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ON THE THEORY OF HOMOGENEOUS AXISYMMETRIC TURBULENCE

By Michio OHJI

This paper intends to display the formulation of the spectrum of homogeneous axisymmetric turbulence in particular. The explicit representation of the second- and third-order wave-number tensors together with the governing dynamical equations are shown. And further, defining the special types of axisymmetric turbulence, that is, quasi-isotropic, axial and transverse turbulence, respectively, we shall see how to interprete the general axisymmetric one as a composition of these elementary fields of motion. Incidentally, Batchelor-Chandrasekhar's formulation of axisymmetric correlation tensors is quoted with some notes.

1. Introduction. A field of homogeneous turbulence is said to be 'axisymmetric' as well if there is present one and only one preferred direction and all its dynamical properties are symmetrical about this direction ($\boldsymbol{s}$, say) in a statistical sense. More precisely speaking, homogeneous axisymmetric turbulence is specialized by the requirement that the joint-probability distribution of the velocity at any n points should be invariant (1) for arbitrary translations, (2) for arbitray rotations about $\boldsymbol{s}$, and (3) for reflexions in planes containing $\boldsymbol{s}$ and normal to $\boldsymbol{s}$, of the configuration of n points. It is, of course, the next simplest form of turbulence to homogeneous isotropic, or in other words, the simplest case of non-isotropy. The possibility of dealing with such a case was early suggested by H. P. Robertson [5], and his plan was realized by G. K. Batchelor [1], to whom S. Chandrasekhar [3], [4] succeeded.

But it seems to the author that further references concerning this problem have seldom appeared during the subsequent period of turbulence research, and this is the reason why the present paper is written. In the following pages, we shall find the basic expressions and equations of homogeneous axisymmetric turbulence, where the axisymmetric spectrum tensors are introduced. The procedure of analysis lies, in principle, within the range of the general theory of homogeneous turbulence which is summarized in Batchelor's monograph [2], for instance. It may, however, still of some use to elucidate again in what respects the axisymmetry makes an appearance of its own. Thus, besides incompressibility of fluid, no special assumption other than homogeneity and axisymmetry of turbulence is made, and we are solely concerned here with the framework of the theory, reserving its elaboration or application for another occasion.

2. The axisymmetric tensors. Let $\boldsymbol{r}=(r_1, r_2, r_3)$ be the space vector separating two arbitrary points $\boldsymbol{x}$ and $\boldsymbol{x}'$, such that $\boldsymbol{r}=\boldsymbol{x}'-\boldsymbol{x}$, and let $\boldsymbol{s}=(s_1, s_2, s_3)$ be a unit vector with fixed direction, where the index i $(=1, 2, 3)$ indicates the i-th component in orthogonal Cartesian coordinate system.

Then, if the turbulence is homogeneous and moreover axisymmetric about the direction $\boldsymbol{s}$, each of true tensors[1] representing the various two-points correlations referred to the separation $\boldsymbol{r}$ must be expressed, according to its order respectively, as follows[2]:

$$Q_i(\boldsymbol{s}\,;\boldsymbol{r}) = Ar_i + Bs_i, \tag{2.1a}$$

$$Q_{ij}(\boldsymbol{s}\,;\boldsymbol{r}) = Ar_ir_j + B\delta_{ij} + Cs_is_j + Ds_ir_j + Er_is_j, \tag{2.1b}$$

$$\begin{aligned} Q_{ijk}(\boldsymbol{s}\,;\boldsymbol{r}) = {} & Ar_ir_jr_k + Br_i\delta_{jk} + Cr_j\delta_{ki} + Dr_k\delta_{ij} \\ & + Es_is_js_k + Fs_i\delta_{jk} + Gs_j\delta_{ki} + Hs_k\delta_{ij} \\ & + Is_ir_jr_k + Jr_is_jr_k + Kr_ir_js_k \\ & + Lr_is_js_k + Ms_ir_js_k + Ns_is_jr_k, \end{aligned} \tag{2.1c}$$

and so on, $A, B, C, \ldots$ being the defining scalars which are regarded, most generally, as arbitrary scalar functions of the scalar products

$$r_ir_i = r^2 \quad \text{and} \quad r_is_i = rm, \tag{2.2}$$

apart from any possible dependency on the time t (m is the cosine of the angle between $\boldsymbol{r}$ and $\boldsymbol{s}$). This is a well-known result from the invariant theory of Cartesian tensors as is proved in [**1**] and [**3**], and such tensors may be simply called the 'axisymmetric $\boldsymbol{r}$-tensors with the axis $\boldsymbol{s}$.'[3]

Now, turning from $\boldsymbol{r}$-space or physical space to $\boldsymbol{\kappa}$-space or wave number space through the three-dimensional Fourier transform relation defined by

$$\chi_{ij\ldots n}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = \left(\frac{1}{2\pi}\right)^3 \int Q_{ij\ldots n}(\boldsymbol{s}\,;\boldsymbol{r})\,e^{-\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}}\,d\boldsymbol{r}, \tag{2.3}$$

where $\iota=\sqrt{-1}$, and $\boldsymbol{\kappa}$ is the wave-number vector, the 'axisymmetric $\boldsymbol{\kappa}$-tensor with the axis $\boldsymbol{s}$' is introduced in association with each $\boldsymbol{r}$-tensor $Q_{ij\ldots n}(\boldsymbol{s}\,;\boldsymbol{r})$.[4]

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1) The adjective 'true' is used here in contradistinction to the adjective 'skew'. It should be noticed that correlation tensors involving an odd number of the axial vectors, such as the vorticities, are skew, since they take the opposite sign to true ones on reflexion in the origin. But, throughout this section, the skew case is excluded. cf. next section, further [**3**] and [**5**].

2) Here, and in the sequel, the customary notation of Cartesian tensors with the summation convention for repeated tensor indices will be followed. δ_{ij} is thus the unit diagonal tensor.

3) In the strict words, we may say 'homogeneous and axisymmetric...' but from now on we shall frequently omit 'homogeneous and' for simplicity's sake. The subsequent pages should be read in this sense.

4) In (2.3), $d\boldsymbol{r}$ stands for a volume element $dr_1\,dr_2\,dr_3$ of $\boldsymbol{r}$-space, and the integration extends over all space. The similar convention will be consistently used hereafter. Indeed, the definition (2.3) becomes a trivial one when the integral on the right hand side does not converge, but by substituting the relevant cumulant tensor in place of the correlation tensor itself if necessary, we can always ensure the existence of the Fourier transforms occur in the theory. See [**2**], p. 21.

We shall often refer this as the axisymmetric spectrum tensor, for which it is evident that the above theorem still holds in a formal sense, so long as we replace the symbol r by κ, and consider $A, B, C, \ldots$ are now arbitrary scalar functions of

$$\kappa_i \kappa_i = \kappa^2 \qquad \text{and} \qquad \kappa_i s_i = \kappa \mu\,, \tag{2.4}$$

μ being the cosine of the angle between $\boldsymbol{\kappa}$ and $\boldsymbol{s}$. To return from $\boldsymbol{\kappa}$-space to $\boldsymbol{r}$-space, the Fourier inverse relation of (2.3) may be utilized, viz.

$$Q_{ij\ldots n}(\boldsymbol{s}\,;\,\boldsymbol{r}) = \int \chi_{ij\ldots n}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})\, e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}\, d\boldsymbol{\kappa}\,. \tag{2.5}$$

The defining scalars of $\chi_{ij\ldots n}(\boldsymbol{s}\,;\,\boldsymbol{\kappa})$ will be written explicitly in terms of those of $Q_{ij\ldots n}(\boldsymbol{s}\,;\,\boldsymbol{r})$ and *vice versa* by carrying out the integration of the right-hand side of (2.3) or (2.5). In view of the considerable complexity of their general expressions, however, we shall not enter into details here.

3. The double velocity correlation tensor. Following the customary notation, let us write

$$R_{ij}(\boldsymbol{r}, t) = \overline{u_i(\boldsymbol{x}, t) \cdot u_j(\boldsymbol{x}+\boldsymbol{r}, t)} \qquad (= \overline{u_i u_j'}, \text{ in short}), \tag{3.1}$$ [5]

for the double velocity correlation tensor. We have then

$$R_{ij}(\boldsymbol{r}) = R_{ji}(-\boldsymbol{r})\,, \tag{3.2}$$

from homogeneity of turbulence, and at the same time continuity of incompressible fluid requires the solenoidal condition with respect to both indices, saying

$$\frac{\partial R_{ij}(\boldsymbol{r})}{\partial r_i} = \frac{\partial R_{ij}(\boldsymbol{r})}{\partial r_j} = 0\,. \tag{3.3}$$

Now, in the case of axisymmetry, it has been proved by Batchelor [**1**] that this tensor should be symmetrical in its indices, viz.

$$R_{ij}(\boldsymbol{s}\,;\,\boldsymbol{r}) = R_{ji}(\boldsymbol{s}\,;\,\boldsymbol{r})\,, \tag{3.4}$$

in virtue of (3.3) and that accordingly it must have a form

$$R_{ij}(\boldsymbol{s}\,;\,\boldsymbol{r}) = R_1 r_i r_j + R_2 \delta_{ij} + R_3 s_i s_j + R_4 (s_i r_j + r_i s_j)\,, \tag{3.5}$$ [6]

where R_1, R_2, R_3 are even, while R_4 is an odd, functions of their arguments r and rm. To be noted, contrary to the case of isotropy, a second-order axisymmetric tensor is by no means necessarily symmetrical in its indices, unless it is solenoidal in both of its indices. Inserting now (3.5) into (3.3), we have

$$4R_1 + r\frac{\partial R_1}{\partial r} + \frac{1}{r}\frac{\partial R_2}{\partial r} - \frac{m}{r^2}\frac{\partial R_3}{\partial m} + m\frac{\partial R_4}{\partial r} + \frac{1-m^2}{r}\frac{\partial R_4}{\partial m} = 0\,, \tag{3.6a}$$

---

[5] Unless specially needed, the symbol of time variable t will be dropped in the followings, for, so far as kinematics is concerned, the role of time can be disregarded at all.

[6] The figures 1, 2, ... attached to the defining scalars are not tensor indices, but no confusion will be caused by such a usage.

and

$$\frac{1}{r}\frac{\partial R_2}{\partial m}+m\frac{\partial R_3}{\partial r}+\frac{1-m^2}{r}\frac{\partial R_3}{\partial m}+4R_4+r\frac{\partial R_4}{\partial r}=0, \tag{3.6b}$$

which are two differential relations between four defining scalars R_1, R_2, R_3 and R_4. Thus, only two of them are independent each other, but unfortunately the fact that these relations (3.6) cannot be expressed in an integrated form makes it quite troublesome to proceed further.

In this respect, however, Chandrasekhar [3] showed how it was convenient to introduce a skew tensor into the course of analysis. Namely, since the curl of any skew tensor becomes a solenoidal true tensor of the same order automatically, there must always exist a second-order skew tensor $q_{ij}(\boldsymbol{r})$ such that

$$R_{ij}(\boldsymbol{r})=\operatorname{curl} q_{ij}(\boldsymbol{r}), \tag{3.7}$$

where in view of (3.3) the curl may be applied to either index. And according to Chandrasekhar, it is seen after some reasonings and tensor manipulations that in the axisymmetric case the relevant skew tensor $q_{ij}(\boldsymbol{s}\,;\boldsymbol{r})$ can be written in the form

$$q_{ij}(\boldsymbol{s}\,;\boldsymbol{r})=q_1\epsilon_{ijk}r_k+q_2s_j\epsilon_{ikl}s_kr_l+\frac{1}{r}\frac{\partial q_1}{\partial m}r_j\epsilon_{ikl}s_kr_l, \tag{3.8}$$ [7]

in which only two arbitrary scalars q_1 and q_2 appear. The four defining scalars of $R_{ij}(\boldsymbol{s}\,;\boldsymbol{r})$ expressed in terms of q_1 and q_2 are:

$$R_1=\frac{1}{r}\frac{\partial}{\partial r}(q_1+q_2)-\frac{m}{r^2}\frac{\partial}{\partial m}(q_1+q_2)-\frac{1}{r^2}\frac{\partial^2 q_1}{\partial m^2}, \tag{3.9a}$$

$$R_2=-r\frac{\partial q_1}{\partial r}-2q_1-r(1-m^2)\frac{\partial q_2}{\partial r}-q_2-m\frac{\partial q_1}{\partial m}+(1-m^2)\frac{\partial^2 q_1}{\partial m^2}$$
$$+m(1-m^2)\frac{\partial q_2}{\partial m}, \tag{3.9b}$$

$$R_3=r\frac{\partial q_2}{\partial r}+q_2-\frac{\partial^2 q_1}{\partial m^2}-m\frac{\partial q_2}{\partial m}, \tag{3.9c}$$

$$R_4=-m\frac{\partial q_2}{\partial r}+\frac{1}{r}\frac{\partial q_1}{\partial m}+\frac{m}{r}\frac{\partial^2 q_1}{\partial m^2}+\frac{m^2}{r}\frac{\partial q_2}{\partial m}, \tag{3.9d}$$

satisfying (3.6) identically and showing that q_1 and q_2 are even functions of r and rm. Especially, if q_1 is independent of m and also $q_2=0$, (3.9) reduce to

$$R_1(r)=\frac{1}{r}\frac{\partial q_1}{\partial r},\quad R_2(r)=-r\frac{\partial q_1}{\partial r}-2q_1\quad\text{and}\quad R_3=R_4=0,$$

which yield

[7] ϵ_{ijk} is the unit alternating tensor which is zero when i, j and k are not all different and is equal to $+1$ or -1, according as i, j and k are in cyclic or acyclic permutation of (1, 2, 3) respectively.

$$4R_1 + r\frac{\partial R_1}{\partial r} + \frac{1}{r}\frac{\partial R_2}{\partial r} = 0,$$

that is, the turbulence is then in fact isotropic.

The mathematical formulation of the axisymmetric double velocity correlation tensor was hereby completed, but from the practical point of view a few words will be added later (sections **5.** and **9.**).

4. The energy spectrum tensor. We shall now turn to $\boldsymbol{\kappa}$-space, and consider the basic kinematics of energy spectrum tensor $\Phi_{ij}(\boldsymbol{\kappa})$, or the Fourier transform of $R_{ij}(\boldsymbol{r})$, for axisymmetric case. First, by definition we have generally

$$\Phi_{ij}(\boldsymbol{\kappa}) = \left(\frac{1}{2\pi}\right)^3 \int R_{ij}(\boldsymbol{r})\, e^{-\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}} d\boldsymbol{r}\,, \tag{4.1a}$$

together with the inverse relation

$$R_{ij}(\boldsymbol{r}) = \int \Phi_{ij}(\boldsymbol{\kappa})\, e^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}} d\boldsymbol{\kappa}\,, \tag{4.1b}$$

which gives, at $\boldsymbol{r} = 0$,

$$R_{ij}(0) = \overline{u_i u_j} = \int \Phi_{ij}(\boldsymbol{\kappa})\, d\boldsymbol{\kappa}\,. \tag{4.1c}$$

To be readily found, the conditions of homogeneity and solenoidality in $\boldsymbol{\kappa}$-space is written as

$$\Phi_{ij}(\boldsymbol{\kappa}) = \Phi_{ji}(-\boldsymbol{\kappa}) = \Phi^*_{ji}(\boldsymbol{\kappa})\,, \tag{4.2}$$ [8]

and

$$\kappa_i \Phi_{ij}(\boldsymbol{\kappa}) = \kappa_j \Phi_{ij}(\boldsymbol{\kappa}) = 0\,, \tag{4.3}$$

these being equivalent to (3.2) and (3.3), respectively. $\Phi_{ij}(\boldsymbol{\kappa})$ is, therefore, an Hermitian tensor which is complex in general except its diagonal elements.

In the axisymmetric turbulence, however, we have at once

$$\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = \Phi_{ji}(\boldsymbol{s}\,;\boldsymbol{\kappa})\,, \tag{4.4}$$

from (3.4), and hence it must be an entirely real tensor having the form

$$\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = \Phi_1 \kappa_i \kappa_j + \Phi_2 \delta_{ij} + \Phi_3 s_i s_j + \Phi_4(s_i\kappa_j + \kappa_i s_j)\,, \tag{4.5}$$

in quite the similar manner as $R_{ij}(\boldsymbol{s}\,;\boldsymbol{r})$. And in view of (4.3) we find

$$\kappa^2 \Phi_1 + \Phi_2 + \kappa\mu\Phi_4 = 0\,, \tag{4.6a}$$

and

$$\mu\Phi_3 + \kappa\Phi_4 = 0\,, \tag{4.6b}$$

by means of which some two of four defining scalars can be readily eliminated. For instance, eliminating Φ_2 and Φ_4, (4.5) becomes

[8] An asterisk indicates the complex conjugate.

$$\Phi_{ij}(s\,;\,\boldsymbol{\kappa})=\Phi_1\kappa_i\kappa_j-(\kappa^2\Phi_1-\mu^2\Phi_3)\delta_{ij}+\Phi_3 s_i s_j-\frac{\mu}{\kappa}\Phi_3(s_i\kappa_j+\kappa_i s_j)\,, \quad (4.7)$$

where Φ_1 and Φ_3 are even functions of κ and $\kappa\mu$. This will reduce to the isotropic expression provided that $\Phi_3=0\ (=\Phi_4)$ and also $\Phi_1\ (=-\Phi_2/\kappa^2)$ does not depend on μ. Thus, the solenoidality needs here no special device such as the use of a skew tensor.

It may be worth noting that further restrictions are necessarily imposed on $\Phi_{ij}(s\,;\,\boldsymbol{\kappa})$. Namely, (1) the integral $\int|\Phi_{ij}(s\,;\,\boldsymbol{\kappa})|\mathrm{d}\boldsymbol{\kappa}$ should be finite; otherwise a certain divergent catastrophe would occur in the correlation tensor, (2) the scalar product $\alpha_i\alpha_j\Phi_{ij}(s\,;\,\boldsymbol{\kappa})$ should be non-negative for an arbitrary choice of the constant vector $\boldsymbol{\alpha}$, since this quantity represents the spectrum density of $\overline{u_i u_j}\alpha_i\alpha_j=\overline{u_\alpha^2}$.[9] This property will be utilized in section 8.

Another relevant quantity of interest is the wave-number magnitude spectrum, or simply κ-spectrum, say, defined by

$$\Psi_{ij}(\kappa)=\int\Phi_{ij}(\boldsymbol{\kappa})\,\mathrm{d}S(\kappa)\,, \quad (4.8)$$

where $\mathrm{d}S(\kappa)$ is a surface element and the integration is over the surface of sphere of radius κ. Clearly, $\Psi_{ij}(\kappa)\mathrm{d}\kappa$ is the contribution to $\overline{u_i u_j}$ from wave-numbers the magnitudes of which lies between κ and $\kappa+\mathrm{d}\kappa$, viz.

$$\overline{u_i u_j}=\int\Phi_{ij}(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}=\int_0^\infty\Psi_{ij}(\kappa)\,\mathrm{d}\boldsymbol{\kappa}\,. \quad (4.9)$$

Now, according to (A.8) of the Appendix, it is seen that

$$\Psi_{ij}(s\,;\,\kappa)=\Psi_a(\kappa)\delta_{ij}+\Psi_b(\kappa)s_i s_j\,, \quad (4.10)$$

and the expressions of $\Psi_a(\kappa)$ and $\Psi_b(\kappa)$ in terms of Φ_1 and Φ_3 are:

$$\Psi_a(\kappa)=-4\pi\kappa^2\Big[\frac{\kappa^2}{2}\langle(1+\mu^2)\Phi_1\rangle_\mu-\langle\mu^2\Phi_3\rangle_\mu\Big]\,, \quad (4.11\mathrm{a})$$

and

$$\Psi_b(\kappa)=-4\pi\kappa^2\Big[\frac{\kappa^2}{2}\langle(1-3\mu^2)\Phi_1\rangle_\mu-\langle(1-2\mu^2)\Phi_3\rangle_\mu\Big]\,, \quad (4.11\mathrm{b})$$

respectively, where the abbreviation

$$\langle\ldots\rangle_\mu=\frac{1}{2}\int_{-1}^{1}(\ldots)\,\mathrm{d}\mu$$

is to be understood. Again, in the isotropic case, these reduce to

$$\Psi_a(\kappa)=-\frac{8}{3}\pi\kappa^2\Phi_1(\kappa) \qquad \text{and} \qquad \Psi_b(\kappa)=0\,.$$

9) These are the physical aspects of the Cramér's theorem established in complex form for general continuous stationary process (cf. [2], p. 25). Also note that as a special case of this theorem, the diagonal elements of $\Phi_{ij}(\boldsymbol{\kappa})$ are proved to be always real and non-negative. An inequality $\Phi_1\kappa^2\leqq\Phi_3$ thus comes from the requirement $\Phi_{ii}(s\,;\,\boldsymbol{\kappa})\geqq 0$.

5. Some relating quantities. Before introducing the dynamical relations, let us derive in this section several quantities particularly accessible to experimental technique or physical interpretation.

(1) The important correlation functions

To begin with, the 'total' correlation function (R, say) is obtained by contracting $R_{ij}(\boldsymbol{s}\,;\boldsymbol{r})$ with respect to i and j, that is,

$$R(r, rm) = R_{ii}(\boldsymbol{s}\,;\boldsymbol{r}) = \overline{u_i u_i'} = r^2 R_1 + 3R_2 + R_3 + 2rmR_4$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\Big[r^3\{2q_1 + (1-m^2)q_2\}\Big] + \frac{\partial}{\partial m}\Big[(1-m^2)\Big(\frac{\partial q_1}{\partial m} + mq_2\Big)\Big], \quad (5.1)$$

this being an invariant for the choice of coordinate axes. One of the correlations suitable for practical measurement, on the other hand, is that between velocity components parallel to the direction of separation $\boldsymbol{r}$. Let R_l^l denote this 'longitudinal' correlation function, then we have

$$R_l^l(r, rm) = \frac{\overline{u_i r_i \cdot u_j' r_j}}{r^2} = \frac{R_{ij}(\boldsymbol{s}\,;\boldsymbol{r}) r_i r_j}{r^2}$$
$$= r^2 R_1 + R_2 + m^2 R_3 + 2rmR_4 = -[2q_1 + (1-m^2)q_2]\,. \quad (5.2)$$

And, subtracting (5.2) from (5.1), there comes the sum of an arbitrary pair of 'lateral' correlation functions, R_n^n and $R_n'^n$ (see Fig. 1). It is thus found that

$$R_n^n + R_n'^n = R - R_l^l = \frac{1}{r}\frac{\partial(r^2 R_l^l)}{\partial r} + \frac{\partial}{\partial m}\Big[(1-m^2)\Big(\frac{\partial q_1}{\partial m} + mq_2\Big)\Big], \quad (5.3)$$

which is to be compared with the well-known isotropic relation

$$2R_n^n = 2R_n'^n = \frac{1}{r}\frac{\partial(r^2 R_l^l)}{\partial r}\,.$$

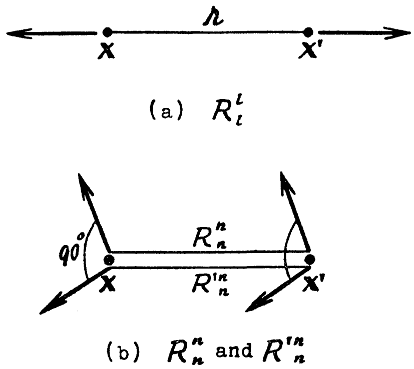

Fig. 1.

(2) The energy tensor and the energy spectrum functions

Considering (3.5), (4.9) and (4.10), the expression (4.1c) of energy tensor can be written as

$$\overline{u_i u_j} = R_{ij}(0) = R_2(0)\,\delta_{ij} + R_3(0)\,s_i s_j$$
$$= \int_0^\infty \Psi_{ij}(\kappa)\,\mathrm{d}\kappa = \delta_{ij}\int_0^\infty \Psi_a(\kappa)\,\mathrm{d}\kappa + s_i s_j\int_0^\infty \Psi_b(\kappa)\,\mathrm{d}\kappa\,, \quad (5.4)$$

showing that

$$R_2(0) = \int_0^\infty \Psi_a(\kappa)\,\mathrm{d}\kappa (= -2q_1(0) - q_2(0))\,, \quad (5.5a)$$

and

$$R_3(0)=\int_0^\infty \Psi_b(\kappa)\,d\kappa\,(=q_2(0))\,, \tag{5.5b}$$

where $R_2(0)$ stands for the value of $R_2(r, rm)$ at $r=0$, and the like. And further, the following expressions are readily obtained:

$$E=\frac{1}{2}\overline{u_i u_i}=\frac{1}{2}R(0)=\frac{1}{2}[3R_2(0)+R_3(0)]=\int_0^\infty \mathcal{E}(\kappa)\,d\kappa\,, \tag{5.6a}$$

$$E_{//}=\frac{1}{2}\overline{u_{//}^2}=\frac{1}{2}\overline{u_i u_j}s_i s_j=\frac{1}{2}[R_2(0)+R_3(0)]=\int_0^\infty \mathcal{E}_{//}(\kappa)\,d\kappa\,, \tag{5.6b}$$

and

$$E_\perp=\frac{1}{2}\overline{u_\perp^2}=\frac{1}{2}(E-E_{//})=\frac{1}{2}R_2(0)=\int_0^\infty \mathcal{E}_\perp(\kappa)\,d\kappa\,, \tag{5.6c}$$

provided that

$$2\mathcal{E}(\kappa)\ =\Psi_{ii}(\kappa)=3\Psi_a(\kappa)+\Psi_b(\kappa)\,, \tag{5.7a}$$

$$2\mathcal{E}_{//}(\kappa)=\Psi_{ij}(\kappa)s_i s_j=\Psi_a(\kappa)+\Psi_b(\kappa)\,, \tag{5.7b}$$

and

$$2\mathcal{E}_\perp(\kappa)=\mathcal{E}(\kappa)-\mathcal{E}_{//}(\kappa)=\Psi_a(\kappa)\,. \tag{5.7c}$$

The symbols // and $\perp$ refer to the directions parallel and perpendicular to the axis $\boldsymbol{s}$, and for the time being we shall call $\mathcal{E}(\kappa)$, $\mathcal{E}_{//}(\kappa)$ and $\mathcal{E}_\perp(\kappa)$ the 'total,' the 'axial' and the 'transverse' (energy) spectrum functions respectively, whereas E, $E_{//}$ and $E_\perp$ will be called the total energy (density) and so on. Evidently, we must have

$$\Psi_a(\kappa)\geqq 0 \quad \text{and} \quad \Psi_b(\kappa)\geqq -\Psi_a(\kappa)\,, \tag{5.8}$$

since each spectrum function is by no means negative. By the way, we can also write

$$\overline{u_i u_j}=2E_\perp\delta_{ij}+2(E_{//}-E_\perp)s_i s_j \tag{5.4'}$$

for the energy tensor.

When $i\neq j$, $\overline{u_i u_j}$ represents the turbulent shearing stress e_{ij}, viz.

$$e_{ij}=e s_i s_j=2(E_{//}-E_\perp)s_i s_j\int_0^\infty \Psi_b(\kappa)\,d\kappa\,, \quad (i\neq j)\,, \tag{5.9}$$

and hence if we take any one of the coordinate axes, x_1-axis, for definiteness, in parallel with $\boldsymbol{s}$, all members of e_{ij} vanish identically, for $s_i=\delta_{1i}$. Let us refer such a coordinate system as the '$\boldsymbol{s}$-system' hereafter. Then we can also say that the $\boldsymbol{s}$-system corresponds to the principal axes of Reynolds stress tensor and that accordingly the vanishing turbulent shear in $\boldsymbol{s}$-system is not a sufficient condition of isotropy.

(3) The one dimensional spectrum functions

If a spacial Fourier analysis of the turbulent velocity is made along a line, the energy spectrum function representing the contributions from every slices in $\boldsymbol{\kappa}$-space perpendicular to that line is obtained. This is the one-dimensional spectrum function conveniently measured in usual experiments. Taking the s-system, and analyzing the velocity variation along x_1-axis, we have

$$\Theta_{ij}(\kappa_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{ij}(r_1,0,0)\,\mathrm{e}^{-\iota\kappa_1 r_1}\mathrm{d}r_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_{ij}(\kappa_1,\kappa_2,\kappa_2)\,\mathrm{d}\kappa_2\mathrm{d}\kappa_3\,,\qquad(5.10)$$

where $\kappa_1=\kappa\mu$ and $\kappa_2^2+\kappa_3^2=\kappa^2(1-\mu^2)$. After simple calculations it is found that

$$\Theta_{11}(\kappa_1)=2\pi\int_{\kappa_1}^{\infty}\left(1-\frac{\kappa_1^2}{\kappa^2}\right)(\Phi_3-\kappa^2\Phi_1)\,\kappa\,\mathrm{d}\kappa\,,\qquad(5.11\mathrm{a})$$

and
$$\Theta_{22}(\kappa_1)=\Theta_{33}(\kappa_1)=2\pi\int_{\kappa_1}^{\infty}\left[\frac{\kappa_1^2}{\kappa^2}\Phi_3-\frac{1}{2}\left(1+\frac{\kappa_1^2}{\kappa^2}\right)\kappa^2\Phi_1\right]\kappa\,\mathrm{d}\kappa\,,\qquad(5.11\mathrm{b})$$

the other elements of $\Theta_{ij}(\kappa_1)$ being all zero.

6. Dynamical equations. As is well established, the basic equations governing the dynamics of $R_{ij}(\boldsymbol{r},t)$ and $\Phi_{ij}(\boldsymbol{\kappa},t)$ are written in the following way:

i.e. in $\boldsymbol{r}$-space,
$$\frac{\partial R_{ij}(\boldsymbol{r},t)}{\partial t}-2\nu\nabla^2R_{ij}(\boldsymbol{r},t)=T_{ij}(\boldsymbol{r},t)+P_{ij}(\boldsymbol{r},t)\,,\qquad(6.1)$$

and in $\boldsymbol{\kappa}$ space,
$$\frac{\partial \Phi_{ij}(\boldsymbol{\kappa},t)}{\partial t}+2\nu\kappa^2\Phi_{ij}(\boldsymbol{\kappa},t)=\Gamma_{ij}(\boldsymbol{\kappa},t)+\Pi_{ij}(\boldsymbol{\kappa},t)\,,\qquad(6.2)$$

where
$$T_{ij}(\boldsymbol{r},t)=\frac{\partial}{\partial r_k}(\overline{u_iu_ku_j'}-\overline{u_iu_k'u_j'})=\int\Gamma_{ij}(\boldsymbol{\kappa},t)\,\mathrm{e}^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{\kappa}\,,\qquad(6.3)$$

$$P_{ij}(\boldsymbol{r},t)=\frac{1}{\rho}\left(\frac{\partial\overline{pu_j'}}{\partial r_i}-\frac{\partial\overline{p'u_i}}{\partial r_j}\right)=\int\Pi_{ij}(\boldsymbol{\kappa},t)\,\mathrm{e}^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{\kappa}\,,\qquad(6.4)$$

p being the pressure, ν the kinematic viscosity, ρ the density and $\nabla^2=\partial^2/\partial r_k\partial r_k$. These are derived from the Navier-Stokes equation of viscous incompressible fluid

$$\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\,\mathrm{grad})\,\boldsymbol{u}=-\frac{1}{\rho}\,\mathrm{grad}\,p+\nu\nabla^2\boldsymbol{u}\,,$$

and are valid for general homogeneous turbulence. Of course, $T_{ij}(\boldsymbol{r})$ or $\Gamma_{ij}(\boldsymbol{\kappa})$ and $P_{ij}(\boldsymbol{r})$ or $\Pi_{ij}(\boldsymbol{\kappa})$ represent the inertia and the pressure actions respectively, both of which stand for the non-linear character of the problem. But from the kinematical considerations, we have

$$T_{ij}(0)=-\frac{\partial}{\partial r_k}\overline{u_iu_ju_k}=0\quad\leftrightarrows\quad\int\Gamma_{ij}(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}=0\,,\qquad(6.5)$$

and
$$P_{ii}(\boldsymbol{r})=0\quad\leftrightarrows\quad\Pi_{ii}(\boldsymbol{\kappa})=0\,.\qquad(6.6)$$

Furthermore, in virtue of the relation

$$\nabla^2 p = -\rho\ \mathrm{div}(\boldsymbol{u}\ \mathrm{grad})\boldsymbol{u}\,,$$

which is the divergence of Navier-Stokes equation, it turns out to be

$$\nabla^2 P_{ij}(\boldsymbol{r}) = -\frac{\partial}{\partial r_k}\left[\frac{\partial T_k(\boldsymbol{r})}{\partial r_i} + \frac{\partial T_{ik}(\boldsymbol{r})}{\partial r_j}\right]$$

$$\leftrightarrows\ \kappa^2 \Pi_{ij}(\boldsymbol{\kappa}) = -\left[\kappa_i \kappa_k \Gamma_{kj}(\boldsymbol{\kappa}) + \kappa_j \kappa_k \Gamma_{ik}(\boldsymbol{\kappa})\right], \tag{6.7}$$

showing the connexion between the pressure action and the inertia action (see [2] 5.2).

Now we shall introduce the condition of axisymmetry and for conciseness' sake consider below in $\boldsymbol{\kappa}$-space exclusively. At the beginning, since $\Gamma_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ as well as $\Pi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ is not solenoidal, there is no reason to suppose it is symmetrical in indices, so we have to assume in general that

$$\Gamma_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = \Gamma_1 \kappa_i \kappa_j + \Gamma_2 \delta_{ij} + \Gamma_3 s_i s_j + \Gamma_4 s_i \kappa_j + \Gamma_5 \kappa_i s_j\,, \tag{6.8}$$

which gives after (6.7)

$$\Pi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = \left[2\mu^2 \Gamma_3 + \kappa\mu(\Gamma_4 + \Gamma_5)\right]\frac{\kappa_i \kappa_j}{\kappa^2} - (\mu\Gamma_3 + \kappa\Gamma_4)\frac{s_i \kappa_j}{\kappa}$$
$$- (\mu\Gamma_3 + \kappa\Gamma_5)\frac{\kappa_i s_j}{\kappa}, \tag{6.9}$$

where the relation (6.6) giving

$$\kappa^2 \Gamma_1 + \Gamma_2 + \mu^2 \Gamma_3 + \kappa\mu(\Gamma_4 + \Gamma_5) = 0 \tag{6.10}$$

has been used. In addition, from the requirement (6.5) we get two integral relations

$$\int_0^\infty \kappa^2 \langle \kappa^2 (1-\mu^2)\Gamma_1 + 2\Gamma_2 \rangle_\mu \, d\kappa = 0\,, \tag{6.11a}$$

and

$$\int_0^\infty \kappa^2 \langle 2\Gamma_2 + (1-\mu^2)\Gamma_3 \rangle_\mu \, d\kappa = 0\,, \tag{6.11b}$$

being together with (6.10) the restrictions imposed on the five defining scalars of $\Gamma_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$. At the same time, (6.8) and (6.9) give immediately

$$\Gamma_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) + \Pi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = -(\Gamma_2 - \mu^2\Gamma_3)\frac{\kappa_i \kappa_j}{\kappa^2} + \Gamma_2 \delta_{ij} + \Gamma_3 s_i s_j$$
$$- \frac{\mu\Gamma_3}{\kappa}(s_i \kappa_j + \kappa_i s_j)\,, \tag{6.12}$$

that is, the right-hand side of the equation (6.2) is, like $\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ itself, symmetrical in the indices and is determined by only two scalar functions as it ought to be so.

Then, integrating (6.2) over the surface of sphere of radius κ, the equation of κ-spectrum $\Psi_{ij}(\kappa)$ is obtained. The result is that

$$\frac{\partial \Psi_{ij}(\kappa, t)}{\partial t} + 2\nu\kappa^2 \Psi_{ij}(\kappa, t) = \Xi_{ij}(\kappa, \mathrm{t})\,, \tag{6.13}$$

where $\quad \Xi_{i}(\kappa, t) = \Xi_a(\kappa, t)\delta_{ij} + \Xi_b(\kappa, t) s_i s_j\,,$ (6.14)

and $\quad \Xi_a(\kappa, t) = 2\pi\kappa^2[\langle(1+\mu^2)\Gamma_2\rangle_\mu + \langle\mu^2(1-\mu^2)\Gamma_3\rangle_\mu]\,,$ (6.15a)

$$\Xi_b(\kappa, t) = 2\pi\kappa^2[\langle(1-3\mu^2)\Gamma_2\rangle_\mu + \langle(2-3\mu^2)(1-\mu^2)\Gamma_3\rangle_\mu]\,. \tag{6.15b}$$

from (A.8). To be noted here, it is found in virtue of (6.11)

$$3\int_0^\infty \Xi_a(\kappa, t)\,d\kappa + \int_0^\infty \Xi_b(\kappa, t)\,d\kappa = 0\,. \tag{6.16}$$

Thus, there comes the following set of equations

$$\frac{\partial \mathcal{E}_{//}(\kappa, t)}{\partial t} + 2\nu\kappa^2\mathcal{E}_{//}(\kappa, t) = \frac{1}{2}[\Xi_a(\kappa, t) + \Xi_b(\kappa, t)]\,, \tag{6.17a}$$

$$\frac{\partial \mathcal{E}_{\perp}(\kappa, t)}{\partial t} + 2\nu\kappa^2\mathcal{E}_{\perp}(\kappa, t) = \frac{1}{2}\Xi_a(\kappa, t)\,, \tag{6.17b}$$

and by integration from $\kappa = 0$ to ∞, we have also

$$\frac{dE_{//}(t)}{dt} + 2\nu\int_0^\infty \kappa^2\mathcal{E}_{//}(\kappa, t)\,d\kappa = -\int_0^\infty \Xi_a(\kappa, t)\,d\kappa\,, \tag{6.18a}$$

$$\frac{dE_{\perp}(t)}{dt} + 2\nu\int_0^\infty \kappa^2\mathcal{E}_{\perp}(\kappa, t)\,d\kappa = \frac{1}{2}\int_0^\infty \Xi_a(\kappa, t)\,d\kappa\,, \tag{6.18b}$$

where use is made of (6.16).

The decay of total energy, on the other hand, is described by the equation

$$\frac{dE(t)}{dt} + 2\nu\int_0^\infty \kappa^2\mathcal{E}(\kappa, t)\,d\kappa = 0\,, \tag{6.19}$$

which tells us that the inertia action (including pressure action) does not affect the total energy decay, while it plays a part in the dynamical processes of partial energies $E_{//}(t)$ and $E_{\perp}(t)$ as is seen in (6.18). Namely, the directional transfer of kinetic energy is well represented by the quantity $\int_0^\infty \Xi_a(\kappa, t)\,d\kappa$ alone. Subtracting now (6.18a) from (6.18b) and comparing with (5.7) and (5.9), we get

$$\frac{de(t)}{dt} + 2\nu\int_0^\infty \kappa^2\Psi_b(\kappa, t)\,d\kappa = -3\int_0^\infty \Xi_a(\kappa, t)\,d\kappa\,. \tag{6.20}$$

This is the equation for the change of turbulent shearing stress.

7. The third-order spectrum tensors and further details of Γ_{ij}. So far we have not entered into the further detailed expression of $\Gamma_{ij}(s\,;\,\boldsymbol{\kappa})$ which may be called the energy-transfer spectrum tensor. But, in general, if we introduce the third-order $\boldsymbol{\kappa}$-tensors $\Upsilon_{(ik)j}(\boldsymbol{\kappa})$ and $\Upsilon_{i(k)}(\boldsymbol{\kappa})$ such that

$$\Upsilon_{(ik)j}(\boldsymbol{\kappa})=\left(\frac{1}{2\pi}\right)^3\int\overline{u_i u_k u_j'}\,\mathrm{e}^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{r}\,, \tag{7.1}$$

and

$$\Upsilon_{i(kj)}(\boldsymbol{\kappa})=\left(\frac{1}{2\pi}\right)^3\int\overline{u_i u_k' u_j'}\,\mathrm{e}^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{r}\,, \tag{7.2}$$

the relation (6.3) in $\boldsymbol{r}$-space is translated into

$$\Gamma_{ij}(\boldsymbol{\kappa})=\iota\kappa_k[\Upsilon_{(ik)j}(\boldsymbol{\kappa})-\Upsilon_{i(kj)}(\boldsymbol{\kappa})]\,. \tag{7.3}$$

In isotropic case this reduces to a single scalar relation, since each of $\Upsilon_{(ik)j}(\boldsymbol{\kappa})$ $(=-\Upsilon_{i(kj)}(\boldsymbol{\kappa})$, in that case) and $\Gamma_{ij}(\boldsymbol{\kappa})$ has only one defining scalar function. In axisymmetric case, however, the corresponding expression becomes somewhat lengthy and complicated as is shown below.

That is, the general properties of third-order spectrum tensors in homogeneous turbulence are:

$$\Upsilon_{i(kj)}(\boldsymbol{\kappa})=\Upsilon_{(kj)i}(-\boldsymbol{\kappa})\,, \tag{7.4}$$

from homogeneity, and

$$\kappa_j\Upsilon_{(ik)j}(\boldsymbol{\kappa})=\kappa_i\Upsilon_{i(kj)}(\boldsymbol{\kappa})=0\,, \tag{7.5}$$

from incompressibility together with the obvious conditions

$$\Upsilon_{(ik)j}(\boldsymbol{\kappa})=\Upsilon_{(ki)j}(\boldsymbol{\kappa})\qquad\text{and}\qquad\Upsilon_{i(kj)}(\boldsymbol{\kappa})=\Upsilon_{i(jk)}(\boldsymbol{\kappa})\,. \tag{7.6}$$

And, taking these into considerations, we can assume for axisymmetric turbulence that[10)]

$$\begin{aligned}\frac{1}{\iota}\Upsilon_{(ik)j}(\boldsymbol{s}\,;\boldsymbol{\kappa})=\;&\Upsilon_1\,[\kappa^2(\kappa_i\delta_{kj}+\kappa_k\delta_{ij})-2\kappa_i\kappa_k\kappa_j]\\&+(\Upsilon_2\kappa_i\kappa_k+\Upsilon_3\delta_{ik}+\Upsilon_4 s_i s_k)(\mu\kappa_j-\kappa s_j)\\&+\Upsilon_5\,[(s_i\kappa_k+\kappa_i s_k)\kappa_j-\kappa^2(s_i\delta_{kj}+s_k\delta_{ij})]\\&+\Upsilon_6\,[(s_i\kappa_k+\kappa_i s_k)s_j-\kappa\mu(s_i\delta_{kj}+s_k\delta_{ij})]\,,\end{aligned} \tag{7.7}$$

and

$$\begin{aligned}\frac{1}{\iota}\Upsilon_{i(kj)}(\boldsymbol{s}\,;\boldsymbol{\kappa})=\;&-\Upsilon_1^-[\kappa^2(\kappa_j\delta_{ki}+\kappa_k\delta_{ij})-2\kappa_i\kappa_k\kappa_j]\\&+(\Upsilon_2^-\kappa_j\kappa_k+\Upsilon_3^-\delta_{jk}+\Upsilon_4^- s_j s_k)(\mu\kappa_i-\kappa s_i)\\&+\Upsilon_5^-[(s_j\kappa_k+\kappa_j s_k)\kappa_i-\kappa^2(s_j\delta_{ki}+s_k\delta_{ij})]\\&-\Upsilon_6^-[(s_j\kappa_k+\kappa_j s_k)s_i-\kappa\mu(s_j\delta_{ki}+s_k\delta_{ij})]\,,\end{aligned} \tag{7.8}$$

where we put

$$\Upsilon_1^-(\kappa,\kappa\mu)=\Upsilon_1(\kappa,-\kappa\mu)\,,\quad\Upsilon_2^-(\kappa,\kappa\mu)=\Upsilon_2(\kappa,-\kappa\mu)\,,\quad\text{and so on,}$$

in view of the fact that μ changes sign with the component κ_i, whereas κ remains in the same sign for replacement of $\boldsymbol{\kappa}$ by $-\boldsymbol{\kappa}$. In this connexion, we shall conveniently divide each scalar function into its even and odd (with respect to μ) parts designated by the superscripts o and e respectively. Namely, we have, for example

10) $\Upsilon_{(ik)j}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ and $\Upsilon_{i(kj)}(\boldsymbol{s}\,;\boldsymbol{\kappa})$ are pure imaginaries, and hence ι appears in (7.7) and (7.8) so that each defining scalars may be real. As to the corresponding expressions in $\boldsymbol{r}$-space, see Chandrasekhar [3].

$$\Upsilon_1 = \Upsilon_1^{e} + \Upsilon_1^{o}, \qquad \Upsilon_1^{-} = \Upsilon_1^{e} - \Upsilon_1^{o},$$

and
$$2\Upsilon_1^{e} = \Upsilon_1 + \Upsilon_1^{-}, \quad 2\Upsilon_1^{o} = \Upsilon_1 - \Upsilon_1^{-}.$$

In these ways, we arrive at *via* (7.3) the following expressions of five Γ's:

$$\Gamma_1 = 2[\kappa(\kappa\Upsilon_1^{e} - \mu\Upsilon_5^{o}) - \mu(\kappa^2\Upsilon_2^{o} + \Upsilon_3^{o})], \tag{7.9a}$$

$$\Gamma_2 = 2\kappa^2[-\kappa(\kappa\Upsilon_1^{e} - \mu\Upsilon_5^{o}) + \mu^2\Upsilon_6^{e}], \tag{7.9b}$$

$$\Gamma_3 = 2\kappa^2(\mu\Upsilon_4^{o} + \Upsilon_6^{e}), \tag{7.9c}$$

$$\Gamma_4 = -\kappa[(\kappa^2\Upsilon_2^{-} + \Upsilon_3^{-}) + \mu^2\Upsilon_4 - 2\mu\Upsilon_6^{o}] \ \ (= -\Gamma_5^{-}), \tag{7.9d}$$

$$\Gamma_5 = \kappa[(\kappa^2\Upsilon_2 + \Upsilon_3) + \mu^2\Upsilon_4^{-} - 2\mu\Upsilon_6^{o}] \ \ (= -\Gamma_4^{-}), \tag{7.9e}$$

showing that Γ_1, Γ_2, Γ_3 are even, and $\Gamma_4 + \Gamma_5$ is an odd. functions of μ, and that the relation (6.10) is, as is expected, automatically satisfied. Furthermore, by using (7.9) in (6,9) it is seen at once that

$$\Pi_{ij}(s\,;\boldsymbol{\kappa}) = 2\mu\Pi^{o}\kappa_i\kappa_j + \kappa(\Pi s_i\kappa_j - \Pi^{-}\kappa_i s_j), \tag{7.10}$$

where
$$\Pi = \kappa^2\Upsilon_2 + \Upsilon_3 + \mu^2\Upsilon_4 - 2\mu\Upsilon_6, \tag{7.11}$$

and it is one of the remarkable feature of axisymmetric turbulence that a single function $\Pi(\kappa, \kappa\mu)$ is suffice to determine the direct action of pressure on the energy spectrum tensor. By the way, the relations

$$\frac{1}{\rho}\left(\frac{1}{2\pi}\right)^3 \int \overline{pu_j'}\, e^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}} d\boldsymbol{r} = \iota\, \Pi(\mu\kappa_j - \kappa s_j), \tag{7.12}$$

and
$$\frac{1}{\rho}\left(\frac{1}{2\pi}\right)^3 \int \overline{p'u_i}\, e^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{r}} d\boldsymbol{r} = \iota\, \Pi^{-}(\mu\kappa_i - \kappa s_i) \tag{7.13}$$

should be noticed here.

Let us now recall the state of affairs in isotropic turbulence. There simply $\Upsilon_1(\kappa) = \Upsilon_1^{-}(\kappa)$ and all other Υ's are zero. So far as the second- and third-order isotropic tensors are concerned, therefore, the governing dynamical equation (either in $\boldsymbol{\kappa}$-space or in $\boldsymbol{r}$-space) reduces to a single scalar relation which makes it possible to evaluate one of them from the knowledge of the other. And first in the fourth-order tensor, there appear more than one defining scalars, the evaluation of which requires some independent physical hypothesis such as the 'hypothesis of zero fourth-order cumulant of the velocity field,'[11] for instance. In the axisymmetric case, on the other hand, the number of defining scalars is six for third-order tensor, while it is two for second-order one. This means that even the complete knowledge of the latter is not sufficient to determine the former without introducing any kind

[11] This is the generalization of the 'hypothesis of normal joint-probability distribution of the velocity field' use of which can be traced back to M. Millionshtchikov (1941). The proposal of this generalized form and an elaborate analysis based upon it has been made recently by T. Tatsumi (*Proc. Roy. Soc.* A, **239** (1957) 16). Also the work of I. Proudman and W. H. Reid (*Philos. Trans.* A, **247** (1954) 163) may be consulted in these connexions.

of special assumption already at this stage. The similar can be said about the pressure term too. That is, in isotropic case it is well known that $\Pi=0$ identically, but here to estimate Π itself is the very question. Thus we cannot help to confront a deeper problem of turbulence theory by one stage earlier, so to speak, than the case of isotropy.

8. Decomposition of axisymmetric turbulence. In section **4.** we have got the expression (4.7) of axisymmetric energy spectrum tensor which may be now rewritten as

$$\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})=\Phi_1(\kappa_i\kappa_j-\kappa^2\delta_{ij})+\Phi_3\left[\mu^2\delta_{ij}+s_is_j-\frac{\mu}{\kappa}(s_i\kappa_j+\kappa_is_j)\right]. \quad (8.1)$$

This shows that an axisymmetric energy spectrum tensor is composed of two parts represented by $\Phi_1(\kappa,\kappa\mu)$ and $\Phi_3(\kappa,\kappa\mu)$, respectively. And each part can be regarded as describing a special type of axisymmetric turbulence, since both of them individually satisfy the conditions of homogeneity and solenoidality by themselves. Accordingly, so far as the energy spectrum tensor is concerned, general axisymmetric turbulence appears *as if* it were a composition of the specified fields of motion. It is, of course, far from being a simple superposition of velocity fields, unless these partial fields are statistically orthogonal each other, and further there must exist a certain complicated interaction of non-linear character between these parts at least during the earlier stage of decay; nevertheless it may be of some interest to speak of the 'decomposition' of axisymmetric turbulence in this context. It is a procedure to express the general axisymmetric field in terms of the elementary ones. The mode of such a decomposition, however, is rather arbitrary, the representation like (8.1) being a mere example. So let us mention in the followings the other modes which seem to be more intuitive in their physical meanings.

To begin with, if we put

$$\Phi_1=-(Z_{\mathrm{I}}+Y_{\mathrm{I}}), \quad \text{and} \quad \Phi_3=-\kappa^2Y_{\mathrm{I}}, \quad (8.2)$$

(8.1) becomes

$$\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})=Z_{\mathrm{I}}(\kappa^2\delta_{ij}-\kappa_i\kappa_j)$$
$$+Y_{\mathrm{I}}[\kappa^2(1-\mu^2)\delta_{ij}-\kappa_i\kappa_j-\kappa^2s_is_j+\kappa\mu(s_i\kappa_j+\kappa_is_j)], \quad (8.3)$$

from which it follows

$$\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})s_is_j=\kappa^2(1-\mu^2)Z_{\mathrm{I}},$$

showing that Y_{I} does not contribute to the axial energy $E_{//}$ at all; in other words, the second term on the right-hand side of (8.3) corresponds to a plane motion perpendicular to $\boldsymbol{s}$.[12] The first term, on the other hand, is similar to the isotropic functional form except that Z_{I} is dependent on μ as well as on κ. Thus, for convenience' sake we shall refer the former as transverse

12) We must distinguish this from a purely two-dimensional motion in which all velocities, even the instantaneous ones, occur uniformly along the direction normal to the plane of motion.

term and the latter as quasi-isotropic term. Or supposing their physical realities, the words 'transverse turbulence' and 'quasi-isotropic turbulence' also will be used.[13] Then from the requirements of the Cramér's theorem (p. 68) applied to each of them we find

$$Z_{\mathrm{I}} \geqq 0 \quad \text{and} \quad Y_{\mathrm{I}} \geqq 0 .$$

And, making use of (8.3) in (4.11) and (5.7) it turns out that

$$\Psi_a(\kappa) = 2\mathcal{E}_\perp(\kappa) = 2\pi\kappa^4[\langle(1+\mu^2)Z_{\mathrm{I}}\rangle_\mu + \langle(1-\mu^2)Y_{\mathrm{I}}\rangle_\mu] , \tag{8.4a}$$

$$\Psi_b(\kappa) = 2[\mathcal{E}_{//}(\kappa) - \mathcal{E}_\perp(\kappa)] = 2\pi\kappa^4[\langle(1-3\mu^2)Z_{\mathrm{I}}\rangle_\mu - \langle(1-\mu^2)Y_{\mathrm{I}}\rangle_\mu] , \tag{8.4b}$$

together with a pair of dynamical equations

$$\frac{\partial Z_{\mathrm{I}}}{\partial t} + 2\nu\kappa^2 Z_{\mathrm{I}} = \frac{\Gamma_2 + (1-\mu^2)\Gamma_3}{\kappa^2} , \tag{8.5a}$$

$$\frac{\partial Y_{\mathrm{I}}}{\partial t} + 2\nu\kappa^2 Y_{\mathrm{I}} = -\frac{\Gamma_3}{\kappa^2} , \tag{8.5b}$$

derived from the tensor equation (6.2). In these equations there appears the cross term Γ_3 representing the interaction between two partial fields of turbulence. Evidently, this term is no longer capable of being decomposed in the same sense as before, and so it should be emphasized again that the notion of decomposition is limited in this respect, too.

Next, we shall have an another representation on putting

$$\Phi_1 = -Z_{\mathrm{II}} + 2\mu^2 X_{\mathrm{II}} \quad \text{and} \quad \Phi_3 = \kappa^2(1+\mu^2)X_{\mathrm{II}} , \tag{8.6}$$

that is,

$$\Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa}) = Z_{\mathrm{II}}(\kappa^2\delta_{ij} - \kappa_i\kappa_j) + X_{\mathrm{II}}[2\mu^2\kappa_i\kappa_j - \kappa^2\mu^2(1-\mu^2)\delta_{ij} + (1+\mu^2)\{\kappa^2 s_i s_j - \kappa\mu(s_i\kappa_j + \kappa_i s_j)\}] . \tag{8.7}$$

In spite of its somewhat lengthy appearance the physical implication of this expression is fairly simple, namely, from (8.7) we have

$$\Phi_{ii}(\boldsymbol{s}\,;\boldsymbol{\kappa}) - \Phi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})s_i s_j = \kappa^2(1+\mu^2)Z_{\mathrm{II}} ,$$

and hence the term of X_{II} corresponds to a rectilinear motion parallel to $\boldsymbol{s}$, having no contribution to the transverse energy $E_\perp$. It may, therefore, be called 'axial turbulence.' Again, in virtue of the Cramér's theorem, the necessary conditions

$$Z_{\mathrm{II}} \geqq 2\mu^2 X_{\mathrm{II}} \geqq 0$$

can be obtained. The related functions and dynamical equations are

$$\Psi_a(\kappa) = 2\mathcal{E}_\perp(\kappa) = 2\pi\kappa^4\langle(1+\mu^2)Z_{\mathrm{II}}\rangle_\mu , \tag{8.8a}$$

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[13] In a strict sense, the proper use of these terminologies should be restricted to the case of non-interacting coexistence, but as an expedient we shall often use them for other general cases as well.

$$\Psi_b(\kappa)=2[\mathcal{E}_{//}(\kappa)-\mathcal{E}_{\perp}(\kappa)]=2\pi\kappa^4[\langle(1-3\mu^2)Z_{\text{II}}\rangle_\mu+2\langle(1-\mu^2)^2X_{\text{II}}\rangle_\mu], \tag{8.8b}$$

and
$$\frac{\partial Z_{\text{II}}}{\partial t}+2\nu\kappa^2Z_{\text{II}}=\frac{\Gamma_2+\mu^2[\Gamma_2+(1-\mu^2)\Gamma_3]}{(1+\mu^2)\kappa^2}, \tag{8.9a}$$

$$\frac{\partial X_{\text{II}}}{\partial t}+2\nu\kappa^2X_{\text{II}}=\frac{\Gamma_3}{(1+\mu^2)\kappa^2}. \tag{8.9b}$$

Two modes of decomposition I and II, say, stated above may be convenient to consider the correspondence to isotropic turbulence as is naturally supposed. Comparing (8.2) and (8.6), however, we see

$$Z_{\text{I}}+Y_{\text{I}}=Z_{\text{II}}-2\mu^2X_{\text{II}} \quad \text{and} \quad Y_{\text{I}}=-(1+\mu^2)X_{\text{II}}, \tag{8.10}$$

which shows that the requirements $Y_{\text{I}}\geqq 0$ and $X_{\text{II}}\geqq 0$ are incompatible unless $Y_{\text{I}}=X_{\text{II}}=0$. Accordingly, we have to say that these two representations are so to speak exclusive each other in the sense that when a given field of turbulence is composed of transverse and quasi-isotropic terms, then the decomposition into axial and quasi-isotropic ones will be physically fictious, where the axial term, conflicting the Cramér's requirements, does not represent a possible field of motion, and *vice versa.* So long as we bear this in mind, interchanging the representations is only a simple algebra after (8.10), but it seems to be preferable to avoid the use of any fictious element in the dynamical theories. Moreover, there is no kinematical reason why we may disregard certain cases in which the spectrum tensor must be differently decomposed for different $\boldsymbol{\kappa}$. In such cases appropriate formulation will become more or less troublesome.

Thus, the third representation expressed in terms of transverse and axial fields will be added. Namely,

$$\Phi_{ij}(s;\boldsymbol{\kappa})=Y_{\text{III}}[\kappa^2(1-\mu^2)\delta_{ij}-\ldots\text{like (8.3)}]+X_{\text{III}}[2\mu^2\kappa_i\kappa_j-\ldots\text{like (8.7)}] \tag{8.11}$$

is the desired expression, where in view of the relations

$$(1-\mu^2)X_{\text{III}}=Z_{\text{I}}=Z_{\text{II}}+(1-\mu^2)X_{\text{II}}, \tag{8.12a}$$

and
$$(1-\mu^2)Y_{\text{III}}=(1-\mu^2)Y_{\text{I}}+(1+\mu^2)Z_{\text{I}}=(1+\mu^2)Z_{\text{II}}, \tag{8.12b}$$

negative values will never occur in both of X_{III} and Y_{III} as in Z_{I} and Z_{II}. We can therefore use (8.11) for every cases, being free from the above inconvenience. To be readily seen, we have now

$$\Psi_a(\kappa)=2\pi\kappa^4\langle(1-\mu^2)Y_{\text{III}}\rangle_\mu, \tag{8.13a}$$

$$\Psi_b(\kappa)=2\pi\kappa^4[2\langle(1-\mu^2)^2X_{\text{III}}\rangle_\mu-\langle(1-\mu^2)Y_{\text{III}}\rangle_\mu], \tag{3.13b}$$

$$\frac{\partial X_{\text{III}}}{\partial t}=\frac{1}{1-\mu^2}\frac{\partial Z_{\text{I}}}{\partial t}, \tag{8.14a}$$

$$\frac{\partial Y_{\text{III}}}{\partial t}=\frac{1+\mu^2}{1-\mu^2}\frac{\partial Z_{\text{II}}}{\partial t}, \tag{8.14b}$$

and the isotropic conditions are that $(1+\mu^2)X_{\mathrm{III}}=Y_{\mathrm{III}}$ and that $(1-\mu^2)X_{\mathrm{III}}$ is a function of κ (and of t) only.

Let us end this section with an illustrative example shown below.

The spectrum of turbulence behind a damping screen.

The theory of damping screen or a plane sheet of woven wire gauze used in the wind-tunnel practice was successfully established by G. I. Taylor and Batchelor **[6]**, and [2] **4. 2.** Suppose that the spectrum tensor of the turbulence is $\Phi_{ij}'(\boldsymbol{\kappa})$ far upstream and is $\Phi_{ij}''(\boldsymbol{\kappa})$ far downstream from the gauze placed at right angles to the main flow. According to their analysis, the connexion between these two tensors can be written concisely as

$$\Phi_{ij}''(\boldsymbol{\kappa})=\sigma_{ki}^{*}\sigma_{lj}\Phi_{kl}'(\boldsymbol{\kappa}), \tag{8'.1}$$

with

$$\sigma_{ij}(\boldsymbol{\kappa})=\frac{(J-a)(\kappa^2\delta_{1j}-\kappa_1\kappa_j)}{\kappa_2{}^2+\kappa_3{}^2}\delta_{1i}+a\delta_{ij}, \tag{8'.2}$$

where the κ_1-axis is directed downstream, and J is related to the aerodynamical characteristics, a and k, of the screen through

$$J=\frac{(\beta+\iota)[2a\beta+\iota(ak-1-a)]}{(\beta-\iota)[2\beta+\iota(k+1+a)]}, \tag{8'.3}$$

on putting $\beta=\kappa_1(\kappa_2{}^2+\kappa_2{}^3)^{-1/2}$.

When the turbulence far upstream is axisymmetric about the direction of the main flow (or specially, is isotropic), then so is the case in the section far downstream, too. For ready comparison with the original work, we shall assume here that

$$\Phi_{ij}'(\boldsymbol{\kappa})=\frac{\mathcal{E}'(\kappa)}{4\pi\kappa^4}(\kappa^2\delta_{ij}-\kappa_i\kappa_j), \tag{8'.4}$$

namely, that the turbulence far upstream is isotropic. Substituting (8'. 4) into (8'. 2), and replacing now δ_{1j} by s_i, κ_1 by $\kappa\mu$, and $\kappa_2{}^2+\kappa_3{}^2$ by $\kappa^2(1-\mu^2)$, respectively, (8'. 1) yields

$$\Phi_{ij}''(\boldsymbol{\kappa})=\frac{\mathcal{E}'(\kappa)}{4\pi\kappa^4(1-\mu^2)}[(a^2+\mu^2JJ^*)(\text{transverse term, like (8.3)})+JJ^*(\text{axial term, like (8.7)})]. \tag{8'.5}$$

In general, k (the resistance coefficient of the gauze) and a (the index of deflection at the gauze, see Fig. 2) are the functions of the incidence angle ψ, but experiments show that under the ordinary conditions we can regard them as practically constant and the empirical relationship

$$a^2(1+k)=1.21 \tag{8'.6}$$

seems to be confirmed. It is thus a straightforward procedure to find the attenuations of axial and transverse energies. In fact, since neither of a and J depends on κ, we have, noting that $E_{/\!/}'=E_{\perp}'=\frac{1}{3}\int_0^\infty\mathcal{E}(\kappa)\,d\kappa$, readily

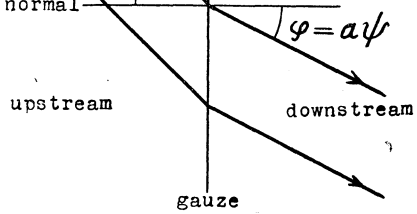

Fig. 2.

$$E_{//}''=\frac{1}{2}<(1-\mu^2)JJ^*>_\mu\int_0^\infty \mathcal{E}'(\kappa)\,d\kappa=\frac{3}{2}<(1-\mu^2)JJ^*>_\mu E_{//}', \qquad (8'.7)$$

and
$$E_\perp''=\frac{1}{4}<a^2+\mu^2JJ^*>_\mu\int_0^\infty \mathcal{E}'(\kappa)\,d\kappa=\frac{3}{4}(a^2+<\mu^2JJ^*>_\mu)E_\perp', \qquad (8'.8)$$

after (5.6),(5.7) and (8.13), and hence

$$\eta_{//}=\frac{E_{//}''}{E_{//}'}=\frac{3}{2}<(1-\mu^2)JJ^*>_\mu$$

$$=\frac{A^2-4a^2}{B^2-4}\Big[1+\frac{3}{2}\Big(\frac{A^2}{A^2-4a^2}-\frac{B^2}{B^2-4}\Big)\Big(1-\frac{4}{B\sqrt{B^2-4}}\operatorname{arctanh}\frac{\sqrt{B^2-4}}{B}\Big)\Big], \qquad (8'.9)$$

$$\eta_\perp=\frac{E_\perp''}{E_\perp'}=\frac{3}{4}(a^2+<\mu^2JJ^*>_\mu)=a^2+\frac{A^2}{8}-\frac{B^2}{8}\eta_{//}, \qquad (8'.10)$$

where $A=ak-1-a$, and $B=k+1+a>2$ (in virtue of (8'.6)). (8'.11)

These are exactly the same as those obtained by original authors.

9. Notes on the expression of the correlation tensor. Returning now to $\boldsymbol{r}$-space, we shall consider again the double velocity correlation tensor. Let us see the expression (3.9) on p. 66. It will be found that like the energy spectrum tensor also the correlation tensor is composed of two parts, namely, q_1- and q_2-terms satisfying the kinematical conditions respectively. We may regard this as the decomposition in $\boldsymbol{r}$-space as it were. And, considering that $\partial q_1/\partial m=0$ together with $q_2=0$ are the conditions of isotropy, the q_1-term can be interpreted as an analogue of the quasi-isotropic field. Whereas it must be the field of transverse turbulence that is represented by the function q_2, because q_2 disappears in the expression of the correlation $\overline{u_{//}u_{//}'}=R_{ij}(\boldsymbol{s};\boldsymbol{r})s_is_j$.[14] Correspondingly, in connexion with the series expansions

$$q_1(r,rm)=q_1(0)+r^2[q_{02}^{(1)}+m^2q_{22}^{(1)}]+r^4[q_{04}^{(1)}+m^2q_{24}^{(1)}+m^4q_{44}^{(1)}]+\dots, \qquad (9.1a)$$

and

$$q_2(r,rm)=q_2(0)+r^2[q_{02}^{(2)}+m^2q_{22}^{(2)}]+r^4[q_{04}^{(2)}+m^2q_{24}^{(2)}+m^4q_{44}^{(2)}]+\dots, \qquad (9.1b)$$

near $r=0$, the relations

$$E_{//}=-q_1(0) \quad \text{and} \quad E_\perp=-q_1(0)-\frac{1}{2}q_2(0) \qquad (9.2)$$

should be noticed (cf. (5.5) and (5.6)).

But, to be obvious, $-q_1(0)$ and $-q_2(0)$ is necessarily non-negative, and accordingly it follows $E_{//}\leqq E_\perp$ (oblate type, say), so long as each elementary field is a physically possible one. This suggests that in the case of $E_{//}>E_\perp$ (oblong type, say) an alternative way of expression would be preferable, the above representation inevitably being fictious from the physical point of view. In other words, so far the expression equivalent to the decomposition II or III of the foregoing section is wanting. The following remarks will be worth noting in this respect.

[14] The correct relations between q_1 and q_2 in $\boldsymbol{r}$-space and Y_{I} and Z_{I} in $\boldsymbol{k}$-space, however, are reserved here.

As is pointed out by Chandrasekhar [3], the general axisymmetric skew tensor of the second-order must be a linear combination of the six fundamental tensors

$$\epsilon_{ijk}r_k, \quad \epsilon_{ijk}s_k, \quad s_j\epsilon_{ikl}s_kr_l, \quad r_j\epsilon_{ikl}s_kr_l, \quad s_i\epsilon_{jkl}s_kr_l \quad \text{and} \quad r_i\epsilon_{jkl}s_kr_l. \tag{9.3}$$

of which, however, only four are linearly independent, and further the contribution from $\epsilon_{ijk}s_k$ can be eliminated by a kind of gauge-transformation in view of the identity

$$\mathrm{curl}^{(j)}(q\epsilon_{ijk}s_k) = \mathrm{curl}^{(j)}\Big(\frac{1}{r}\frac{\partial q}{\partial m}s_j\epsilon_{ikl}s_kr_l\Big) + \mathrm{curl}^{(j)}\Big[\Big(\frac{1}{r}\frac{\partial q}{\partial r} - \frac{m}{r^2}\frac{\partial q}{\partial m}\Big)r_j\epsilon_{ikl}s_kr_l\Big], \tag{9.4}$$

for an arbitrary function q of r and rm.[15] Mathematically, therefore, three tensors of (9.3) suffice to construct the second-order skew tensor $q_{ij}(\boldsymbol{s};\boldsymbol{r})$ in general for the purpose of deriving an associated axisymmetric true tensor $\mathrm{curl}^{(j)}q_{ij}(\boldsymbol{s};\boldsymbol{r})$. Thus, taking the first, the third and the fourth of (9.3), we have

$$q_{ij}(\boldsymbol{s};\boldsymbol{r}) = q_1\epsilon_{ijk}r_k + q_2s_j\epsilon_{ikl}s_kr_l + q_3r_j\epsilon_{ikl}s_kr_l, \tag{9.5}$$

and by means of the fact that the symmetry in indices of $R_{ij}(\boldsymbol{s};\boldsymbol{r})$ requires

$$q_3 = \frac{1}{r}\frac{\partial q_1}{\partial m}, \tag{9.6}$$

we arrive at the assumption (3.8). But, in principle, the choice of fundamental tensors is certainly left at our disposal; other combinations will yield other representations. Although these various representations are all identical in the mathematical sense, their physical implications are not the same.

Now, we need only to find out the adequate combination of fundamental tensors (9.3), corresponding to the desired mode of decomposition in $\boldsymbol{r}$-space. It seems, nevertheless, we shall gain little from the attempt to do so, since the condition of indices-symmetry has no longer such a tractable form as (9.6) for any assumption other than (9.5). We must, therefore, be contented with the previous representation which is to be properly applied to axisymmetric turbulence of oblate type. For the case of oblong type, the care should be taken of its fictious character.

10. Acknowledgements. In conclusion, the author's best thanks are due to Professor Hikoji Yamada of Kyushu University for continual encouragement and valuable instructions. He is also grateful to Professor Tomomasa Tatsumi of Kyoto University who has been kind enough to help him in the plan of this work.

[15] The symbol $\mathrm{curl}^{(j)}$ indicates that the curl-operation should be applied to the index j.

Appendix

Evaluation of the surface integral

We shall here suppose that

$$\chi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})=A\kappa_i\kappa_j+B\delta_{ij}+Cs_is_j+Ds_i\kappa_j+E\kappa_is_j \tag{A. 1}$$

is a second-order axisymmetric tensor in general, and show the evaluation of the surface integral $\int\chi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})\,\mathrm{d}S(\kappa)$ introduced in section **4.**

In the first place, let us consider the rigid rotation of coordinate axes from the κ_i-system to $\boldsymbol{s}$-system (p. 70) in which κ_1', κ_2' and κ_3' are the coordinates. The transformation is clearly represented by

$$\left.\begin{aligned}&\kappa_i=\gamma_{ij}\kappa_j'\,,\\ \text{where}\qquad &\gamma_{ij}=\gamma_{ji}\qquad\text{and}\qquad\gamma_{ik}\gamma_{jk}=\delta_{ij}\,,\end{aligned}\right\} \tag{A. 2}$$

but, since the positive κ_1'-axis is directed in parallel with $\boldsymbol{s}$, we have

$$\gamma_{1i}=\gamma_{i1}=s_i\,, \tag{A. 3}$$

and accordingly it follows

$$\gamma_{i2}\gamma_{j2}+\gamma_{i3}\gamma_{j3}=\gamma_{ik}\gamma_{jk}-\gamma_{i1}\gamma_{j1}=\delta_{ij}-s_is_j\,. \tag{A. 4}$$

Then, introducing the spherical polar coordinates $(\kappa,\,\theta,\,\phi)$ through

$$\kappa_1'=\kappa\cos\theta=\kappa\mu\,,\qquad\kappa_2'=\kappa\sin\theta\cos\phi\,,\qquad\kappa_3'=\kappa\sin\theta\sin\phi\,, \tag{A. 5}$$

the surface element $\mathrm{d}S(\kappa)$ of the sphere of radius κ becomes

$$\mathrm{d}S(\kappa)=\kappa^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=-\kappa^2\mathrm{d}\mu\,\mathrm{d}\phi\,, \tag{A. 6}$$

and we have

$$\int A\kappa_i\kappa_j\,\mathrm{d}S(\kappa)=\kappa^2\int_{-1}^{1}\int_0^{2\pi}A\gamma_{ik}\gamma_{jl}\kappa_k'\kappa_l'\,\mathrm{d}\phi\,\mathrm{d}\mu\,,$$

and the like. But, we find readily

$$\left.\begin{aligned}&\int_0^{2\pi}\kappa_1'\mathrm{d}\phi=2\pi\kappa\mu\,,\qquad\int_0^{2\pi}\kappa_2'\mathrm{d}\phi=\int_0^{2\pi}\kappa_3'\mathrm{d}\phi=0\,,\\ &\int_0^{2\pi}\kappa_1'^2\mathrm{d}\phi=2\pi\kappa^2\mu^2\,,\qquad\int_0^{2\pi}\kappa_2'^2\mathrm{d}\phi=\int_0^{2}\kappa_3'^2\mathrm{d}\phi=\pi\kappa^2(1-\mu^2)\,,\\ &\int_0^{2\pi}\kappa_k'\kappa_l'\mathrm{d}\phi=0\qquad(\text{for }k\neq l)\,.\end{aligned}\right\} \tag{A. 7}$$

From these together with (A. 3) and (A. 4), considering that A, B, ... are independent of ϕ, there comes the result

$$\left.\begin{aligned}\frac{1}{4\pi\kappa^2}\int\chi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})\,\mathrm{d}S(\kappa) &= \Big[\frac{\kappa^2}{2}\langle(1-\mu^2)A\rangle_\mu+\langle B\rangle_\mu\Big]\delta_{ij}\\ &+\Big[-\frac{\kappa^2}{2}\langle(1-3\mu^2)A\rangle_\mu+\langle C\rangle_\mu+\kappa\langle\mu(D+E)\rangle_\mu\Big]s_i s_j\,,\\ \text{where we write}\qquad \int_{-1}^{1}(\ldots)\,\mathrm{d}\mu &= 2\langle\ldots\rangle_\mu\,.\end{aligned}\right\}\quad (\mathrm{A}.8)$$

Another way to obtain the same result is as follows:
In view of the condition of axisymmetry, we can assume *ab initio* that

$$\int\chi_{ij}(\boldsymbol{s}\,;\boldsymbol{\kappa})\,\mathrm{d}S(\kappa)=A_a(\kappa)\,\delta_{ij}+A_b(\kappa)\,s_i s_j\,.$$

Then we have

$$\int\chi_{ii}\,\mathrm{d}S=\int[\kappa^2A+3B+C+\kappa\mu\,(D+E)]\,\mathrm{d}S(\kappa)=3A_a+A_b\,,$$

together with

$$s_i s_j\int\chi_{ij}\,\mathrm{d}S=\int[\kappa^2\mu^2A+B+C+\kappa\mu(D+E)]\,\mathrm{d}S(\kappa)=A_a+A_b\,.$$

Solving these (algebraic) equations with respect to A_a and A_b, and considering that the integrands are now scalar quantities, we get readily (A.8) again.

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ERRATA

Sand Transport along a Model Sandy Beach by Wave Action

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Page 1, line 26 from the bottom, for "Dr. Iwagki" read "Dr. Iwagaki".

Page 21, Fig.-11, for the oridinate of the figure "$\dfrac{S}{L_0 H_0 \dfrac{gH_0}{v_s^2}} \times 10^8$"

read "$\dfrac{S}{L_0 H_0 \dfrac{gH_0}{v_s^2}} \times 10^6$".

Page 24, Appendix

	For	Read
Colomn 10	$Qs / L_0 \sqrt{g H_0^3}$	$(Qs / L_0 \sqrt{g H_0^3}) \times 10^4$
11	$S_b / L_0 H_0 \dfrac{g H_0}{v_s^2}$	$\left(S_b / L_0 H_0 \dfrac{g H_0}{v_s^2} \right) \times 10^6$
12	$S / L_0 H_0 \dfrac{g H_0}{v_s^2}$	$\left(S / L_0 H_0 \dfrac{g H_0}{v_s^2} \right) \times 10^6$