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ON THE APPROXIMATE EXPRESSION OF STOKES WAVES

By Hikoji YAMADA

Résumé. The approximate solution[1] due to T. V. Davies—his first approximation which is very simple and integrable—is examined and proved to be a good approximation for almost all the range of steepness. For this purpose two accurate solutions, the one of the highest amplitude and the other of the steepness 0.0827, are calculated and in comparison with these the degree of approximation is decided.

§ 1. **Lévi-Civita's formulation of the problem.**——As we use in the followings the formulation due to Lévi-Civita,[2] it is convenient to describe in this section its outline briefly, as far as we require it.

The permanent waves which travel on the calm water of infinite depth, being their wave length L and proceeding to left with velocity U, are observed from the rectangular axes O-xy which follows the waves with the same velocity. The wave form on the surface stops and instead water flows from left to right stationary, the velocity at the infinite depth being horizontal and of magnitude U. Origin of the coordinates is at a wave crest, x-axis is horizontal and directed to right, y-axis vertical and upwards.

Our waves are irrotational, and to any point $z=x+iy$ in water there is the complex potential $W=\varphi+i\psi$, which is a holomorphic function of the position z. With this potential W

$$\frac{1}{U}\frac{dW}{dz}=qe^{-i\theta}, \tag{1}$$

where q is the magnitude, measured with U as unit, and θ is tlhe angle, measured from the horizontal direction upwards, of the flow velocity. q never vanishes in water and then, when we introduce a field function $\Omega(z)$ by the definition

$$\Omega(z)=i\log\left(\frac{1}{U}\frac{dW}{dz}\right)=\theta+i\tau, \tag{2}$$

$$\tau=\log q, \tag{3}$$

Ω is everywhere holomorphic in water, being its real part the flow direction and the imaginary part logarithm of the magnitude of velocity. To know this function is the principal part of the solution of our problem.

Here we transform a part of W-plane, which corresponds to a wave length ($-UL/2\leq\varphi<UL/2$, $\psi\leq 0$; $\varphi=\psi=0$ corresponding to a crest) onto a unit circle $|\zeta|\leq 1$ of $\zeta=\xi+i\eta$ plane by the equation

$$\zeta = e^{-z\pi i \frac{W}{UL}}, \quad W = i\frac{UL}{2\pi}\log\zeta\,; \tag{4}$$

the correspondence being one to one. Then through W a wave length domain of z-plane corresponds also to unit circle $|\zeta|\leq 1$ one to one, and any function of z becomes a function of ζ, especially $\Omega(z)$ to $\Omega(\zeta)$. $\Omega(z)$ is a periodic function of a wave length period and it results into a wholly holomorphic $\Omega(\zeta)$ in the domain $|\zeta|<1$; moreover, as we are dealing with symmetrical waves, $\Omega(\zeta)$ is pure imaginary on the real axis $\zeta=\xi\,(-1<\xi<1)$.

When we can find $\Omega(\zeta)$ by one way or another, we obtain from (2) and (4) the relation

$$dz = \frac{iL}{2\pi\zeta}e^{i\Omega(\zeta)}d\zeta\,, \tag{5}$$

which gives by integration $z=z(\zeta)$ or $\zeta=\zeta(z)$. $\Omega(\zeta)$ can then be rewritten into $\Omega(z)$, which is our aimed field quantity. Our problem has thus been transformed into the determination of $\Omega(\zeta)$.

When y tends to $-\infty$ *i.e.*, ψ to $-\infty$, θ and τ, and accordingly Ω, vanish, and then $\Omega(\zeta)$ has to vanish at $\zeta=0$. $\Omega(\zeta)$ is thus a holomorphic function which vanishes at the origin, and fulfils on the circumference of unit circle $\zeta=e^{i\sigma}$ $(-\pi<\sigma\leq\pi)$ the surface condition. By the surface condition $\Omega(\zeta)$ is defined. Of course there remains one parameter free, if limited, which may be identified *e.g.*, with the wave steepness.

The surface condition which means that along the surface stream line the statical pressure must be that of atmosphere is expressed by Bernoulli's theorem as

$$\frac{1}{2}U^2q^2+gy=\text{const}, \tag{6}$$

and is rewritten, by differentiation along the arc of that stream line, into

$$\frac{2g}{U^2}\frac{dy}{ds}=-2q\frac{dq}{ds},$$

where arc element ds and the depth element dy are combined by the relation $dy=ds\cdot\sin\theta$, and so we have

$$q\frac{dq}{ds}=-\frac{g}{U^2}\sin\theta\,. \tag{7}$$

Now we have to bring this relation on the circumference of the unit circle $\zeta=e^{i\sigma}$ $(-\pi<\sigma\leq\pi)$. On the water surface dz is evidently equal to $ds\cdot e^{i\theta}$ and by virtue of (5)

$$ds=-\frac{L}{2\pi q}d\sigma\,, \tag{8}$$

and when we regard θ and q defined as functions of σ the surface condition (7) becomes finally to

$$q^2\frac{dq}{d\sigma}=p\sin\theta \qquad (q=e^{\tau(\sigma)})\,, \tag{9}$$

$$p = \frac{gL}{2\pi U^2}, \tag{10}$$

which is assigned to these functions $\theta(\sigma)$, $q(\sigma)$.

When we find $\Omega(\zeta)$ whose real and imaginary parts θ, τ on the unit circle satisfy the condition (9) our problem is solved, and this formulation is the one due to Lévi-Civita. To each p, the eigenvalue, which is less than 1 but sufficiently close to 1, belongs a certain unique solution; to smaller p belongs steeper waves; there is the smallest p which corresponds to the steepest waves with angular crests. These characteristics are well known but, owing to the non-linearity of the boundary condition (9), full determination of the waves is difficult and has been deferred.

§ 2. Accurate solutions for some special values of steepness.——The scheme for the numerical determination of waves, which we have given recently,[3] is as follows. As long as the angnlar crests do not appear $\Omega(z)$ is analytic on the free surface and then $\Omega(\zeta)$ holomorphic in the domain $|\zeta| \leq 1$. Accordingly the power series which represents $\Omega(\zeta)$:

$$\Omega(\zeta) = i \sum_{n=1}^{\infty} a_n \zeta^n \tag{11}$$

can be cut short to an appropriate polynomial with good approximation up to as far as $|\zeta| = 1$. As $\Omega(\zeta)$ has pure imaginary values on the real axis, as mentioned above, it is evident that all a_n have purely real values and that on the circumference of unit circle

$$\theta(\sigma) = -\sum_{n=1} a_n \sin n\sigma, \quad \tau(\sigma) = \sum_{n=1} a_n \cos n\sigma. \tag{12}$$

The coefficients a_n have to be so determined as to satisfy the surface condition, and here we adopt an iterative calculation: we replace the infinite series (11) by a polynomial of a certain degree N, and try to determine the numerical constants a_n $(n = 1, 2, \cdots, N)$ to satisfy the surface condition as good as possible. The starting set of values of a_n is to be selected arbitrarily but adequately. One cycle of the iterative procedure consists of two steps: first the determination of $\tau(\sigma)$ from $\theta(\sigma)$ through the surface condition (9), which we use now in the form:

$$q^3(\sigma) = q^3(0) + 3p \int_0^{\sigma} \sin\theta(\sigma')\, d\sigma', \tag{9'}$$

and second the determination of $\theta(\sigma)$ by virtue of the expression (12).

To complete the cycle another relation is necessary, which enables us to calculate $q(0)$ from p or *vice versa*, the one or the other being given at the beginning. At present we take $q(0)/q(\pi) \equiv \rho$ as given and make use of the relation:[4]

$$\int_0^L q^2\, dx = L, \tag{13}$$

which holds for the surface values as the most convenient one. Of the line-elements of the water surface:

$$\left.\begin{aligned} dx &= -\frac{L}{2\pi q(\sigma)}\cos\theta(\sigma)\,d\sigma \\ dy &= -\frac{L}{2\pi q(\sigma)}\sin\theta(\sigma)d\sigma\,, \end{aligned}\right\} \tag{14}$$

which follows from (8), we use the former into (13) and get to the required equation:

$$\int_0^\pi q(\sigma)\cos\theta(\sigma)\,d\sigma = \pi\,. \tag{13'}$$

When σ is equal to π (9′) gives

$$p = \frac{1}{3}q^3(\pi)(1-\rho^3)\Big/\int_0^\pi \sin\theta(\sigma')\,d\sigma'\,, \tag{15}$$

and inserting this into (9′) we obtain

$$\left(\frac{q(\sigma)}{q(\pi)}\right)^3 = \rho^3 + (1-\rho^3)\int_0^\sigma \sin\theta(\sigma')\,d\sigma'\Big/\int_0^\pi \sin\theta(\sigma')\,d\sigma'\,. \tag{9''}$$

This last equation, used into (13′), fixes the values of $q(\pi)$, and then p and $q(\sigma)$ are determined by (15) and (9″). The Fourier decomposition of $\tau(\sigma) = \log q(\sigma)$ is as usual: N coefficients are determined making use of the values of $\tau(\sigma)$ at N points with equal intervals.

With this scheme of calculation the highest waves have been calculated.[3] The wave crest ($\zeta = 1$) of this case is a singular point and in the neighborhood of this point $\Omega(\zeta)$ is approximate to

$$\Omega_0(\zeta) = \frac{i}{3}\log(1-\zeta)\,. \tag{16}$$

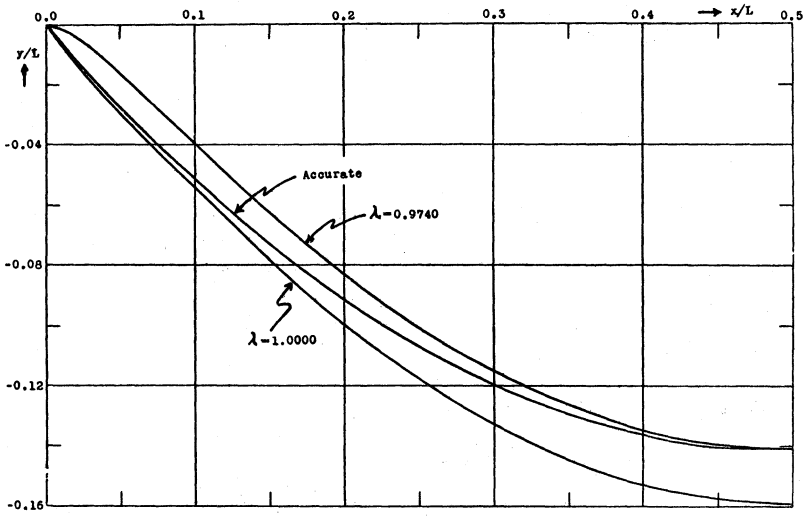

Fig. 1.

We put then, instead of (11),

$$\Omega(\zeta) = \Omega_0(\zeta) + i \sum_{n=1}^{\infty} a_n \zeta^n , \tag{11'}$$

and the iteration process above stated is used for the numerical determination of a_n. After a certain device for the refinement of approximation 12 terms expansion has given the result which satisfies the surface condition (6) *i.e.*

$$q^2 = 4\pi p \frac{(-y)}{L} \tag{6'}$$

within an error of several per mille of the magnitude of each term. Some of the numericals which we now require are reproduced in Table 4 columns 2, 3 and inscribed in Fig. 1.

For the examination of the degree of approximation of any proposed expression we are in need of at least one more accurate solution of moderate steepness. To obtain such one we start with an approximate expression[1]

$$[\Omega_0(\zeta)]_{\zeta=e^{i\sigma}} = \theta_0(\sigma) + i_0(\sigma) = \left[\frac{i}{3} \log(1-\lambda\zeta)\right]_{\zeta=e^{i\sigma}} \qquad (\lambda = 0.7) . \tag{17}$$

Table 1.

a_1	+ 0.00029
a_2	− 0.00395
a_3	− 0.00153
a_4	− 0.00027
a_5	+ 0.00028
a_6	+ 0.00045
a_7	+ 0.00048
a_8	+ 0.00040
a_9	+ 0.00030
a_{10}	+ 0.00025
a_{11}	+ 0.00017
a_{12}	+ 0.00004
p	0.9339
$U\Big/\sqrt{\frac{gL}{2\pi}}$	1.0348
H/L	0.08267

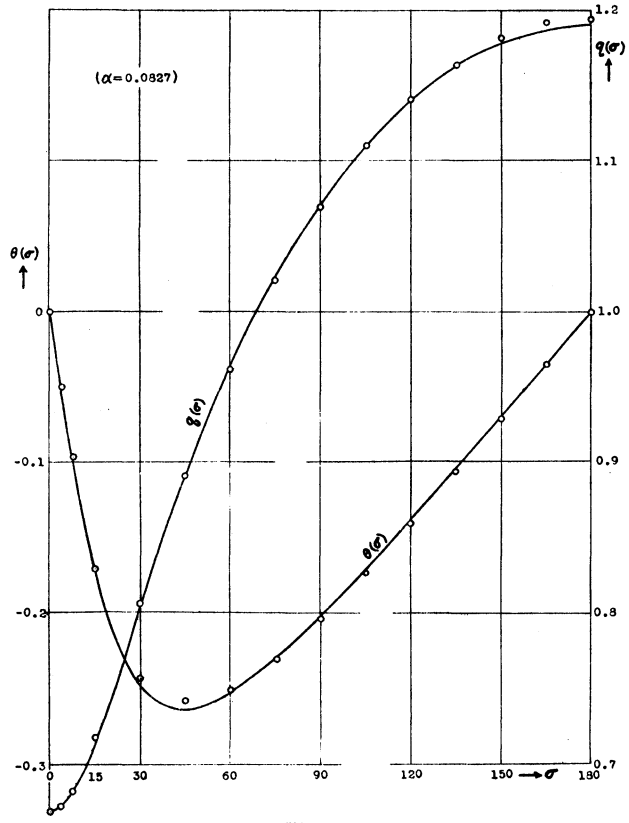

Fig. 2.

Table 2.

σ	θ	q	$-x/L$	$-y/L$	$\dfrac{q^2(\sigma)-q^2(0)+4\pi p\dfrac{y}{L}}{q^2(\sigma)-q^2(0)}\times 1.000$
0	0.0000	0.6674 (0.6673)	0.0000	0.00000	—
7.5	0.0966	0.6808		0.00154	0.0
15	0.1714	0.7148 (0.7148)	0.0607	0.00552	−1.5
22.5	0.2204	0.7588		0.01115	−3.9
30	0.2483	0.8056 (0.8057)	0.1142	0.01737	−1.5
37.5	0.2627	0.8510		0.02375	0.0
45	0.2637	0.8929 (0.8929)	0.1616	0.03000	−0.3
42.5	0.2603	0.9308		0.03589	−0.5
60	0.2530	0.9654 (0.9656)	0.2049	0.04150	−1.0
67.5	0.2426	0.9967		0.04669	0.2
75	0.2301	1.0247 (1.0248)	0.2455	0.05155	−0.5
82.5	0.2164	1.0498		0.05597	−0.2
90	0.2018	1.0724 (1.0725)	0.2843	0.06007	−0.4
97.5	0.1864	1.0926		0.06376	0.3
105	0.1703	1.1105 (1.1105)	0.3218	0.06714	−0.1
112.5	0.1539	1.1263		0.07012	0.2
120	0.1374	1.1401 (1.1401)	0.3583	0.07281	−0.1
127.5	0.1206	1.1521		0.07513	0.2
135	0.1035	1.1624 (1.1622)	0.3943	0.07717	0.2
142.5	0.0863	1.1709		0.07883	0.5
150	0.0691	1.1777 (1.1776)	0.4297	0.08024	0.0
157.5	0.0520	1.1831		0.08128	0.5
165	0.0346	1.1868 (1.1867)	0.4649	0.08207	0.0
172.5	0.0173	1.1891		0.08249	0.5
180	0.0000	1.1898 (1.1897)	0.4999	0.08267	−0.2

This approximation gives $\rho=q(0)/q(\pi)=0\,5609$; we take this value fixed and calculate by the iteration process the accurate values of $\Omega(\zeta)$ starting from $\Omega_0(\zeta)$. Expansions of the form (11′), instead of (11), are used here also, because of the simple calculations only. Four cycles of iterations were sufficient for the fifth cycle gave no noticeable changes in numericals. The results is as seen in Table 1, and the surface velocity and the wave form calculated therewith are in Table 2 (numbers in parentheses are the fifth approximation), and inscribed into Figs. 2 and 3.

§ 3. **Approximate solution by T. V. Davies.**——If we change the surface condition (9) a little and write instead

$$q^2\frac{dq}{d\sigma}=p'\sin 3\theta\,, \tag{18}$$

the formalism of § 1 gives as the exact solution a simple expression

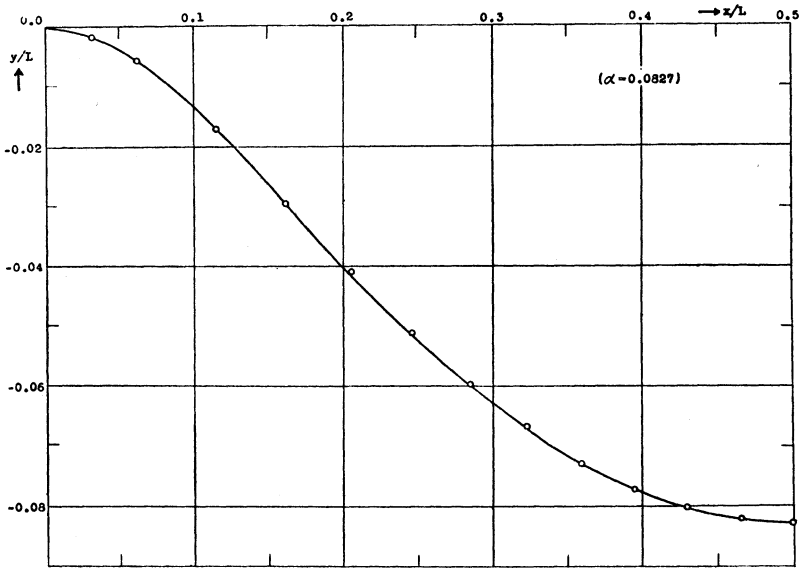

Fig. 3.

$$\Omega_0(\zeta) = \frac{i}{3} \log(1 - \lambda\zeta), \tag{17'}$$

which has been used in last section as the zeroth approximation to the Stokes' waves. This is the first approximation due to T. V. Davies,[1] who has rewritten the surface condition (9) by means of the expansion

$$\sin\theta = \frac{1}{3}\sin 3\theta + \frac{4}{81}\sin^3 3\theta + \frac{16}{729}\sin^5 3\theta + \cdots, \tag{19}$$

or, when we content with the first approximation, by the relation

$$\sin\theta = \frac{\kappa}{3}\sin 3\theta, \tag{19'}$$

where κ is a constant appropriately chosen and dependent on the steepness of the waves. As the second and higher approximations are not tenable to practical purposes, we here take up the first approximation (17′) and study its approximation in some details, which is our main object in this paper. Evidently

$$p' = \frac{1}{3}\kappa p. \tag{20}$$

When the steepness is small and consequently the slope of water surface is small everywhere, the proportionality (19′) of $\sin\theta$ and $\sin(3\theta)$ is rather accurate with the value of κ nearly equal to 1. $\Omega_0(\zeta)$ is then expected to be a good approximation for small steepness, and to deviate gradually with increasing steepness.

On the unit circle $\zeta = e^{i\sigma}$ $(-\pi < \sigma \leq \pi)$

$$\left.\begin{aligned} &\Omega_0(e^{i\sigma}) = \theta_0(\sigma) + i\tau_0(\sigma)\,; \\ &\theta_0(\sigma) = -\frac{1}{3}\vartheta(\sigma)\,, \quad \sin\vartheta(\sigma) = -\frac{\lambda\sin\sigma}{\sqrt{1+\lambda^2-2\lambda\cos\sigma}}\,; \\ &\tau_0(\sigma) = \log(1+\lambda^2-2\lambda\cos\sigma)^{1/6} \\ &\qquad\qquad i.e. \quad q_0(\sigma) = (1+\lambda^2-2\lambda\cos\sigma)^{1/6}\,; \end{aligned}\right\} \tag{21}$$

and using these expressions into (18) we know that the eigenvalue p' is equal to 1/3. Then from (20) we have

$$p = \frac{1}{\kappa} \tag{22}$$

and, resorting to (10), the velocity of waves is

$$U = \sqrt{\kappa}\cdot\sqrt{\frac{gL}{2\pi}}. \tag{23}$$

The field of particle velocity is described by $\Omega_0(\zeta)$, and the relation connecting ζ-plane to z-plane is given by (5), which by means of (17′) becomes to

$$d\left(\frac{z}{L}\right) = \frac{i}{2\pi\zeta}e^{i\Omega_0(\zeta)}d\zeta = \frac{i}{2\pi\zeta}\frac{d\zeta}{(1-\lambda\zeta)^{1/3}},$$

and can be integrated easily, introducing the complex variable

$$\eta_1 = (1-\lambda\zeta)^{1/3}, \tag{24}$$

which is equal to the complex velocity $qe^{-i\theta}$, into

$$\frac{z}{L} = \text{const.} + \frac{i}{2\pi}\{\log(1-\eta_1) + \bar{\omega}\log(\omega-\eta_1) + \omega\log(\bar{\omega}-\eta_1)\}\,; \tag{25}$$

here 1, ω and $\bar{\omega}$ are the three cubic roots of 1. This integrability is a character of special simplicity, which rarely occurs in the wave problems of finite amplitude, and inserting $e^{i\sigma}$ for ζ (25) is nothing but the equation of water surface form, $\sigma = 0$ and π corresponding to wave crest and trough.

Table 3.

λ	α	κ	$\sqrt{\kappa}$	λ	α	κ	$\sqrt{\kappa}$
0.0	0.00000	1.0000	1.0000	0.7018	0.08273	1.0619	1.0305
0.1	0.01063	1.0015	1.0007	0.8	0.09853	1.0883	1.0432
0.2	0.02137	1.0037	1.0018	0.9	0.11887	1.1329	1.0644
0.3	0.03231	1.0081	1.0041	0.965	0.13765	1.1833	1.0878
0.4	0.04371	1.0170	1.0085	0.970	0.13958	1.1892	1.0905
0.5	0.05564	1.0276	1.0137	0.974	0.14121	1.1943	1.0928
0.6	0.06838	1.0415	1.0205	0.980	0.14387	1.2028	1.0967
0.7	0.08248	1.0617	1.0304	1.0	0.15903	1.2589	1.1220

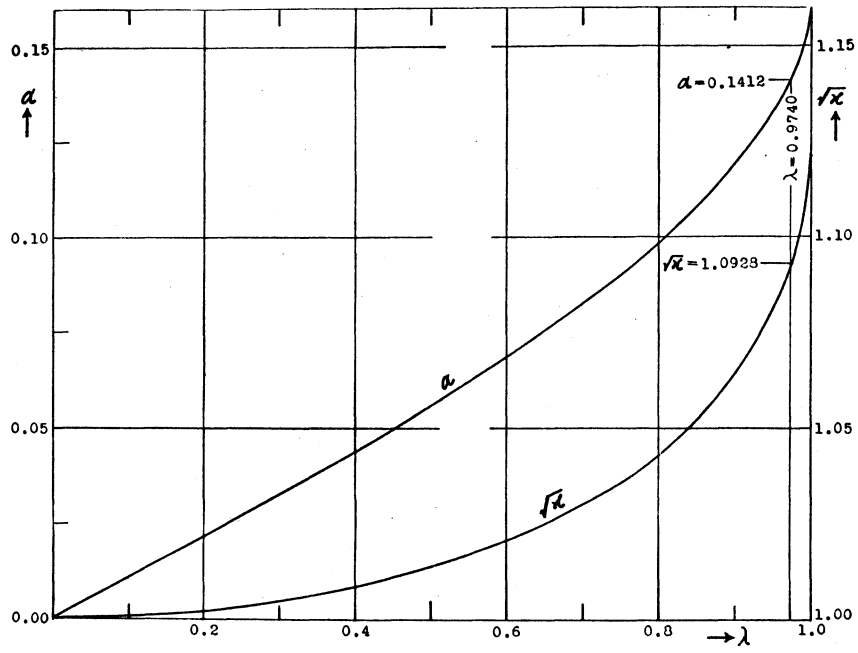

Fig. 4.

The steepness α of the waves is, by definition,

$$\alpha = \text{imag. part}\left\{\left(\frac{z}{L}\right)_{\zeta=0} - \left(\frac{z}{L}\right)_{\zeta=1}\right\}, \tag{26}$$

and given, by virtue of (25), by the formula

$$\alpha = \text{imag. part}\frac{1}{2\pi}\left\{\begin{matrix}\log(1-a) + \bar{\omega}\log(\omega - a) + \omega\log(\bar{\omega} - a) \\ -\log(1-b) - \bar{\omega}\log(\omega - b) - \omega\log(\omega - b)\end{matrix}\right\}$$

$$= \frac{1}{4\pi}\left\{3\log\frac{1-b}{a-1} - 2\sqrt{3}\left[\arctan\frac{\sqrt{3}\,a}{2+a} - \arctan\frac{\sqrt{3}\,b}{2+b}\right]\right\}, \tag{26'}$$

where

$$a = (1+\lambda)^{1/3} = q(\pi), \quad b = (1-\lambda)^{1/3} = q(0). \tag{27}$$

(26′) defines α as a function of λ, which is tabulated in Table 3 column 2 and inscribed in Fig. 4. Inversely for a given steepness α corresponding value of λ can be obtained by this table or figure, and then, making use of the relation (25), the velocity field $\Omega_0(z)$ is determined from $\Omega_0(\zeta)$. Above all, for $\zeta = e^{i\sigma}$ (24) and (25) determine the wave form $z(\sigma)/L = x(\sigma)/L + iy(\sigma)/L$, the parametric expression of wave form.

§ 4. **Examination of the foregoing solution.**——Exact solution $\Omega(\zeta)$ and the approximate $\Omega_0(\zeta)$ are both the holomorphic functions in the unit circle $|\zeta| < 1$, and then their difference $\Omega(\zeta) - \Omega_0(\zeta)$ is also holomorphic in it.

Therefore the real and imaginary parts $\theta-\theta_0$, $\tau-\tau_0$ of it are harmonic in the unit circle domain, and the maximum values of these differences are attained at points on the circumference of this circle. The maximum error of our solutions are then to be found on the unit circle and our examination of the approximation can be restricted only on this circle.

After this notice we examine $\theta_0(\sigma)$ and $\tau_0(\sigma)$ in the cases where accurate solutions are already found, *i.e.*, the highest waves and the waves of steepness $\alpha=0.0827$.

Table 4.

σ	accurate		$\lambda=1.0000$		$\lambda=0.9740$	
	$-x/L$	$-y/L$	$-x/L$	$-y/L$	$-x/L$	$-y/L$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.75	0.0314	0.0177	—	—	0.0290	0.0079
7.5	0.0502	0.0277	0.0495	—	0.0499	0.0165
15	0.0808	0.0431	0.0855	0.0476	0.0826	0.0321
30	0.1310	0.0658	0.1370	0.0731	0.1349	0.0565
45	0.1747	0.0830	0.1814	0.0926	0.1796	0.0754
60	0.2152	0.0969	0.2217	0.1085	0.2204	0.0909
75	0.2536	0.1082	0.2602	0.1211	0.2586	0.1037
90	0.2907	0.1175	0.2963	0.1318	0.2955	0.1142
105	0.3268	0.1250	0.3320	0.1407	0.3311	0.1227
120	0.3622	0.1310	0.3665	0.1475	0.3656	0.1294
135	0.3970	0.1355	0.4001	0.1528	0.4001	0.1348
150	0.4316	0.1387	0.4337	0.1563	0.4332	0.1383
165	0.4659	0.1406	0.4671	0.1585	0.4677	0.1407
180	0.5001	0.1412	0.5000	0.1590	0.5000	0.1412
σ	θ	q	θ	q	θ	q
0	0.5236	0.0000	0.5236	0.0000	0.0000	0.2963
3.75	0.4988	0.4322	0.5127	0.4030	0.3852	0.4114
7.5	0.4803	0.5409	0.5018	0.5076	0.4357	0.5088
15	0.4500	0.6729	0.4800	0.6391	0.4467	0.6374
30	0.3992	0.8317	0.4363	0.8029	0.4200	0.7998
45	0.3514	0.9347	0.3927	0.9148	0.3821	0.9109
60	0.3075	1.0099	0.3491	1.0000	0.3415	0.9957
75	0.2653	1.0673	0.3054	1.0678	0.2997	1.0632
90	0.2252	1.1122	0.2618	1.1224	0.2574	1.1176
105	0.1860	1.1473	0.2182	1.1664	0.2148	1.1614
120	0.1478	1.1744	0.1745	1.2009	0.1720	1.1957
135	0.1103	1.1944	0.1309	1.2272	0.1291	1.2218
150	0.0734	1.2085	0.0873	1.2455	0.0861	1.2400
165	0.0366	1.2167	0.0436	1.2563	0.0431	1.2508
180	0.0000	1.2194	0.0000	1.2600	0.0000	1.2544

(i) **Highest waves.**——When $\lambda=1$ $\Omega_0(\zeta)$ gives a jump of $\pi/3$ *rad.* of the surface flow direction at $\zeta=1$ and water surface exhibits an angular point of an angle $2\pi/3$ *rad.* there, which is an exact nature of the highest waves. The steepness of this waves is calculated by the formula (26′) with $\lambda=1$. Its value $\alpha=0.1590$, which differs from the accurate value[3] $\alpha=0.1412$ about 13%, is not of negligible error in general. If we take $\Omega_0(\zeta)$ which gives the accurate steepness $\alpha=0.1412$ as the highest waves, then $\lambda=0.9740$ and the wave crests are rounded over, contrary to the remarkable characteristic of the angular crest. For either of these waves with $\lambda=1$ and 0.9740 the surface velocities and wave forms are calculated by (21), (25) and compared with the accurate solution quoted above in Table 4, the wave forms being also compared in Fig. 1. We see resemblances but not quantitative accordances up to such a high case.

(ii) **Waves of steepness $\alpha=0.0827$.**——The steepness of ordinary ocean waves is of the order 0.05, and, as we noticed above, present solution is the more accurate the more the steepness decreases. If, therefore, the solution is sufficiently accurate for the waves of $\alpha=0.0827$ it is accurate through the range of steepness of practical importance. As we will see in the following, really it is the case.

$\alpha=0.0827$ corresponds to $\lambda=0.7018$ (Table 3) and with this value of λ (21), (25) give the surface velocity and wave form which are indicated in Table 5 and inscribed in Figs. 2 and 3 with small white circles, each compared to the accurate solution of § 2, which are denoted by full lines. The accordance is remarkably good and we see then that the Stokes' waves can be represented by the formulas:

Table 5.

σ	accurate		$\lambda=0.7018$		σ	accurate		$\lambda=0.7018$	
	$-x/L$	$-y/L$	$-x/L$	$-y/L$		θ	q	θ	q
0	0.0000	0.00000	0.0000	0.00000	0	0.0000	0.6674	0.0000	0.6681
3.75	—	—	0.0155	0.00038	3.75	—	—	0.0508	0.6718
7.5	—	0.00154	0.0309	0.00153	7.5	0.0966	0.6808	0.0974	0.6823
15	0.0607	0.00552	0.0605	0.00557	15	0.1714	0.7148	0.1712	0.7178
30	0.1142	0.01735	0.1140	0.01713	30	0.2483	0.8056	0.2433	0.8074
45	0.1616	0.03000	0.1614	0.02952	45	0.2637	0.8929	0.2593	0.8909
60	0.2049	0.04150	0.2049	0.04079	60	0.2530	0.9654	0.2508	0.9616
75	0.2455	0.05155	0.2456	0.05098	75	0.2301	1.0247	0.2306	1.0205
90	0.2843	0.06007	0.2846	0.05952	90	0.2018	1.0724	0.2040	1.0690
105	0.3218	0.06714	0.3222	0.06675	105	0.1703	1.1105	0.1736	1 1086
120	0.3583	0.07281	0.3588	0.07259	120	0.1374	1.1401	0.1409	1.1400
135	0.3943	0.07717	0.3947	0.07708	135	0.1035	1.1624	0.1067	1.1638
150	0.4297	0.08024	0.4300	0.08025	150	0.0691	1.1777	0.0716	1.1806
165	0.4649	0.08207	0.4651	0.08206	165	0.0346	1.1868	0.0359	1.1906
180	0.4999	0.08267	0.5000	0.08273	180	0.0000	1.1896	0.0000	1.1939

$$\left.\begin{aligned} \eta_1 &= (1-\lambda\zeta)^{1/3}, \\ \Omega &= \theta + i\tau = i\log\eta_1, \\ \frac{z}{L} &= \text{const.} + \frac{i}{2\pi}\{\log(1-\eta_1) + \bar{\omega}\log(\omega-\eta_1) + \omega\log(\bar{\omega}-\eta_1)\}. \end{aligned}\right\} \quad (28)$$

So far we have been independent of the constant κ. When, however, we want to obtain the wave velocity, this constant has to be determined beforehand, as is shown in (23). Of course (19′) is not satisfied by any constant κ and has to be interpreted in a certain average. Among many processes of averaging we here adopt the one which is weighted with the free surface area. That is to say, we multiply both sides of (19′) with the arc element ds of the surface stream line and integrate through a wave length:

$$2\int_{x=-L/2}^{0} \sin\theta_0(s)\,ds = 2\cdot\frac{\kappa}{3}\int_{x=-L/2}^{0} \sin 3\theta_0(s)\,ds. \quad (29)$$

In this expression the left hand side is nothing but twice the height A of waves, and then reducing the height into the steepness $\alpha = A/L$ we have

$$\alpha = \frac{\kappa}{3L}\int_{\sigma=\pi}^{0} \sin 3\theta_0(\sigma)\cdot\frac{ds}{d\sigma}\cdot d\sigma;$$

on the right hand side $ds/d\sigma = -L/2\pi q_0$ by (8) and $\theta_0(\sigma)$, $q_0(\sigma)$ are as given in (21). Introducing these expressions the integration is easily accomplished:

$$\alpha = \frac{\kappa\lambda}{6\pi}\int_0^{\pi} \frac{\sin\sigma\,d\sigma}{(1+\lambda^2-2\lambda\cos\sigma)^{2/3}}$$

$$= \frac{\kappa}{4\pi}\{(1+\lambda)^{2/3}-(1-\lambda)^{2/3}\}, \quad (29')$$

and it follows:

$$\kappa = \frac{4\pi\alpha}{(1+\lambda)^{2/3}-(1-\lambda)^{2/3}}. \quad (30)$$

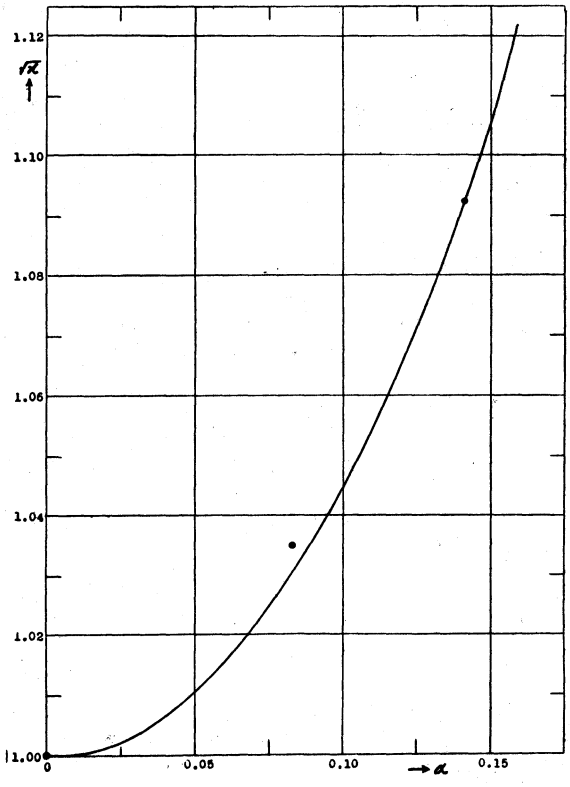

Fig. 5.

Making use of the expression (26′) into (30) we can obtain $\kappa(\lambda)$. As we require $\sqrt{\kappa}$ also, κ and $\sqrt{\kappa}$ are tabulated in Table 3 and the latter inscribed in Fig. 4. From Table 3

columns 2 and 4 we finally obtain $\sqrt{\kappa}$ as a function of α, this function being equal to $U/\sqrt{gL}/2\pi$. We express this relation in Fig. 5, in which the accurate values from § 2 are also inscribed with small black circles. The accordance is very good and it seems that the wave velocity is approximated by (26′) and (30) with sufficient accuracy over the whole range of steepness.

The method of averaging above adopted is supported by another consideration. The Bernoulli's theorem applied between the wave crest and trough gives

$$\frac{1}{2}U^2q^2(0)+gA=\frac{1}{2}U^2q^2(\pi)\,.$$

This becomes evidently into

$$\alpha=\frac{A}{L}=\frac{U^2}{2gL}\{q^2(\pi)-q^2(0)\}\,, \tag{31}$$

and making use of (23) and (27) this is nothing but the relation (30). That is to say, the determination of $\kappa(\lambda)$ above adopted is in accordance with the Bernoulli's relation in a certain special but important case.

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