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By Toshiro SUHARA

1. Introduction It is a matter of importance in considering the strength of structures to evaluate the stress concentration, plastic deformation and their strength at the geometrically discontinuous part, which is often observed in a welded structure such as the ship-hull. As the first step, the author evaluated approximately the elastic stress in beams with a rectangular section that changes discontinuously as shown in Fig. 1 in this report.

The calculation was made on the basis of so-called " Two-beam theory." This two beam theory gives fairly good approximation at the part other than the neighborhood of the root of notch in case when the connecting part of two beams has little stiffness,[1] but the error grows large if this theory is applied as it is by dividing the continuous body into two beams.

The author, therefore, adopted the calculation of the two-beam theory, considering a point load at the root of notch which he introduced as representing the stress concentration in order to correct the above-mentioned error and then determined the magnitude of the point load from " the principle of the minimum potential energy of the entire beam," in which the energy function near the root of notch was obtained by the two-dimensional elasticity theory.

The result thus obtained showed favorable coincidence with the experimental value except the neighborhood close to the discontinuous point (as far as about one tenth of the beam's depth).

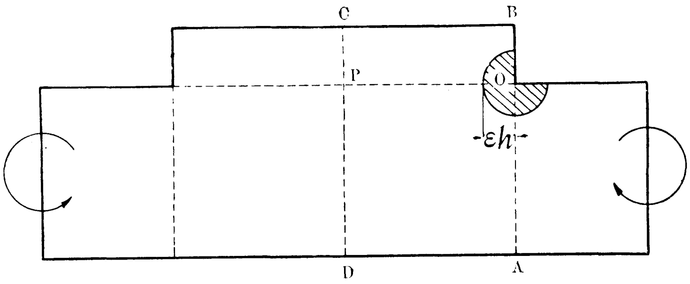

Fig. 1. Beam with discontinuous cross section

Fig. 1 gives the case where the depth of the beam at the central part is half as long as its length. Since we can, however, generally consider that the local stress disapears at the section separated from the root of notch as far as the distance equal to the depth of the beam, the author chose this form for convenience. Moreover, he believes that it is possible to expand the above-mentioned method for the aualysis of various cases : when the beams have complicated sections or ribs, or when the beam is connected with the projecting part which is made of different material.

Nomenclature

M = externally applied moment present in beam 2 at the discontinuous section.

$n_1 h, n_2 h$ = depth of beam I and II respectively where $n_1 + n_2 = 1$.

I_1, I_2 = moment of inertia of beam I and II per unit thickness of beam.
σ_m = horizontal normal stress σ_x at the point P (Fig. 1).
p, q = normal and shear stresses on the connecting surface of two beams.
δ_1, δ_2 = horizontal displacement of the lower edge of beam I and the upper edge of beam II.
y_1, y_2 = virtical displacement of the neutral axis of beam I and II.
P^*, S^* = concentrated virtical and horizontal forces at the point of discontinuity.

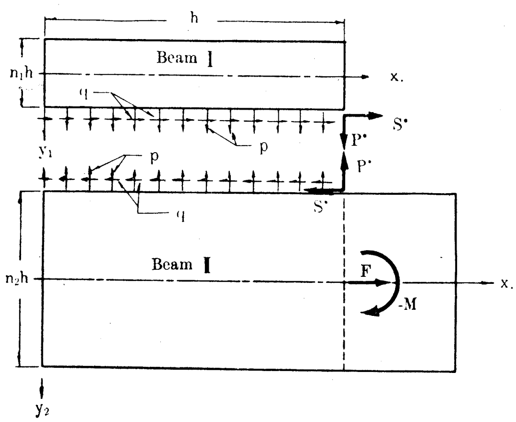

Fig. 2. Loadings of beam I and II

2. Two beam theory. The beam is assumed to act like two simple beams loaded by the normal pressure p and tangential stress q on their connecting surface and the load P^*, S^* at the point of discontinuity. (Fig. 2)

The moment equations of each beam are similar to that obtained by Cornell,[1]

$$\left.\begin{aligned} EI_1\frac{d^2y_1}{dx^2} &= -\frac{n_1h}{2}\int_x^h q\,d\xi+\int_x^h p(\xi-x)\,d\xi-\frac{n_1h}{2}S^*+(h-x)P^* \\ EI_2\frac{d^2y_2}{dx^2} &= -\frac{n_2h}{2}\int_x^h q\,d\xi-\int_x^h p(\xi-x)\,d\xi-\frac{n_2h}{2}S^*-(h-x)P^*+M \end{aligned}\right\}\quad\cdots(1)$$

An approximate expression for normal stress p in these equations is[2]

$$p=\frac{2kE}{h}(y_2-y_1),\quad (k=1.23). \qquad\cdots\cdots(2)$$

Using (2) and differentiating (1) twice with respect to x, we have

$$\left.\begin{aligned} D^4y_1 &= \frac{6h}{En_1^2}Dq+\frac{24k}{n_1^3}(y_2-y_1), \\ D^4y_2 &= \frac{6h}{En_2^2}Dq-\frac{24k}{n_2^3}(y_2-y_1), \end{aligned}\right\}\quad\cdots(3)$$

where $D=d/d\xi, \quad \xi=x/h.$

Further we have another equation from the condition of continuity of the displacements of two beams across the connecting surface. An approximate expressions of the displacements u_1 and u_2 of the lower and upper edge of beams 1 and 2 by the shearing stress q along the edges are[3]

$$u_1=\frac{n_1h}{E}q,\quad u_2=\frac{n_2h}{E}q. \qquad\cdots\cdots(4)$$

Then the total displacements δ_1 and δ_2 are obtained by adding to (4) the displacements by bending and normal extention.

$$\left.\begin{aligned}\delta_1 &= \int_0^x\int_x^h \frac{q}{E n_1 h} d\xi_1 d\xi_2 - \int_0^x \frac{n_1 h}{2}\frac{d^2 y_1}{dx^2} d\xi + \frac{S^*}{E n_1 h} + \frac{n_1 h}{E} q\,, \\ \delta_2 &= -\int_0^x\int_x^h \frac{q}{E n_2 h} d\xi_1 d\xi_2 + \int_0^x \frac{n_2 h}{2}\frac{d^2 y_2}{dx^2} d\xi - \frac{S^*}{E n_1 h} - \frac{n_2 h}{E} q + \frac{F}{E n_2 h} x\,.\end{aligned}\right\} \cdots(5)$$

Since δ_1 is equal to δ_2, we get

$$\frac{h}{E} q = -\int_0^x\int_x^h \left(\frac{1}{n_1 h} + \frac{1}{n_2 h}\right) q\, d\xi_1 d\xi_2 + \int_0^x \frac{h}{2}\left(n_1 \frac{d^2 y_1}{dx^2} + n_2 \frac{d^2 y_2}{dx^2}\right) d\xi$$

$$-\left(\frac{1}{E n_1 h} + \frac{1}{E n_2 h}\right) S^* x + \frac{F}{E n_2 h} x\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

Differentiating equation (6) twice with respect to x we have

$$\frac{h}{E} D^2 q = \left(\frac{1}{n_1 h} + \frac{1}{n_2 h}\right) q + \frac{h}{2}(n_1 D^3 y_1 + n_2 D^3 y_2)\,. \qquad \cdots\cdots\cdots(7)$$

From (4) and (7)

$$D^4\left[D^6 - \frac{4}{n_1 n_2} D^4 + 24k\left(\frac{1}{n_1^3} + \frac{1}{n_2^3}\right) D^2 - \frac{24k}{n_1^4 n_2^4}\right] y_{1 \text{ or } 2} = 0\,, \qquad \cdots\cdots\cdots(8)$$

which is the same equation as Cornell deduced.

As the beams are geometrically symmetrical the deflection and shear equations are of the form,

$$y_1 = c_1\xi^2 + f_1 \cosh\alpha\xi + g_1 \sinh\beta\xi \sin\gamma\xi + k_1 \cosh\beta\xi \cos\gamma\xi\,, \qquad \cdots(9)$$

$$y_2 = c_2\xi^2 + f_2 \cosh\alpha\xi + g_2 \sinh\beta\xi \sin\gamma\xi + k_2 \cosh\beta\xi \cos\gamma\xi\,, \qquad \cdots(10)$$

$$q = b_3 \sinh\alpha\xi + e_3 \sinh\beta\xi \cos\gamma\xi + f_3 \cosh\beta\xi \sin\gamma\xi\,, \qquad \cdots(11)$$

where α, β and γ are given by (8).

Substituting (9) (10) and (11) into the equations (1), (3) and (6) we have seven relationships of the coefficients and four equations of the boundary conditions from which the eleven constants in (9)～(11) should be determined as linear functions of P^* and S^*.

$$\left.\begin{aligned} & b_3 = \frac{\alpha^3}{\alpha^2 - \Delta^2}\frac{E}{2h} f\,, && f = n_1 f_1 + n_2 f_2\,, \\ & e_3 = \frac{E}{2h}\left[\frac{-\mu g + \pi k}{\nu^2 + \lambda^2}\right], && g = n_1 g_1 + n_2 g_2\,, \\ & f_3 = \frac{E}{2h}\left[\frac{\mu k + \pi g}{\nu^2 + \lambda^2}\right], && k = n_1 k_1 + n_2 k_2\,, \\ \text{where}\quad & \Delta^2 = \frac{1}{n_1 n_2}\,, \\ & \mu = -\gamma(3\beta^2 - \gamma^2)(\beta^2 - \gamma^2 - \Delta^2) + 2\gamma\beta^2(\beta^2 - 3\gamma^2)\,, \\ & \nu = \beta^2 - \gamma^2 - \Delta^2\,, \\ & \pi = -\beta(3\gamma^2 - \beta^2)(\beta^2 - \gamma^2 - \Delta^2) + 2\gamma^2\beta(3\beta^2 - \gamma^2)\,, \\ & \lambda = 2\gamma\beta\,. \end{aligned}\right\} \cdots(12)$$

$$
\left.
\begin{aligned}
& c_1 = c_2 , \\
& f_2 = \frac{\Gamma n_1^2 \alpha^4 + 1}{\Gamma n_2^2 \alpha^4 + 1} f_1 , \\
& g_2 = \frac{(\mu_1 \pi_1 + \nu_1 \lambda_1) g_1 + (\nu_1 - \lambda_1) k_1}{\mu_1^2 + \nu_1^2} , \\
& k_2 = \frac{(\mu_1 \pi_1 + \nu_1 \lambda_1) k_1 - (\nu_1 - \lambda_1) g_1}{\mu_1^2 + \nu_1^2} , \\
\text{where} \quad & \Gamma = \frac{n_1 n_2}{24 k} , \\
& \mu_1 = (\beta^4 - 6 \beta^2 \gamma^2 + \gamma^4) \Gamma n_2^2 + 1 , \\
& \nu_1 = 4 \beta \gamma \Gamma n_2^2 (\beta^2 - \gamma^2) , \\
& \pi_1 = (\beta^4 - 6 \beta^2 \gamma^2 + \gamma^4) \Gamma n_1^2 + 1 , \\
& \lambda_1 = 4 \beta \lambda \Gamma n_1^2 (\beta^2 - \gamma^2) .
\end{aligned}
\right\} \cdots (13)
$$

$$
\left.
\begin{aligned}
& b_3 \frac{\cosh \alpha}{\alpha} + \frac{e_3}{\beta^2 + \gamma^2} (\gamma \sinh \beta \sin \gamma + \beta \cosh \beta \cos \gamma) \\
& \quad + \frac{f_3}{\beta^2 + \gamma^2} (\beta \sinh \beta \sin \gamma - \gamma \cosh \beta \cos \gamma) \\
& = - \frac{S^*}{h} + \frac{E}{h^2} n_1 n_2 \left[6 \frac{M}{E} + (n_1^2 - n_1 n_2 + n_2^2) h^2 \frac{F}{E n_2 h} \right] , \\
& f_0 \frac{\cosh \alpha}{\alpha^2} + \frac{g_0}{(\beta^2 + \gamma^2)^2} \{ (\beta^2 - \gamma^2) \sinh \beta \sin \gamma - 2 \beta \gamma \cosh \beta \cos \gamma \} \\
& \quad + \frac{k_0}{(\beta^2 + \gamma^2)^2} \{ 2 \beta \gamma \sinh \beta \sin \gamma + (\beta^2 - \gamma^2) \cosh \beta \cos \gamma \} \\
& = - \frac{n_1^2}{2 k E h} \left[(n_1 + 3 n_2) M + \frac{n_2^2}{2} (n_2 - n_1) h^2 \frac{F}{n_2 h} \right] , \\
& f_0 \frac{\sinh \alpha}{\alpha} + \frac{g_0}{\beta^2 + \gamma^2} (- \gamma \sinh \beta \cos \gamma + \beta \cosh \beta \sin \gamma) \\
& \quad + \frac{k_0}{\beta^2 + \gamma^2} (\beta \sinh \beta \cos \gamma + \gamma \cosh \beta \sin \gamma) \\
& = - \frac{P^*}{2 k E} , \\
& c_1 = c_2 = 6 \left[\frac{M}{E h} - \frac{n_1 F}{2 E} \right] , \\
& \text{where} \quad f_0 = f_2 - f_1 , \quad g_0 = g_2 - g_1 \quad \text{and} \quad k_0 = k_2 - k_1 .
\end{aligned}
\right\} \cdots (14)
$$

3. The elastic energy stored near the root of notch. When a sector with its vertex angle β and two radial edges of stress free is subjected to extension in its plane (Fig. 3), the stresses produced near the corner are expressed approximately as follows.[4][5]

$$
\left.
\begin{aligned}
& \sigma_r^* = r^{\lambda_0 - 1} [f''(\theta) + (\lambda_0 + 1) f(\theta)] , \\
& \sigma_\theta^* = r^{\lambda_0 - 1} [\lambda_0 (\lambda_0 + 1) f(\theta)] , \\
& \tau_{r\theta}^* = r^{\lambda_0 - 1} [- \lambda_0 f'(\theta)] \\
& f(\theta) = A \left[\frac{1}{\lambda_0 - 1} \{ (\lambda_0 - 1) \sin (\lambda_0 + 1) \theta - (\lambda_0 + 1) \sin (\lambda_0 - 1) \theta \} \right. \\
& \quad \left. + \frac{\cos \lambda_0 \beta + \cos \beta}{\sin \lambda_0 \beta + \sin \beta} \{ \cos (\lambda_0 + 1) \theta - \cos (\lambda_0 - 1) \theta \} \right] ,
\end{aligned}
\right\} \cdots (15)
$$

and in the present case $\beta=\frac{3}{2}\pi$ and $\lambda_0=0.545$. Similarly the corresponding displacements can be obtained. Then the strain evergy V_s stored in the small region of radius εh and $\frac{\pi}{2}<\theta<\frac{3}{2}\pi$ (hatched region in Fig 3) is calculated by

$$V_s=\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\int_0^{\varepsilon h}\left\{\frac{1}{2E}(\sigma_r^{*2}+\sigma_\theta^{*2})-\frac{\nu}{E}\sigma_r^*\sigma_\theta^*+\frac{1}{2G}\tau_{r\theta}^{*2}\right\}r\,dr\,d\theta\,. \qquad \cdots\cdots\cdots(16)$$

It is easily seen from (15) and (16) that V_s is the function of one variable $A\varepsilon^{\lambda_0}$ $(\equiv w)$.

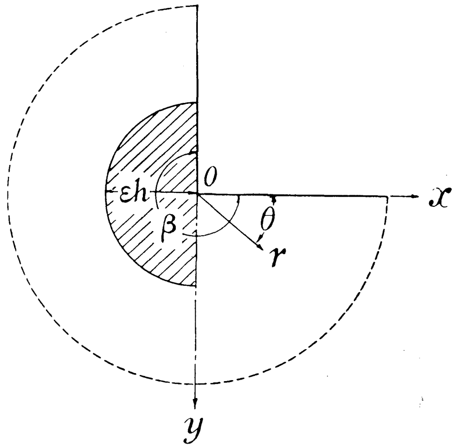

Fig. 3. Infinite sector of both radial edges free

4. Determination of P^* and S^*. From the assumption of previous articles P^* and S^* are given by

$$\left.\begin{aligned}P^*&=\int_0^{\varepsilon h}\sigma_\theta^*(\pi)\,dr\\ S^*&=\int_0^{\varepsilon h}\tau_{r\theta}^*(\pi)\,dr\end{aligned}\right\} \qquad \cdots\cdots\cdots(17)$$

which are also the functions of variable w.

Now the energy of the beam ABCD (Fig. 1) is equal to the work done by the stresses σ_x and τ_{xy} along the discontinuous section $0A$, that is

$$V=\frac{1}{2}\int_{0A}\{(u_B+u^*)(\sigma_B+\sigma^*)+(v_B+v^*)(\tau_B+\tau^*)\}\,dy \qquad \cdots\cdots\cdots(18)$$

where u_B, v_B are horizontal and virtical displacements and σ_B, τ_B are horizontal normal stress and shear stress along the section $0A$ and are the functions of P^* and S^* which are deduced from the equations (9), (10) and (11), and u^*, v^*, σ^* and τ^* are the local displacements and stresses in the vicinity of the notch. Neglecting the second- order effects, equation (18) is simplified as

$$V=\frac{1}{2}\int_{0A}u_B\sigma_B\,dy+\frac{1}{2}\delta_2F^*+\frac{1}{2}\int_0^{\varepsilon h}(u^*\sigma^*+v^*\tau^*)\,dy \qquad \cdots\cdots\cdots(19)$$

where $$F^*=\int_0^{\varepsilon h}\sigma_B^*\left(\frac{\pi}{2}\right)dy\,,$$

and the third term of the right hand side is equal to V_s. From the principle of minimum potential energy we have

$$\frac{\partial V}{\partial w}=0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots(20)$$

from which the value of w can be calculated.

5. Analytical and Experimental Results. Analytical results were obtained for the specimen of $n_1=1/4$ and $n_2=3/4$ under pure bending and the results were put in the form of stress ratios based on the horizontal normal stress σ_m at the point P. (thick lines in Figs. 4~6) The fine lines in these figures show the results of calculation based on the ordinary two-beam theory in which the stress-concentration at the discontinuity is not taken into consideration and the chain lines show the photoelastic results. It must be noted in Fig. 4 that the horizontal normal stresses along the lower

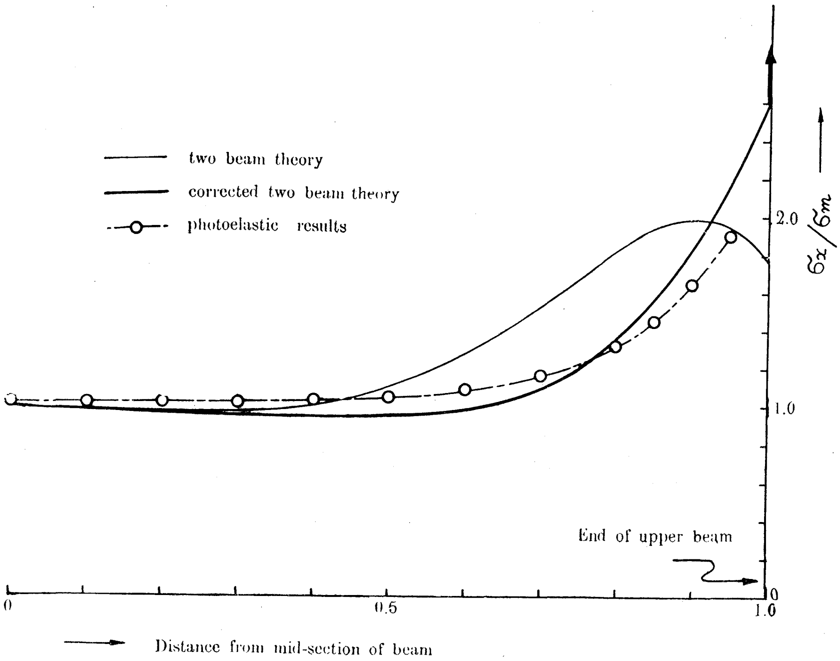

Fig. 4. Horizontal normal stress σ_x along the connecting surface of two beams.

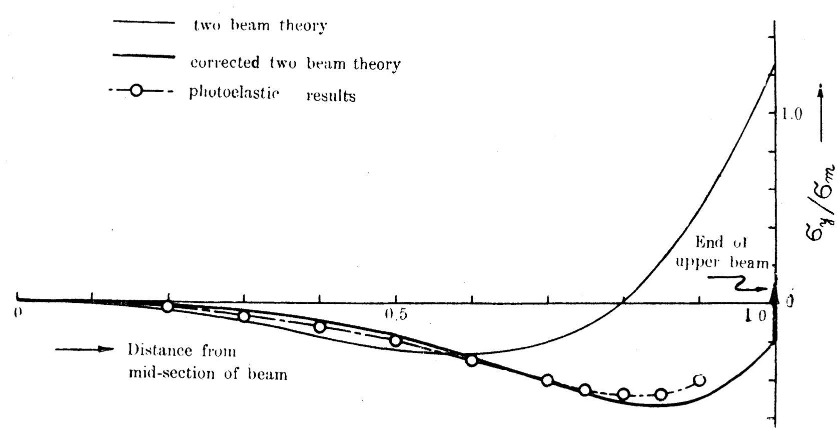

Fig. 5. Virtical normal stress σ_y on the connecting surface of two beams.

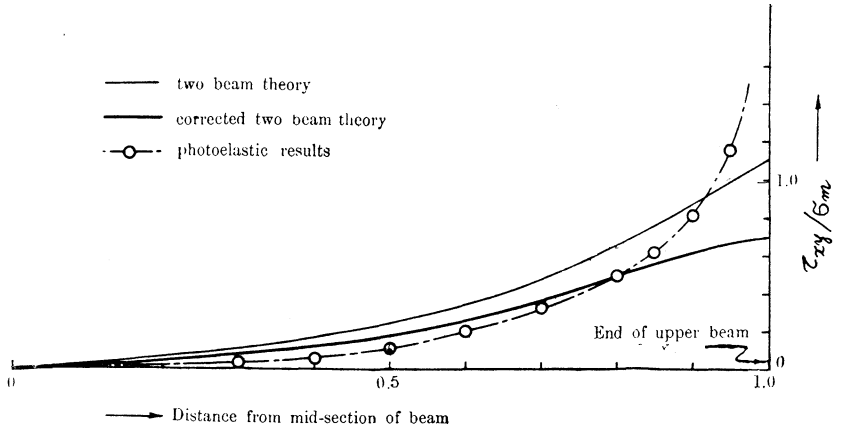

Fig. 6. Shearing stress τ_{xy} on the connecting surface of two beams.

edge of beam I and upper edge of beam II are not equal to each other, and we defined the stress σ_x along the connecting surface as the mean of them. In general, the results thus obtained showed favorable coincidence with the experimental value the main discrepancies being in the neighborhood close to the discontinuous point (as far as about one tenth of the beam's depth).

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