九州大学学術情報リポジトリ
Kyushu University Institutional Repository

ON APPROXIMATE EXPRESSIONS OF SOLITARY WAVE

YAMADA, Hikoji
Research Institute for Applied Mechanics, Kyushu University

https://doi.org/10.5109/7162225

出版情報：Reports of Research Institute for Applied Mechanics. 6 (21), pp.35-47, 1958. 九州大学応用力学研究所
バージョン：
権利関係：

KYUSHU UNIVERSITY

Reports of Research Institute for Applied Mechanics
Vol. VI, No. 21, 1958

ON APPROXIMATE EXPRESSIONS OF SOLITARY WAVE

By Hikoji YAMADA

Abstract.——The approximate solution due to Boussinesq is examined and proved to be a good approximation—the best one among the ones now existing—up to considerable height of the wave. For this purpose two accurate numerical solutions, the one of the extreme height and the other of the height (relative to the depth of water) 0.4808 are calculated and in comparison with these the degree of approximation is decided. The solution recently proposed by B. A. Packham is also discussed at some length.

Introduction.——J. W. Daily and S. C. Stephens(1) of M. I. T. Hydrodynamics Laboratory have found the remarkable fact that the solitary wave of permanent type produced in an experimental flume is closely approximated by the classical Boussinesq's formula(2) and deviates appreciably from other solutions such as that due to Lord Rayleigh,(3) that due to J. McCowan,(4) or that due to A. Weinstein.(5) The reason for this discrepancy between experimental facts and the seemingly improved solutions is to be sought either in the assumption that the experimental wave is hydrodynamically *not* the one of the theory, or in the unexpected fact that the Boussinesq's solution is the best one so far obtained.

Analytical discussion of the comparative accuracies of present solutions is difficult because of their different standpoints, nonlinearity of the problem prohibiting the easy transformability of the standpoints to each other. We have, however, given recently a method of Fourier series expansion(6) which enables, somewhat tedious it may be, the numerical determination of the coefficients up to the required degree of approximation. By means of this method we have calculated the highest wave, which differs a little from the McCowan's one.(7) By the same method we can obtain an accurate solution in the neighborhood of the experimental wave height, and by means of these we are able to decide numerically the comparative accuracies of theoretical solutions and the correspondence between theory and experiment.

---

(1) *c.f.* Coastal Engineering, vol. 3 (1952), div. 1, chap. 2; Council on Wave Research, Calif. Univ.

(2) First appeared in the Comptes Rendus de L'Acad. des Sci. Paris, June 19, 1871.

(3) Phil. mag., vol. 1, p. 257 (1876).

(4) Phil. mag., vol. 32, p. 45 (1891).

(5) First reported in the Comptes Rendus de L'Acad. dei Lincei, vol. 3, no. 8, p. 463 (1926).

(6) Rep. Res. Inst. for Appl. Mech. Kyushu Univ. (this Journal), vol. 5, no. 18, p. 53 (1957).

(7) Phil. mag. (5), vol. 38, p. 351 (1894).

§ 1. Method of numerical solution.

——Here we describe the outline of our method of solution, which follows after the method of T. Lévi-Civita.(8) The coordinate-system $O\text{-}xy$ of the physical plane $z=x+iy$ is fixed to the permanent solitary wave which runs with velocity U to left, and consequently the wave form stands still relative to the axes and water flows steadily from left to right, the velocity at infinite distance being U. x-axis coincides with the bottom of the canal and is directed to right, y-axis is vertical and upwards, origin O being just under the centre of crest O_1; the depth of water at infinite distance in both directions is equal and H. Wave height is denoted by A.

Our wave is irrotational and the complex potential $W=\varphi+i\psi$, where φ is the potential and ψ the stream function, of the flow is the function, after adequate selection of constants to let coincide origin to origin, of the physical plane z in such a way that the real and imaginary axes correspond each to each and the water surface to the level line $\psi=UH$. Our main object is the establishment of the analytical form $W(z)$ of this function or its derivative

$$\frac{1}{U}\frac{dW}{dz}=qe^{-i\theta}; \tag{1}$$

here q is the magnitude and θ the angle of inclination of flow velocity, the former measured with U as unit and the latter from Ox-direction upwards.

We transform this W-plane on an auxiliary plane $\zeta=\xi+i\eta$ by the relation

$$\frac{1-\zeta}{1+\zeta}=\cosh\left(\frac{\pi W}{2UH}\right)$$

$$\text{i.e.}\qquad \zeta=-\tanh^2\left(\frac{\pi W}{4UH}\right). \tag{2}$$

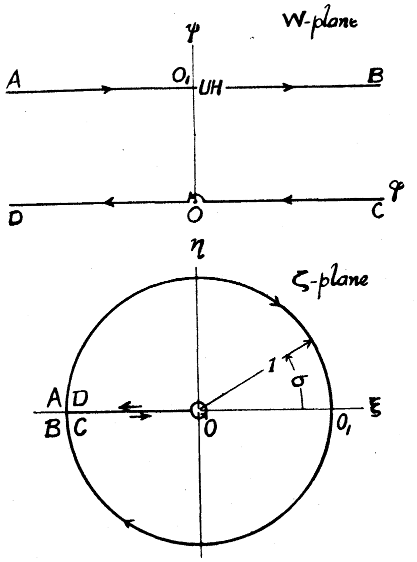

Fig. 1.

By this transformation the water region $(O\leq\psi\leq UH)$ is mapped onto the unit circle of ζ-plane as is shown in Fig. 1, where the correspondence of boundaries being shown by the same alphabetical letters and arrows as usual in such a case. The unit circle region has a cut along the negative real axis from origin to $\zeta=-1$. We determine the flow field of z-plane by means of this cut unit circle of ζ-plane.

Following Lévi-Civita we introduce the Ω-function by the definition

$$\Omega=i\log\left(\frac{1}{U}\frac{dW}{dz}\right), \tag{3}$$

---

(8) Math. Annalen, vol. 93, p. 264 (1925).

which, by means of (1), is equivalent to

$$\Omega=\theta+i\tau \ (\tau=\log q), \quad \text{or} \quad e^{-i\Omega}=qe^{-i\theta}. \tag{4}$$

As the flow velocity does vanish at nowhere in water $\Omega(z)$ is a holomorphic function of z in the water region, and then through (2) we have $\Omega(\zeta)$ defined on the cut unit circle and holomorphic in it.

We are dealing with a symmetrical wave and then it is easily understood that $\tau-i\theta=-i\Omega$ in the unit circle has values conjugate to each other at two conjugate points ζ and $\bar{\zeta}$, and has real values on the real axis, not only on its positive part but also on the negative part. By the Schwarz principle therefore $-i\Omega$, and then Ω itself, is holomorphic throughout the interior of the unit cicle without any cut, which is the property identical to that of Lévi-Civita's case, permitting us to employ his theory without any alteration in our case.

From (2) we have by differentiation

$$\frac{1}{U}\frac{dW}{d\zeta}=\frac{2Hi}{\pi}\cdot\frac{1}{\sqrt{\zeta}\,(1+\zeta)} \quad (-\pi<\arg\zeta\leq\pi), \tag{5}$$

and the left hand side being changed by (3) into

$$\frac{1}{U}\frac{dW}{d\zeta}=\frac{1}{U}\frac{dW}{dz}\cdot\frac{dz}{d\zeta}=e^{-i\Omega}\frac{dz}{d\zeta},$$

this reduces into

$$dz=\frac{2Hi}{\pi}\frac{e^{i\Omega(\zeta)}}{\sqrt{\zeta}\,(1+\zeta)}d\zeta, \tag{6}$$

and gives by quadrature the transformation function $z=z(\zeta)$, or $\zeta=\zeta(z)$, if we can determine $\Omega(\zeta)$ in a way or another. $\Omega(z)$, the required velocity field, would then be obtained by mere substitution and so our problem be settled. On the circumference of unit circle $\zeta=e^{i\sigma}$ $(-\pi<\sigma\leq\pi)$ (6) reduces into

$$ds=-\frac{H}{\pi}\frac{d\sigma}{q\cos\frac{\sigma}{2}}, \tag{6'}$$

ds being the arc element along the surface stream line on the z-plane.

The surface stream line is generally an analytic function of arc length s, and consequently of σ, along which the surface condition being fulfilled. The surface condition is nothing but the fact that the surface stream line contacts with a constant atmospheric pressure all along its length, and by means of the Bernoulli's theorem it can be expressed in an equation:

$$q^2+\frac{2gy}{U^2}=\text{const}. \tag{7}$$

on the water surface; differentiating this along the arc length we have

$$q\frac{dq}{ds} = -\frac{g}{U^2}\sin\theta\,, \tag{7'}$$

and using (6′) into this the surface condition is rewritten into

$$\cos\frac{\sigma}{2}\cdot q^2\frac{dq}{d\sigma} = p\sin\theta\,, \tag{8}$$

$$p = 1/\pi M^2, \qquad M = U/\sqrt{gH}\,, \tag{9}$$

which is our condition to be applied on the circumference of the unit circle $|\zeta| = 1$.

In this way our problem is reduced to the determination of a holomorphic $\Omega(\zeta)$ in the unit circle, which fullfils the boundary condition (8), the symmetry condition $\Omega(\zeta) = -\overline{\Omega(\bar{\zeta})}$ above stated and finally $\Omega(-1) = 0$ which is evident. The problem has certainly a solution(9) when p is an arbitrary value less than $1/\pi$ and sufficiently near to it, and we have proposed a method, in its essential part the power series expansion of $\Omega(\zeta)$:

$$\Omega(\zeta) = i\sum_{n=0}^{\infty} a_n\zeta^n \tag{10}$$

is adopted as a useful device for the numerical determination of solution. The coefficients a_n's are all real, so as to realise the symmetrical character required. On the circumference $\zeta = e^{i\sigma}$

$$\theta(\sigma) = -\sum_{n=1}^{\infty} a_n\sin(n\sigma)\,, \qquad \tau(\sigma) = \sum_{n=0}^{\infty} a_n\cos(n\sigma)\,. \tag{11}$$

We are now in a position to determine the numerical values of the a_n's through the boundary condition (8). At first we fix the number $N+1$ of a_n's to be taken in, neglecting the others, and attempt to determine these $a_0, a_1, \cdots, a_N$ by an iteration process as good as possible.(10) One cycle of iteration consists of two steps: the first is the derivation of $\tau(\sigma)$ from $\theta(\sigma)$ through the surface condition, which is now used in the form

$$q(\sigma)^3 = q(0)^3 + 3p\int_0^{\sigma}\sin\theta(\sigma')\sec\frac{\sigma'}{2}\,d\sigma'\,, \tag{12}$$

where we assign to $q(0)$ a constant value once for all cycles, and p has to be determined through the condition $\tau(\pi) = 0$, which follows from $\Omega(-1) = 0$, used in (12):

$$3p\int_0^{\pi}\sin\theta(\sigma')\sec\frac{\sigma'}{2}\,d\sigma' = 1 - q(0)^3\,. \tag{12'}$$

The second step is the determination of $\theta(\sigma)$ which is harmonic conjugate to $\tau(\sigma)$ above obtained. For this we expand $\tau(\sigma)$ into a Fourier cosine series

---

(9) *c.f.* K. O. Friedrichs and D. H. Hyers, Comm. Pure Appl. Math., vol. 7, no. 3, p. 517 (1954).

(10) *c.f.* G. Birkhoff *et al.*, Proc. Symp. Appl. Math., vol. 4, p. 117 (1953).

with coefficients $a_0, a_1, \cdots, a_N$ by virtue of $N+1$ values of it at the points $\sigma = n\pi/N$ $(n = 0, 1, \cdots, N)$. With these coefficients a_n's the required $\theta(\sigma)$ can be written down at once by the formula (11).

Starting from an arbitrary but adequately chosen θ the iterative process converges practically after a number of cycles. The set of final values of θ, τ and p is the all that is required, and every quantity of interest can be inferred from them without difficulty. For examples the wave velocity and height are given by

$$U = \left(\frac{gH}{\pi p}\right)^{1/2}, \quad a = \frac{A}{H} = (2\pi p)^{-1}, \tag{13}$$

and wave profile (x, y) is obtained by mere quadrature of the relation

$$\frac{dx + i dy}{H} = -\frac{1}{\pi}\frac{e^{i\theta(\sigma)}}{q(\sigma)} \sec\frac{\sigma}{2}\, d\sigma, \tag{14}$$

which is given by (6) with $\zeta = e^{i\sigma}$.

Our solution $(\theta, \tau; p)$ satisfies the surface condition and the conjugate relation simultaneously, the former, however, approximately for the satisfaction of it is exact only at a number of discrete points $\sigma = n\pi/N$ $(n = 0, 1, \cdots, N)$. As an indication of the degree of approximation, therefore, we can take the error in the satisfaction of the original surface condition (7), which is written with a new ordinate $y_1 = y - (A + H)$ in the form:

$$q(\sigma)^2 - q(0)^2 - 2\pi p\left(\frac{-y_1}{H}\right) = 0. \tag{15}$$

§ 2. **Two accurate solutions.**——One of our two accurate solutions is that of the extreme height and has been given recently.(6) Some of the results are

$$a = 0.8284, \quad M = 1.287, \tag{16}$$

all the other numericals being useless in the present paper. The error in the fulfilment of surface condition (15) in this case is at most 1% of the representative term $q(\sigma)^2$.

The other solution is chosen in the neighborhood of $a = 0.5$, which is the wave studied experimentally,(1) and calculated as follows. To secure rapid convergence we started with the Boussinesq's solution as the first approximation. With the notations

$$\frac{x}{H} = \xi, \quad \frac{y}{H} = \eta; \quad \sqrt{\frac{3a}{4}} = \mu, \tag{17}$$

his solution is

$$\eta = a\,\mathrm{sech}^2(\mu\xi), \quad M = \sqrt{1+a}; \tag{18}$$

consequently by Bernoulli's theorem

$$q = \left\{1 - \frac{2a}{1+a}\mathrm{sech}^2(\mu\xi)\right\}^{1/2}, \tag{19}$$

$$\tan\theta = \frac{d\eta}{d\xi} = -2a\mu \tanh(\mu\xi)\,\mathrm{sech}^2(\mu\xi)\,. \tag{20}$$

For the transformation of this solution into ζ-plane we require the correspondence between $\mu\xi$ and σ, which is aquired by the insertion of the line-element of surface stream line:

$$ds = H\sqrt{d\xi^2 + d\eta^2} = H\sqrt{1+\tan^2\theta}\,d\xi$$

into the relation (6′):

$$q(\mu\xi)\sqrt{1+\tan^2\theta}\,d\xi = -\frac{1}{\pi}\frac{d\sigma}{\cos\frac{\sigma}{2}}\,,$$

which gives by integration

$$\tan\left(\frac{\pi+\sigma}{4}\right) = \exp\left\{-\frac{\pi}{2\mu}\int_0^{\mu\xi} q\sqrt{1+\tan^2\theta}\,d(\mu\xi)\right\}. \tag{21}$$

By this relation (19), (20) are converted into $q(\sigma)$ and $\theta(\sigma)$.

We take $a = 0.5000$, and consequently $q(0) = 1/\sqrt{3}$. Values of σ are calculated for equal interval points of $\mu\xi$ by numerical quadrature, and then by inverse interpolation values of $\mu\xi$ are given for equal interval points of σ. Our first approximation $q_1(\sigma)$, $\theta_1(\sigma)$ are then easily found.

For the convenience only of calculation we have introduced an auxiliary holomorphic function in the unit circle:

$$\Omega_0(\zeta) = \frac{i}{3}\log\frac{1-\lambda\zeta}{1+\lambda}\,, \tag{22}$$

which gives on the circumference

$$\left.\begin{aligned} &\Omega_0(e^{i\sigma}) = \theta_0(\sigma) + i\tau_0(\sigma)\,;\\ &\sin(3\theta_0(\sigma)) = \frac{\lambda}{1+\lambda}\frac{\sin\sigma}{q_0(\sigma)^3}\,,\\ &q_0(\sigma) = \left\{1 - \frac{2\lambda}{(1+\lambda)^2}(1+\cos\sigma)\right\}^{1/6}. \end{aligned}\right\} \tag{23}$$

The constant λ is related to $q_0(0)$ by the relation

$$\lambda = \{1 - q_0(0)^3\}\{1 + q_0(0)^3\}^{-1}\,, \tag{24}$$

and as we have $q(0)$ fixed to the value $1/\sqrt{3}$, it would be natural to take λ so as to give $q_0(0) = 1/\sqrt{3}$. Our auxiliary function is, however, arbitrary in nature and consequently λ can also be so. We used in the practice

Table 1

a_0	− 0.14512
a_1	+ 0.02302
a_2	+ 0.11570
a_3	+ 0.02254
a_4	+ 0.03873
a_5	+ 0.00535
a_6	+ 0.01740
a_7	− 0.00022
a_8	+ 0.00977
a_9	− 0.00201
a_{10}	+ 0.00670
a_{11}	− 0.00259
a_{12}	+ 0.00292
p	0.22022 (0.22049)
M	1.2023 (1.2015)
a	0.4818 (0.4812)

$q_0(0)=0.52653$ (corresponding to $\lambda=0.74525$), which we happened to have and is sufficiently near to $q_0(0)=1/\sqrt{3}$.

The process of iteration is as given in preceding section except that we expand now $\Omega(\zeta)-\Omega_0(\zeta)$ into a power series about the origin:

$$\Omega(\zeta)=\Omega_0(\zeta)+i\sum_{n=0}^{\infty}a_n\zeta^n. \tag{25}$$

After several cycles of iteration a_n's are as given in Table 1, and $\theta(\sigma)$, $q(\sigma)$, calculated therewith are tabulated in Table 2 columns 2 and 3. The numbers in brackets there are values one cycle before, and we see good convergence of the iteration process.

Wave profile $(x/H, y/H)$ is obtained from values of (q, θ) by means of (14), which is the numbers in Table 2 columns 4 and 5, and inscribed in

Table 2

σ (deg.)	θ	q	$-x/H$	$-y_1/H$	$\dfrac{q^2+\frac{2\pi p y_1}{H}-q^2(0)}{q^2}\times 100$
0	0.0000 (0.0000)	0.5774	0.0000	0.0000	0.0
3.75	0.0151	0.5780			
7.5	0.0291 (0.0292)	0.5794	0.0721	0.0011	0.3
15	0.0556 (0.0558)	0.5823			
22.5	0.0843 (0.0845)	0.5873	0.2162	0.0092	−0.3
30	0.1104 (0.1108)	0.5965			
37.5	0.1311 (0.1314)	0.6073	0.3600	0.0250	0.2
45	0.1494 (0.1498)	0.6182			
52.5	0.1663 (0.1667)	0.6308	0.5044	0.0467	0.0
60	0.1799 (0.1803)	0.6449			
67.5	0.1907 (0.1911)	0.6594	0.6513	0.0734	0.0
75	0.1996 (0.1999)	0.6751			
82.5	0.2048 (0.2049)	0.6920	0.8040	0.1042	0.3
90	0.2069 (0.2070)	0.7079			
97.5	0.2096 (0.2097)	0.7237	0.9672	0.1385	−0.2
105	0.2111 (0.2111)	0.7425			
112.5	0.2062 (0.2062)	0.7626	1.1480	0.1770	0.6
120	0.1985 (0.1984)	0.7794			
127.5	0.1948 (0.1948)	0.7958	1.3585	0.2195	−1.0
135	0.1904 (0.1904)	0.8191			
142.5	0.1745 (0.1744)	0.8451	1.6216	0.2696	1.1
150	0.1536 (0.1534)	0.8631			
157.5	0.1433 (0.1430)	0.8804	1.9978	0.3280	−1.6
165	0.1331 (0.1327)	0.9181			
172.5	0.0878 (0.0875)	0.9721	2.7549	0.4203	3.2
176.25	0.0472 (0.0473)	0.9925			
180	0.0000 (0.0000)	1.0000	∞	0.4808	0.2

Fig. 2 as accurate one. Its accuracy is as seen in Table 2 column 6, where errors amount to several % at most, and almost equal to that of the extreme wave aboved mentioned. The height and velocity of wave are given as

$$a = 0.4808 \text{ (or } 0.4818), \quad M = 1.2023, \tag{26}$$

and it is interesting to note that the point (0.481, 1.202) is just on the experimental (a, M) curve of the M. I. T. Hydrodynamical Laboratory.

§ 3. Discussion of approximate solutions (1).——Boussinesq's solution is accurate when the wave height is small, and becomes rough as the height increases, as the deduction of the solution shows.(11) In fact, $q = 0$ at the crest has the consequences

$$a = 1, \quad M = \sqrt{2}, \tag{27}$$

which are remote from the accurate values (16). Moreover it has a round crest contrary to the theory, and can not be seen as an approximation. When $a \fallingdotseq 0.5$, however, experimental evidence is in favor of the Boussinesq's and the accordance between them is, as alluded above, pretty good. Is then the solution very accurate in such a case of height?

Our accurate solution with $q(0) = 0.5774$ has $a = 0.4808$. Two Boussinesq's corresponding to this are the one with $q(0) = 0.5774$ ($a = 0.5000$) and the other with $a = 0.4808$ ($q(0) = 0.5922$), which are shown in Table 3. Of these two solutions the latter is remarkably coincident with our accurate one, as is seen in Fig. 2. Its wave velocity $(1 + 0.4808)^{1/2} = 1.217$ is also very near to the accurate value 1.202. Thus we know that at $a \fallingdotseq 0.5$, and consequently at $a \leq 0.5$, the approximation of the Boussinesq's solution is very good not only from the experimental evidence, but also from theoretical point of view.

Table 3

$\begin{pmatrix} a = 0.5000 \\ q(0) = 0.5774 \end{pmatrix}$		$\begin{pmatrix} a = 0.4808 \\ q(0) = 0.5922 \end{pmatrix}$	
$\dfrac{x}{H}$	$\dfrac{y_1}{H}$	$\dfrac{x}{H}$	$\dfrac{y_1}{H}$
0.0000	0.0000	0.0000	0.0000
0.0817	0.0013	0.0833	0.0012
0.1633	0.0050	0.1665	0.0048
0.3266	0.0195	0.3331	0.0188
0.4899	0.0424	0.4996	0.0408
0.6532	0.0722	0.6661	0.0694
0.8165	0.1068	0.8327	0.1027
0.9798	0.1442	0.9992	0.1387
1.1431	0.1826	1.1657	0.1756
1.3064	0.2205	1.3323	0.2120
1.4697	0.2565	1.4988	0.2467
1.6330	0.2900	1.6653	0.2789
1.7963	0.3204	1.8319	0.3081
1.9596	0.3475	1.9984	0.3341
2.1229	0.3713	2.1649	0.3570
2.2862	0.3919	2.3315	0.3769
2.4495	0.4096	2.4980	0.3939
2.6128	0.4247	2.6645	0.4084
2.7761	0.4375	2.8311	0.4207
∞	0.5000	∞	0.4808

In Fig. 2 the wave profiles for $a = 0.4808$ due to Rayleigh and to McCowan are also inscribed, and

---

(11) *c.f.* G. H. Keulegan and G. W. Patterson, Jour. Res. Nat. Bur. Stand., vol. 24, p. 47 (1940)

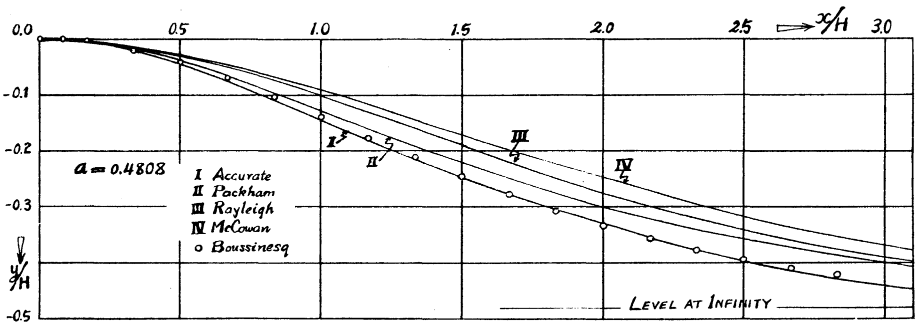

Fig. 2.

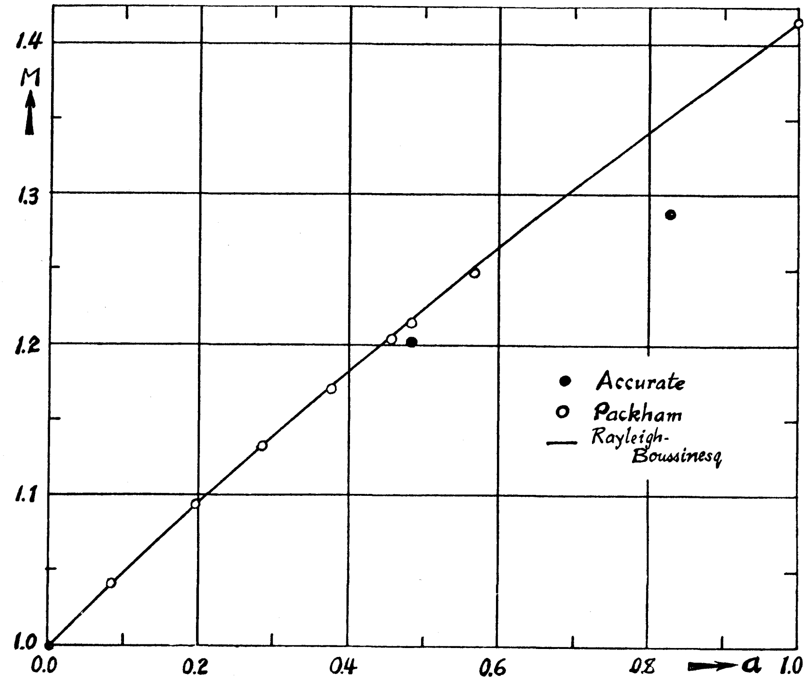

Fig. 3.

it is seen that these approximate solutions are worse than that of Boussinesq. In Fig. 3 the wave velocities are exhibited as functions of height. This function by Boussinesq (being identical with the Rayleigh's one) is $M=\sqrt{1+a}$, and it seems that the accurate values are in the neighborhood of this line when a is less than about 0.5.

§ **4. Discussion of approximate solutions** (2).——Recently another approximate solution is proposed by B. A. Packham,(12) which is interesting from

---

(12) B. A. Packham, Proc. Roy. Soc. London (A), vol. 213, p. 238 (1952).

theoretical point of view and deserves numerical investigation. He uses the approximation due to T. V. Davies[13]:

$$\sin\theta = \frac{\kappa}{3}\sin(3\theta), \tag{28}$$

with appropriate constant κ, into the surface condition (7′), and then by means of the relation $d\varphi = Uq\,ds$ it becomes

$$e^{3\tau}\frac{d\tau}{d\varphi} = \frac{g\kappa}{3U^3}\sin(3\theta). \tag{29}$$

It is evident that this relation is the more accurate when the smaller is the wave height. This relation is the boundary condition for the required field quantity $\Omega = \theta + i\tau$, which is now defined in the domain $0 \leq \psi \leq UH$ of the W-plane, at one boundary $\psi = UH$.

The solution adequate to our problem is given by Packham as

$$\Omega(W) = \frac{i}{3}\log\left\{1 - \sin^2(2\Lambda)\,\mathrm{sech}^2\left(\frac{\Lambda W}{UH}\right)\right\}, \tag{30}$$

the eigenvalue being

$$\frac{g\kappa}{3U^3} = \frac{2\Lambda}{3UH}\cot(2\Lambda) \qquad \text{i.e.} \qquad M = \sqrt{\frac{\kappa}{2\Lambda}\tan(2\Lambda)}\,; \tag{31}$$

Λ is a parameter and

$$\Lambda = \frac{\pi\lambda}{6}, \qquad 0 \leq \lambda \leq 1, \tag{32}$$

corresponding $\lambda = 0$ to uniform flow and $\lambda = 1$ to the highest wave.

From the definition (3) we have

$$dz = \frac{1}{U}e^{i\Omega}dW, \tag{33}$$

and integrating both sides of this along the imaginary axis:

$$i(A + H) = \frac{1}{U}\int_0^{iUH} e^{i\Omega(W)}dW,$$

from which it follows:

$$\begin{aligned} a = A/H &= -1 + \frac{1}{\Lambda}\int_0^{\Lambda}\{1 - \sin^2(2\Lambda)\sec^2\xi\}^{-1/3}d\xi \\ &= -1 + \frac{1}{\Lambda\cos^{2/3}(2\Lambda)}\int_0^{\Lambda}\{1 - \tan^2(2\Lambda)\tan^2\xi\}^{-1/3}d\xi. \end{aligned} \tag{34}$$

Expanding the integrand into a binomial series and integrating termwise it gives

---

(13) T. V. Davies, Proc. Roy. Soc. London (A), vol. 208, p. 475 (1951).

$$a(\lambda) = -1 + \frac{1}{\Lambda \cos^{2/3}(2\Lambda)} \sum_{n=0}^{\infty} \alpha_{2n} I_{2n} \tan^{2n}(2\Lambda) , \tag{34'}$$

where

$$\alpha_0 = 1 , \quad \alpha_2 = 1/3 , \quad \alpha_4 = 2/9 , \quad \alpha_6 = 14/81 , \quad \alpha_8 = 35/243 ,$$
$$\alpha_{10} = 91/729 , \quad \alpha_{12} = 728/6561 , \quad \alpha_{14} = 1976/19683 , \quad \textit{etc.}$$

and

$$I_0 = \Lambda , \quad I_{2n} = (2n-1)^{-1} \tan^{2n-1}\Lambda - I_{2n-2} .$$

By means of this series $a(\lambda)$ is calculated for several values of λ and tabulated in Table 4 column 3. When λ approaches to 1 the convergency is slow and the value for $\lambda = 1$ in the Table is calculated afresh by another series of good convergence. The extreme height $a = 1.000$ is identical with that of the Boussinesq's case and remote from the accurate value; the approximation at such a height being as false as the Boussinesq's one, in spite of the improvement that the Packham's wave has an angular crest of $2\pi/3$ *rad.*, which is a character of exact solution.

Table 4

λ	Λ	a	κ	M
0.0000	0.00000	0.0000	1.0000	0.00000
0.2	0.10472	0.0149	1.0074	0.10472
0.45	0.23562	0.0818	1.0402	0.23562
0.65	0.34034	0.1939	1.0919	0.34034
0.75	0.39270	0.2847	1.1320	0.39270
0.82	0.42935	0.3726	1.1692	0.42935
0.87	0.45553	0.4555	1.2033	0.45553
0.8828	0.46222	0.4806	1.2134	0.46222
0.92	0.48171	0.5670	1.2476	0.48171
1.0000	0.52360	1.0000	1.4142	0.52360

$\lambda = 0.8828$ corresponds to $a = 0.4806$, and this is almost equal to $a = 0.4808$, to which accurate wave are determined. We calculate, therefore, the wave form of $\lambda = 0.8828$, and to this purpose (33) for the value $W = \varphi + iUH$ is to be used:

$$dz(\varphi) = \frac{1}{U} e^{i\Omega(\varphi + iUH)} d\varphi = \frac{e^{i\theta}}{Uq} d\varphi , \tag{35}$$

where $z(\varphi)$ is the parametric expression of the surface profile, the parameter φ being real and $\varphi = 0$ corresponding to the crest. Integration of (35) by an infinite series, analogous to that of (34'), can of course be accomplished, but owing to its slow convergence it is not useful for the present purpose. We adopt here the method of numerical integration.

From (30) we have

$$e^{-3i\Omega} = 1 - \sin^2(2\Lambda)\,\mathrm{sech}^2\left(\frac{\Lambda W}{UH}\right) = \frac{\cos(4\Lambda) + \cosh\left(\dfrac{2\Lambda W}{UH}\right)}{1 + \cosh\left(\dfrac{2\Lambda W}{UH}\right)} ,$$

and then, inserting $W = \varphi + iUH$ into this and rewriting a little, it becomes:

$$e^{-3i\Omega(\varphi+iUH)}=q^3\cos(3\theta)-iq^3\sin(3\theta)$$
$$=\frac{-2+3\cos^2(2\Lambda)+2\cos^3(2\Lambda)\cosh\varphi'+\cosh^2\varphi'+i2\sin^3(2\Lambda)\sinh\varphi'}{\{\cos(2\Lambda)+\cosh\varphi'\}^2},$$

and we reach finally, after some calculations, to

$$\left.\begin{aligned}\frac{1}{q(\varphi')}&=\left\{\frac{\cos(2\Lambda)+\cosh\varphi'}{\cos(6\Lambda)+\cosh\varphi'}\right\}^{1/6},\\ \sin(3\theta)&=-\frac{2\sin^3(2\Lambda)\sinh\varphi'}{[\{\cos(6\Lambda)+\cosh\varphi'\}\{\cos(2\Lambda)+\cosh\varphi'\}^3]^{1/2}}\end{aligned}\right\}\qquad(36)$$

where $\varphi'=2\Lambda\varphi/UH$.

For $\lambda=0.8828$ ($\Lambda=0.4622$) are calculated q^{-1} and $\sin(3\theta)$ by (36), at every point φ' taken at equal and appropriate distances; from $\sin(3\theta)$ then θ and lastly $e^{i\theta}$ calculated. With the pairs of values (q^{-1}, $e^{i\theta}$) taken into (35) *i.e.*

$$\frac{z(\varphi')}{H}=\frac{1}{2\Lambda}\int_0^{\varphi'}\frac{1}{q}e^{i\theta}d\varphi',\qquad(35')$$

where the origin of z being now at the crest, we easily accomplish the numerical quadrature and obtain z/H or $(x/H, y/H)$, which is the wave profile. The result is tabulated in Table 5 and inscribed in Fig. 2 as the curve II. We see an approximation pretty good but certainly worse than the Boussinesq's. For a little smaller height, however, the order of approximation should be better and comparable with the Boussinesq's.

Table 5

φ'	$\frac{x}{H}$	$\frac{y_1}{H}$
0.0	0.0000	0.0000
0.1	0.1828	0.0054
0.2	0.3611	0.0210
0.4	0.6958	0.0709
0.6	1.0021	0.1279
0.8	1.2869	0.1822
1.0	1.5561	0.2305
1.2	1.8139	0.2722
1.4	2.0632	0.3076
1.6	2.3059	0.3374
1.8	2.5435	0.3623
2.0	2.7770	0.3830
2.2	3.0073	0.4001
2.4	3.2350	0.4143
2.6	3.4606	0.4260
2.8	3.6845	0.4356
∞	∞	0.4806

So far the constant κ is indeterminate. When we want, however, to know the wave velocity, this contant has to be determined beforehand. Of course (28) is not satisfied by any constant κ and has to be interpreted in a certain average. Among several processes of averaging we adopt here the one which is weighted with the free surface area. That is to say, we multiply both sides of (28) with the arc element ds of the surface stream line and integrate through the half wave:

$$\int_{x=-\infty}^{0}\sin\theta\,ds=\frac{\kappa}{3}\int_{x=-\infty}^{0}\sin(3\theta)\,ds.\qquad(37)$$

The left hand side is equal to the wave height A, and then have

$$a=\frac{A}{H}=\frac{\kappa}{3H}\int_{-\infty}^{0}\sin(3\theta)\left(\frac{d\varphi'}{ds}\right)^{-1}d\varphi'=\frac{\kappa}{6\Lambda}\int_0^{\infty}\frac{-\sin(3\theta)}{q}d\varphi',$$

which, by means of (36), is

$$a=\frac{\kappa}{3\Lambda}\sin^3(2\Lambda)\int_0^\infty\frac{\sinh\varphi' d\varphi'}{\{\cos(6\Lambda)+\cosh\varphi'\}^{2/3}\{\cos(2\Lambda)+\cosh\varphi'\}^{4/3}}. \tag{37'}$$

The integral on the right hand side can be accomplished by the successive substitutions:

$$\cosh\varphi'=1+\{1+\cos(2\Lambda)\}\tan^2\xi, \qquad \cos^2\xi=\eta;$$

and after a little calculation we arrive at

$$a=\frac{\kappa}{4\Lambda}\tan(2\Lambda)[1-\{1-2\cos(2\Lambda)\}^{2/3}], \tag{37''}$$

from which follows:

$$\kappa(\lambda)=\frac{4\Lambda a(\lambda)}{\tan(2\Lambda)[1-\{1-2\cos(2\Lambda)\}^{2/3}]}. \tag{38}$$

Thus $\kappa(\lambda)$ can be calculated by means of the values of $a(\lambda)$, tabulated in Table 4. The result is given in the same table column 4.

We can then calculate the wave velocity at once. The formula is (31), and the result is tabulated in Table 4 column 5. The velocity M as a function of the height a is the correspondence between the columns 3 and 5, and is inscribed in Fig. 3 with small white circles. All the circles lie quite well on the Boussinesq's line $M=\sqrt{1+a}$, and then the order of approximation of wave velocity thus determined is almost identical with that of Boussinesq's (or Rayleigh's), through all the wave-heights.

The determination of κ here adopted is supported by another consideration. The Bernoulli's theorem applied between the crest and the base of wave profile gives

$$\frac{1}{2}U^2q(0)^2+gA=\frac{1}{2}U^2q(\infty)^2,$$

which is rewritten into

$$a=\frac{U^2}{2gH}\{q(\infty)^2-q(0)^2\}. \tag{39}$$

Using the values:

$$q(0)=\left\{\frac{1+\cos(6\Lambda)}{1+\cos(2\Lambda)}\right\}^{1/6}=\{2\cos(2\Lambda)-1\}^{1/3}, \qquad q(\infty)=1$$

obtained from (36) into (39) we have finally

$$a=\frac{\kappa}{4\Lambda}\tan(2\Lambda)[1-\{2\cos(2\Lambda)-1\}^{2/3}],$$

and this is nothing but the relation (37''). Thus our determination of κ is in accordance with the Bernoulli's relation between a pair of important points.

(Received Dec. 2, 1957)