma'. \tag{26}$$

Now the iterative procedure: we take an approximate function $\theta(\sigma)$, given in some arbitrary way, as the starting approximation, and by means of this calculate $Q(0)$ by (26), and then $Q(\sigma)$ by (24), or by

$$Q(\sigma)=(1-\rho^3)^{-1/3}\left\{\int_0^\sigma \frac{\sin\theta(\sigma')}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma'+\rho^3\int_\sigma^\pi \frac{\sin\theta(\sigma')}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma'\right\}^{1/3}. \tag{27}$$

As this $Q(\sigma)$ is connected to $\tau(\sigma)$ by (23), we have the values of $\tau(\sigma)$ determined but an additive constant. From $\tau(\sigma)$ we obtain $\theta(\sigma)$ which is the conjugate function of $\tau(\sigma)$:

$$\theta(\sigma)=\text{const.}-\frac{1}{2\pi}\int_0^{2\pi}\{\tau(\sigma')-\tau(\sigma)\}\cot\frac{\sigma'-\sigma}{2}d\sigma', \tag{28}$$

and if we prefer to the power series expansion (16), $\tau(\sigma)$ has to be expressed in a *Fourier* series:

$$\tau(\sigma)=\sum_{n=1}^{\infty}a_n\cos(n\sigma)+\begin{matrix}\textit{undetermined}\\ \textit{constant}\end{matrix}, \tag{29}$$

and $\theta(\sigma)$ is then given at once:

$$\theta(\sigma)=-\sum_{n=1}^{\infty}a_n\sin(n\sigma). \tag{30}$$

The undetermined constant of $\tau(\sigma)$ has no influence on $\theta(\sigma)$, for such a constant of $\theta(\sigma)$ is determined by the condition $\theta(0)=0$.

The value of $\theta(\sigma)$ thus obtained is the second approximation, and is generally different from the starting first approximation.(9) We repeat above calculation then taking the second approximation as the starting one and the third approximation for $\theta(\sigma)$ is obtained, and so on. Such cycles of calculation have to be repeated until the difference between the starting and resulting ones of a cycle is within the permissible error, and then $\theta(\sigma)$ and $Q(\sigma)$ are determined.

(9) Iterative procedure in this sense converges actually, as will be seen in the following example, and also in the examples already reported. If this procedure were divergent, the iteration in the inverse sense, using (19) instead of (24), would be hoped for convergence.

To fix the value of eigenvalue $p$ we may use any one relation which contains $q(\sigma)$, for by use of the equation (23) or

$$q(\sigma)=(3p)^{1/3}\cdot Q(\sigma)\,, \tag{23'}$$

we can bring the eigenvalue into an equation as a sole unknown quantity. Here we take up as such a relation the expression of wave-length. As the wave form is given by

$$dz=ds\,e^{i\theta}=-\frac{L}{4K}\frac{e^{i\theta}}{q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}d\sigma\,, \tag{31}$$

which is nothing but the equation (18), we take its real part:

$$-\frac{L}{2}=\int_0^{-L/2}dx=-\frac{L}{4K}\int_0^{\pi}\frac{\cos\theta}{q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}d\sigma\,, \tag{31'}$$

from which follows the aimed equation at once:

$$(3p)^{1/3}=\frac{1}{2K}\int_0^{\pi}\frac{\cos\theta(\sigma)}{Q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}d\sigma\,; \tag{32}$$

$p$ can be obtained by mere quadrature.

Lastly $\tau(\sigma)$, or $q(\sigma)$, is obtained by means of (23), or (23′), and $\Omega(\zeta)$ for any inner point $\zeta$ of the unit circle is expressed by the *Schwarz-Poisson*'s formula:

$$\Omega(\zeta)=ia_0+\frac{1}{2\pi}\int_0^{2\pi}\theta(\sigma)\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}d\sigma\,. \tag{33}$$

When *Fourier* expansion (29) is used $\Omega(\zeta)$ can be written down at once by (15).

**§ 5. A numerical example.** The case of $k=\sin 80^\circ=0.98481$ ($K=3.1534$, $K'=1.5828$) is taken up as an example. By (3) and (8)

$$\frac{H}{L}=0.2510\left(\frac{L}{H}=3.984\right),\qquad \zeta_B=-0.7041\,.$$

Here we calculate the highest wave, and, as in the preceding papers, we employ

$$\Omega_0(\zeta)=\frac{i}{3}\log\frac{1-\zeta}{2} \tag{34}$$

*i. e.*

$$\theta_0(\sigma)+i\tau_0(\sigma)=\frac{\pi-\sigma}{6}+\frac{i}{3}\log\sin\left(\frac{\sigma}{2}\right) \tag{34'}$$

$$(\pi\geqq\sigma\geqq 0)$$

as a rough approximation for $\Omega(\zeta)$ in the neighborhood of the singular point $\zeta=1$, which corresponds to the angular crest.

As the starting value of $\theta(\sigma)$ we have used the solution for the highest solitary wave, which we have reported recently, expecting in vain that our present solution is near to it. $\Omega$ is expressed:

$$\Omega(\zeta) = \Omega_0(\zeta) + i \sum_{n=0}^{\infty} a_n \zeta^n, \tag{35}$$

and then

$$\left.\begin{aligned} \theta(\sigma) &= \theta_0(\sigma) - \sum_{n=1}^{\infty} a_n \sin(n\sigma), \\ \log Q(\sigma) &= \tau_0(\sigma) + a_0' + \sum_{n=1}^{\infty} a_n \cos(n\sigma), \\ a_0 &= a_0' + \frac{1}{3}\log(3p). \end{aligned}\right\} \tag{35'}$$

In the course of the iteration calculus $\log Q(\sigma)$ has, as stated above, been used in place of $\tau(\sigma)$.

We take $a$'s up to $a_{12}$ and assume the others zero. Seven cycles of iteration were necessary for convergence, and results are as given in Table 1. Determination of $p$ and $a_0$ is done along the scheme given above. Numerical integration by *Simpson*'s rule is used, the division of the interval being 7.5°; at $\sigma = 0$ $Q(\sigma)$ vanishes and some cautions were necessary for the evaluation of (32). From $p$-value it follows:

$$U = \sqrt{\frac{\pi}{2pK} \cdot \frac{gL}{2\pi}} = 1.050 \sqrt{\frac{gL}{2\pi}}$$

or

$$U = \sqrt{\frac{1}{2pK'} \cdot gH} = 0.8361\sqrt{gH}.$$

Table 1.

| | |
|---|---|
| $(a_0')$ | (+ 0.0080) |
| $a_0$ | + 0.1094 |
| $a_1$ | − 0.0442 |
| $a_2$ | + 0.0246 |
| $a_3$ | − 0.0023 |
| $a_4$ | + 0.0054 |
| $a_5$ | + 0.0005 |
| $a_6$ | + 0.0023 |
| $a_7$ | + 0.0009 |
| $a_8$ | + 0.0014 |
| $a_9$ | + 0.0009 |
| $a_{10}$ | + 0.0010 |
| $a_{11}$ | + 0.0009 |
| $a_{12}$ | + 0.0005 |
| $p$ | 0.4519 |

$\theta(\sigma)$ and $q(\sigma)$ calculated by (35′) and (23′) are shown in Table 2; numbers in parentheses in the second column are the starting values of the last cycle and we see good convergence. There is tabulated also the wave form $(x/L, y/L)$ calculated by (31), whose integrals are improper by reason of $q(0) = 0$. The wave form is inscribed also in Fig. 3, in comparison with the deep water wave, and we see more resemblance of the two forms than in our preoccupied vision.

One test of the accuracy of the solution is the degree of satisfaction of the surface condition (16), which in this case of extreme height becomes to

$$q^2 + 8pK\frac{y}{L} = 0. \tag{16'}$$

With the values in Table 2 ($\sigma = 180°$), the two terms on the left side are 1.4566 and − 1.4562, and the sum of them 0.0004 is equal to zero within the accuracy of numerical integration.

This accuracy of numerical integration can be known comparing the integral values:

$$\int dz = \int \left[ i\frac{L}{4K} \frac{e^{i\Omega}}{\sqrt{\zeta^2 + \frac{k^2}{4}\zeta(1-\zeta)^2}} \right] d\zeta \tag{36}$$

Table 2.

| $\sigma$ (deg.) | $\theta$ (rad.) | $q$ | $-x/L$ | $-y/L$ |
|---|---|---|---|---|
| 0 | 0.5236 (0.5236) | 0.0000 | 0.0000 | 0.00000 |
| 7.5 | 0.4923 (0.4923) | 0.4435 | | |
| 15 | 0.4698 (0.4699) | 0.5525 | 0.0491 | 0.02678 |
| 22.5 | 0.4544 (0.4545) | 0.6277 | | |
| 30 | 0.4385 (0.4386) | 0.6893 | 0.0796 | 0.04170 |
| 37.5 | 0.4196 (0.4197) | 0.7405 | | |
| 45 | 0.4020 (0.4020) | 0.7829 | 0.1067 | 0.05379 |
| 52.5 | 0.3865 (0.3865) | 0.8209 | | |
| 60 | 0.3710 (0.3711) | 0.8567 | 0.1327 | 0.06440 |
| 67.5 | 0.3538 (0.3539) | 0.8896 | | |
| 75 | 0.3366 (0.3367) | 0.9189 | 0.1589 | 0.07407 |
| 82.5 | 0.3205 (0.3206) | 0.9464 | | |
| 90 | 0.3041 (0.3042) | 0.9736 | 0.1864 | 0.08316 |
| 97.5 | 0.2863 (0.2864) | 0.9994 | | |
| 105 | 0.2680 (0.2681) | 1.0235 | 0.2161 | 0.09190 |
| 112.5 | 0.2495 (0.2496) | 1.0466 | | |
| 120 | 0.2300 (0.2301) | 1.0697 | 0.2496 | 0.10043 |
| 127.5 | 0.2093 (0.2093) | 1.0922 | | |
| 135 | 0.1870 (0.1870) | 1.1140 | 0.2894 | 0.10885 |
| 142.5 | 0.1632 (0.1633) | 1.1350 | | |
| 150 | 0.1371 (0.1372) | 1.1551 | 0.3397 | 0.11705 |
| 157.5 | 0.1079 (0.1080) | 1.1739 | | |
| 165 | 0.0756 (0.0757) | 1.1903 | 0.4080 | 0.12430 |
| 172.5 | 0.0393 (0.0393) | 1.2025 | | |
| 180 | 0.0000 (0.0000) | 1.2069 | 0.5000 | 0.12775 |

taken between any two points of special interest. As the integral along a closed contour which lies wholly in the water region vanishes identically, we must have (*c.f.* Fig. 1*a* and 2*d*)

$$\frac{1}{L}\int_{A}^{B'}\Big[\qquad\qquad\Big]d\zeta=\Re\left\{\frac{1}{L}\int_{0}^{C'}\Big[\qquad\qquad\Big]d\zeta\right\}, \tag{37}$$

and

$$-\frac{i}{L}\int_{A}^{0}\Big[\qquad\qquad\Big]d\zeta=-\frac{i}{L}\int_{B'}^{C'}\Big[\qquad\qquad\Big]d\zeta+\Im\left\{\frac{1}{L}\int_{C'}^{0}\Big[\qquad\qquad\Big]d\zeta\right\}, \tag{37'}$$

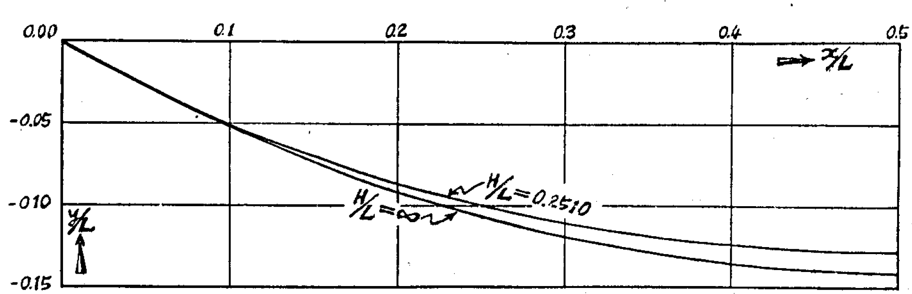


Fig. 3

where $\Re$ and $\Im$ denoting the real and imaginary parts of the expression in { }. The right-hand side of (37) is 1/2 by (31′), and its left-hand side gives the numerical value 0.4999. The left-hand side of (37′) is 0.3480 and the two terms on the right-hand side are 0.12775 and 0.22031, their sum being 0.3481. Thus the error seems to be several tenth of one per mille. Integrals are all done by *Simpson's* rule of 7.5° division with proper cautions at the improper ends.

§ 6. **Mass transport.** The time duration $T(\psi)$, which is necessary for a water particle on a streamline $\psi$ to go through one wave-length distance, is given by[10]

$$T(\psi)=\int_{\varphi=-\frac{UL}{2}}^{\frac{UL}{2}}\frac{ds}{Uq}=\int_{-\frac{UL}{2}}^{\frac{UL}{2}}\frac{d\varphi}{U^2q^2}. \qquad (38)$$

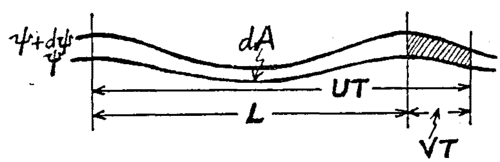


Fig. 4.

In this duration the wave advances the disttance $UT(\psi)$, *i.e.* the coordinates of reference in which waves have the velocity $U$ recedes the the distance $UT(\psi)$ backwards. In this coordinates then particles advance the distance $UT(\psi)-L$ meanwhile, and the velocity of this advance is, in mean,

$$V(\psi)=\frac{UT(\psi)-L}{T(\psi)}. \qquad (39)$$

Next we denote by $dA$ the water quantity included in a wave-length, and by $dQ(\psi)$ the water quantity of advance per unit time (mass transport), both bounded by two streamlines $\psi$ and $\psi+d\psi$. Then by the definition of $V(\psi)$

$$dQ(\psi)=\frac{dA}{L}\cdot V(\psi),$$

and by the definitions of stream function and $T(\psi)$

$$dA=d\psi\cdot T(\psi).$$

Combing these two equations we have

$$dQ(\psi)=\frac{V(\psi)T(\psi)}{L}d\psi=\left\{\frac{U}{L}T(\psi)-1\right\}d\psi, \qquad (40)$$

and then the expression of the total mass transport $Q$ follows at once:

$$Q=\int_{-UH}^{0}\left\{\frac{U}{L}T(\psi)-1\right\}d\psi, \qquad (41)$$

which, by means of (38), can be written into

$$Q=\frac{1}{UL}\int\int d\varphi\, d\psi(q^{-2}-1), \qquad (41')$$

(10) In the followings *c.f.* *F. Ursell*, Proc. Camb. Phil. Soc., vol. 49 (1953), pp. 145–150.

where the integral is to be extended over one wave-length domain in the $W$-plane (hatched area in Fig. 1$b$).

On the other hand, extending the integral over the same area, we have

$$LD = \int\int \frac{\partial(x, y)}{\partial(\varphi, \psi)} d\varphi\, d\psi = \int\int \frac{1}{U^2 q^2} d\varphi\, d\psi\,, \tag{42}$$

or

$$\frac{D}{L} = \frac{1}{(UL)^2} \int\int \frac{1}{q^2} d\varphi\, d\psi\,, \tag{42'}$$

and introducing this into (41′) we have

$$Q = U(D-H)\,, \tag{43}$$

which is the result, anticipated above, combining $D$, $H$ and $Q$.

In the preceding example

$$D\left(-\frac{L}{2}\right) = \int_0^{-L/2} (A+y)\,dx = A\left(-\frac{L}{2}\right) + L^2 \int_0^{-1/2} \left(\frac{y}{L}\right) d\left(\frac{x}{L}\right),$$

and the last integral has to be evaluated by means of values in the last two columns in Table 2. This is done making use of the *Gauss*' interpolation formula for every set of three successive points, and gave the value $0.04417\,L^2$. Then $D/L = 0.2597$, and $Q/UL = 0.0087$ by (43).

§ 7. **Distribution of eigenvalue $p$.** We have seen that the permanent periodic waves in a canal of uniform depth constitute a two-parametric family of solutions, parameters being $k$, the modulus, and $p$, the eigenvalue. For a given $k$ $(0 \leq k \leq 1)$ there is a maximum and a minimum values of $p$, maximum belonging to infinitesimal amplitude and minimum to the highest one.

For the infinitesimal waves $H$ is equal to $D$, and the well-known velocity formula:

Table 3.

| $\theta$ | $k(=\sin\theta)$ | $L/H$ | $p_{max}$ | $p_{min}$ |
|---|---|---|---|---|
| deg. 0 | 0.0000 | 0.0000 | 1.0000 | 0.8381 |
| 3 | 0.0523 | 0.7246 | 0.9992 | |
| 20 | 0.3420 | 1.2937 | 0.9698 | |
| 45 | 0.7071 | 2.0000 | 0.8504 | |
| 70 | 0.9397 | 3.0920 | 0.6490 | |
| 80 | 0.9848 | 3.984 | 0.5427 | 0.4519 |
| 85 | 0.9962 | 4.869 | 0.4772 | |
| 87 | 0.9986 | 5.519 | 0.4449 | |
| 88 | 0.9994 | 6.039 | 0.4255 | |
| 89 | 0.9998 | 6.920 | 0.4013 | |
| 90 | 1.0000 | ∞ | 0.3183 | 0.1921 |

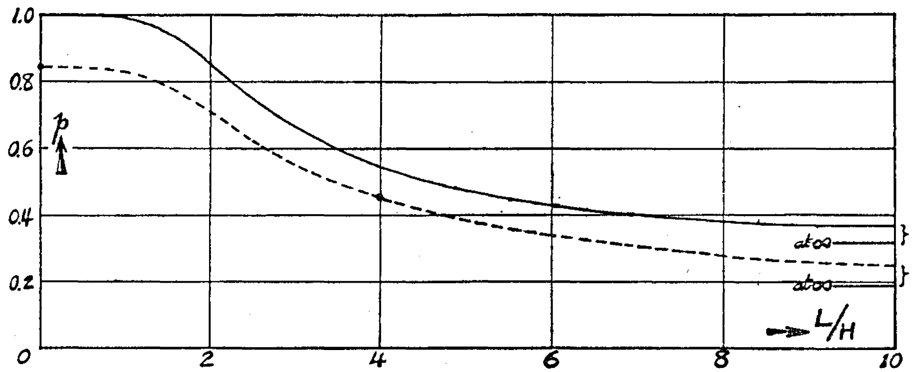


Fig. 5.

$$U^2 = \frac{gL}{2\pi} \tanh\left(2\pi \frac{H}{L}\right) \tag{44}$$

holds. By use of this in (20) maximum $p$ is found:

$$p = \frac{\pi}{2K} \coth\left(\pi \frac{K'}{K}\right). \tag{44'}$$

To obtain minimum $p$ as a function of $k$ it is necessary to solve the problem of highest wave everytime for every value of $k$ following the model example in section 5. Up to the present only three values are known, and these values are given in Table 3, together with the highest values calculated by (44').

With these values the domain for the existence of permanent waves are sketched in Fig. 5, in there $L/H$ being used instead of $k$ for the sake of direct physical meaning.

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