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By Hikoji YAMADA

Résumé.— Solitary wave problem is transformed into the one of holomorphic function $\Omega(\zeta)$ in a unit circle $|\zeta|<1$, whose real and imaginary parts θ, τ satisfy a certain nonlinear "surface condition" on the circumference of unit circle. This is almost identical formulation as that of deep water due to Lévi-Civita. Then the highest wave is determined by numerical iteration procedure, whose one cycle consisting of two steps: the first the derivation of τ from θ by surface condition and the second the determination of θ, the harmonic conjugate function of τ, Fourier series with 12 components being used as an available function for adjustment. Further improvement of approximation has been undertaken to satisfy the surface condition within a percent error.

§ 1. Transformation to unit circle. Harmonic functions or holomorphic functions are most easily dealt with in a circular region, and in the Lévi-Civita's theory[1] of deep water waves the water region was transformed into a unit circle $|\zeta|\leq 1$ of an auxiliary plane ζ to define a holomorphic function $\Omega=\theta+i\tau$, which describes the velocity field, in this region by the "surface condition" assigned to the values on the circumference. If such a transformation to unit circle is possible by means of a simple analytical expression, in the occasion of solitary wave in a uniform canal, we shall hope for a wide perspective of the problem and calculation much lightened. In the following we show such a simple transformation, that makes calculation on the unit circle possible, and in this formulation of the problem we take up in the first place the determination of highest possible solitary wave in a uniform canal.

The coordinates system O-xy of the physical plane $z=x+iy$ follows after the permanent solitary wave with common velocity U to left, and consequently the wave form stands still relative to the axes and water flows steadily from left to right, the velocity at infinite distance being U, x-axis coincides with the bottom of canal and directed to right, y-axis vertical and upwards, origin O being just under the centre of crest O_1; the depth of water at infinite distance in both directions is equal and H.

[1] T. Lévi-Civita, Math. Annalen, vol. 93, p. 264- (1925); recently we have calculated the highnest waves on deep water by his formulation of the problem, and in our paper reporting the results (this vol. of this journal) the outline of his formulation is described as far as we required. This paper will be quoted in the followings as (D).

Our wave is irrotational and the complex potential $W=\varphi+i\psi$, where φ is the potential function and ψ the stream function, of the flow corresponds, after adequate selection of constants to let coincide origin to origin, to the physical plane z as is indicated qualitatively in Fig. 1. If we know this correspondence *i.e.* the functional form $W(z)$ analytically we have

$$\frac{1}{U}\frac{dW}{dz}=qe^{-i\theta}; \tag{1}$$

here q is the magnitude and θ the angle of inclination of flow velocity, the former measured with U as unit and the later from Ox-direction upwards. Then $W(z)$ is our main object and the velocity field can be obtained by mere differentiation. It is, however, difficult to get it directly and we go round by the introduction of unit circle region in a certain auxiliary plane.

When we make a correspondence between W and an auxiliary plane Z by the relation

$$Z=\cosh\left(\frac{\pi}{2}\cdot\frac{W}{UH}\right), \tag{2}$$

the flow region $(0\leq\psi\leq UH)$ is mapped on a right half plane of Z as shown in Fig. 1 and 2, the correspondence of boundaries being shown by the same alphabetical letters and arrows as usual in such a case. On the other hand the relation

$$\zeta=\frac{1-Z}{1+Z}, \quad i.e. \quad Z=\frac{1-\zeta}{1+\zeta} \tag{3}$$

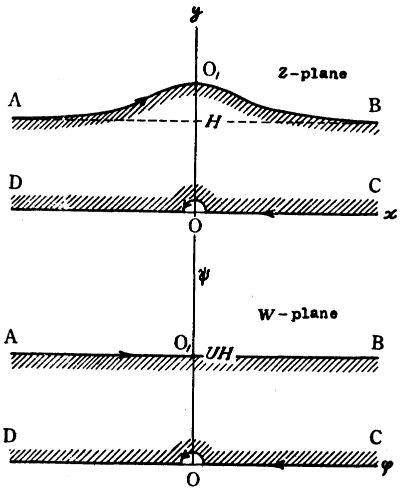

Fig. 1.

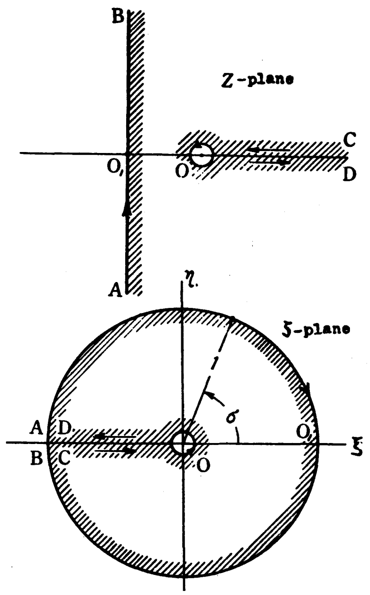

Fig. 2.

maps the Z region above stated onto the unit circle of ζ, and the correspondence of boundaries is as shown in the same way lin Fig. 2. Then the equation

$$\frac{1-\zeta}{1+\zeta}=\cosh\left(\frac{\pi}{2}\cdot\frac{W}{UH}\right) \tag{4}$$

maps the water region of W directly onto unit circle of ζ.

In this mapping the unit circle region of ζ is cut along the negative real axis from origin to $\zeta=-1$, and each edge of the cut corresponds to the bottom of the canal ($\psi=0$), the under edge to the right half ($\varphi>0$) and the upper edge to the left half ($\varphi<0$) of the bottom. To any two points $-\varphi+i\psi$ and $\varphi+i\psi$ which are placed symmetrically about ψ-axis correspond each to each two points $\xi+i\eta$ and $\xi-i\eta$ which are placed symmetrically about real axis $\eta=0$, *i.e.* two complex conjugate points; specially the imaginary axis ($\varphi=0$, $0\leq\psi\leq UH$) of the W-plane is mapped on the real axis ($\eta=0$, $0\leq\xi\leq 1$) of the ζ-plane.

Now as our wave is symmetrical[2] every stream line is also symmetrical about y-axis in the z-plane. From this fact it follows that to every pair of symmetrical points about y-axis belongs a couple of complex conjugate velocities, and also a pair of potentials of equal magnitude and opposite signs. Transforming this correspondence into W-plane, to every pair of symmetrical points about ψ-axis belongs a couple of complex conjugate velocities, and transforming again into ζ-plane, to every pair of complex conjugate points belongs a couple of complex conjugate velocities. Thus we see that the flow state mapped on the unit circle $|\zeta|\leq 1$ is symmetrical about real axis; above all, the flow velocity is horizontal on the total real axis ($-1\leq\xi\leq 1$, $\eta=0$).

§2. Formulation of problem. Following Lévi-Civita we introduce the Ω-function by the definition

$$\Omega=i\log\left(\frac{1}{U}\frac{dW}{dz}\right). \tag{5}$$

Inserting (1) into this equation we have

$$\Omega=\theta+i\tau \quad (\tau=\log q), \qquad \text{or} \qquad e^{-i\Omega}=qe^{-i\theta}; \tag{6}$$

and, as the flow velocity does vanish at nowhere in water, is holomorphic function of z in the water region. Through the transformation of z into ζ we have $\Omega(\zeta)$ defined on the unit circle region (with the cut) and holomorphic in the interior of it.

As we observed in preceding section the values of Ω *i.e.* $\theta+i\tau$ at two points in the unit circle $|\zeta|\leq 1$, which are conjugate to each other, have the real parts of equal magnitude and opposite signs and the imaginary

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[2] *c.f.* K. O. Friedrichs and D. H. Hyers, Comm. Pure Appl. Math., vol. 7, no. 3, p. 517 (1954).

parts of equal magnitude and sign. In other words $-i\Omega=\tau-i\theta$ are complex conjugate to each other at every pair of complex conjugate points of ζ. Above all $-i\Omega$ is real on the real axis: on the edges of the cut the values of upper edge coincide with those of lower edge perfectly all along the cut, and on the real axis the values are real throughly. Therefore by the Schwarz principle $-i\Omega$ and consequently Ω itself, is holomorphic function in the unit circle (without any cut) $|\zeta|<1$.

We know then that our Ω is of the same properties as that of Lévi-Civita's case and that his theory can be employed without alteration in our case as follows. From (4) we have by differentiation

$$-\frac{2}{(1+\zeta)^2}=\sinh\left(\frac{\pi}{2}\frac{W}{UH}\right)\cdot\frac{\pi}{2H}\cdot\frac{1}{U}\frac{dW}{d\zeta},$$

i.e.
$$\frac{1}{U}\frac{dW}{d\zeta}=i\frac{2H}{\pi}\frac{1}{\sqrt{\zeta}\,(1+\zeta)}(-\pi<\arg\zeta\leq\pi), \tag{7}$$

and the left hand side of this being changed by (5) into

$$\frac{1}{U}\frac{dW}{d\zeta}=\frac{1}{U}\frac{dW}{dz}\frac{dz}{d\zeta}=e^{-i\Omega}\cdot\frac{dz}{d\zeta},$$

(7) reduces to

$$dz=i\frac{2H}{\pi}\frac{e^{i\Omega(\zeta)}}{\sqrt{\zeta}\,(1+\zeta)}d\zeta. \tag{8}$$

Therefore if $\Omega(\zeta)$ can be determined in a way or another the fundamental relation between ζ and z can be found by integration, and then $\Omega(z)$ *i.e.* the velocity field, or $W(z)$ *i.e.* the potential field in the physical plane is obtainable by mere substitution.

As the circumference of unit circle $\zeta=e^{i\sigma}(-\pi<\sigma\leq\pi)$ corresponds to the water surface $\Omega(e^{i\sigma})=\theta(\sigma)+i\tau(\sigma)$ is generally a continuous function of σ, the point on the circumference, and fulfils the surface condition. This surface condition is, in the physical plane, the fact that the water surface contacts with atmosphere and consequently there prevails a certain constant statical pressure. The surface is a streamline $\psi=UH$ and by Bernoulli's theorem the surface condition can be written in the form:

$$q^2+\frac{2gy}{U^2}=\text{const.}, \tag{9}$$

which we use in a differential form, differentiation being along the arc of surface streamline:

$$q\frac{dq}{ds}=-\frac{g}{U^2}\frac{\partial y}{\partial s}=\frac{g}{U^2}\sin\theta; \tag{9'}$$

ds is counted positive when the increment is against stream and $\partial y/\partial s=-\sin\theta$. Transfer of (9') on the circumference of unit circle is accomplished by rewriting ds by $d\sigma$, the relation between them can be obtained from (8) when we write $\zeta=e^{i\sigma}$:

$$dz = -ds\,e^{i\theta}, \quad ds = \frac{H}{\pi} \frac{1}{q \cos\frac{\sigma}{2}} d\sigma . \tag{10}$$

The required surface condition on the unit circle is then

$$\cos\frac{\sigma}{2} q^2 \frac{dq}{d\sigma} = p \sin\theta , \tag{11}$$

where the parameter p is defined:

$$p = \frac{1}{\pi M^2}, \quad M^2 = \frac{U^2}{gH}. \tag{11'}$$

The essentials of our solution are thus concentrated to the construction of a holomorphic function $\Omega(\zeta) = \theta + i\tau$ in a unit circle, which fulfils (i) the surface condition (11) on the circumference; and (ii) fulfils the symmetry condition $\theta(\bar{\zeta}) = -\theta(\zeta)$ and $\tau(\bar{\zeta}) = \tau(\zeta)$, where the superimposed bar denoting complex conjugate quantity; and finally (iii) $\tau(\zeta) = 0$ when $\zeta = -1$, which is evident. It is known that when p is an arbitrary value less than π^{-1} and sufficiently near to it a certain solution exists,[2] and several approximate solutions have ever been proposed.[3] As among these solutions, however, remarkable differences exist further investigation of solution is necessary. Leaving the general discussions along our formulation, which will reveal, we hope, some facts of importance, till later on, we here calculate the highest possible wave, its shape, its speed etc, numerically in the same way as it has been done in our paper (D) with regard to the deep water waves.

§3. **Calculation scheme for highest wave.** Power series expansion of $\Omega(\zeta)$ will be adequate for the general case where $\Omega(\zeta)$ is holomorphic not only in but on the circumference of unit circle. In the extreme case of the highest crest, however, the point $\zeta = 1$ is a singular point at which $\theta(\sigma)$ along the circular arc jumps:

$$\lim_{\sigma \to 0+} \theta(\sigma) = \pi/6, \quad \lim_{\sigma \to 0-} \theta(\sigma) = -\pi/6 .$$

Therefore we introduce, as in (D), an appropriate expression $\Omega_0(\zeta)$ which is in the neighborhood of $\zeta = 1$ an approximate solution:

$$\Omega_0(\zeta) = \frac{i}{3} \log(1-\zeta) - \frac{i}{3} \log 2 , \tag{12}$$

where the first term on the right side is identical with that of (D) and the second term is added so as $\Omega_0(\zeta)$ vanishes at $\zeta = -1$; on the circumference it becomes to

[3] *c.f.* Boussinesq, J. Math. France, vol. 17, p. 55 (1872); Lord Rayleigh, Phil. Mag. (5), vol. 1, p. 257 (1876); J. McCowan, Phil. Mag. (5), vol. 32, p. 45 (1891), vol. 38, p. 351 (1894).

$$\theta_0(\sigma) + i\,\tau_0(\sigma) = \begin{Bmatrix} \dfrac{\pi-\sigma}{6}\,(\pi \geq \sigma > 0) \\ \dfrac{-\pi-\sigma}{6}\,(0 > \sigma > -\pi) \end{Bmatrix} + \frac{i}{3}\log\left|\sin\frac{\sigma}{2}\right|. \tag{12'}$$

When we put

$$\Omega(\zeta) = \Omega_0(\zeta) + \Omega_r(\zeta)\,, \tag{13}$$

Ω_r is holomorphic in and on the unit circle, except a point $\zeta = 1$, where it is continuous, as is explained in (D). Then we expand $\Omega_r(\zeta)$ into power series

$$\Omega_r(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n\,, \tag{14}$$

and this series can be approximated by the sum of finite terms to any desired degree of accuracy in and on the unit circle, which is a necessary condition for numerical computation. As is explained above $\Omega(\zeta)$ is pure imaginary on the real axis and $\Omega_0(\zeta)$ has, as is easily provable, the same property, so that $\Omega_r(\zeta)$ has also that property. Therefore the coefficients b_n's in (14) are all pure imaginary and it can be expressed in the form

$$\Omega_r(\zeta) = i\sum_{n=0}^{\infty} a_n \zeta^n \tag{14'}$$

with real coefficients a_n's; above all on the circumference

$$\left.\begin{aligned} \theta(\sigma) &= \theta_0(\sigma) + \theta_r(\sigma)\,, \quad \theta_r(\sigma) = -\sum_{n=1}^{\infty} a_n \sin n\sigma\,; \\ \tau(\sigma) &= \tau_0(\sigma) + \tau_r(\sigma)\,, \quad \tau_r(\sigma) = \sum_{n=0}^{\infty} a_n \cos n\sigma\,. \end{aligned}\right\} \tag{15}$$

With the assumed form of solution symmetry requirement is satisfied, and the condition at $\zeta = -1$ is expressed in the form $\tau_r(\pi) = 0$.

We are now in a position to determine the coefficients a_n's through the surface condition (11). As the direct substitution of (15) into (11) results in so much complicated expression that we can at most determine first several coefficients only, we adopt here also the numerical iterative method used in (D). At first we fix the number $N+1$ of coefficients $a_0, a_1, \cdots, a_N$ to be taken in and neglect the others, then we attempt to determine these $N+1$ constants as good as possible. The iterative method consists of cycles, each of which having two steps. First step is the derivation of $\tau_r(\sigma)$ from $\theta_r(\sigma)$ through the surface condition, which we use now in the form:

$$q^3(\sigma) = 3p\int_0^{\sigma} \sin\theta(\sigma')\sec\frac{\sigma'}{2}\,d\sigma'\,, \tag{17}$$

where $\theta(\sigma) = \theta_0(\sigma) + \theta_r(\sigma)$ of course. As $q(\sigma)$ is velocity magnitude on the surface it must satisfy the condition $q(\pi) = 1$, which is equivalent to $\tau_r(\pi) = 0$ alluded to above, and then from (17)

$$1 = 3p \int_0^{\pi} \sin\theta(\sigma') \sec\frac{\sigma'}{2} d\sigma', \tag{18}$$

which determines the value of parameter *i.e.* eigenvalue p of this cycle. Making use of this value (17) determines the velocity magnitude, and by the relation $\tau_r(\sigma) = \log q(\sigma) - \tau_1(\sigma)$ we obtain $\tau_r(\sigma)$ of this cycle. Be noted in passing that $\tau_r(0+) = (1/3)\log(3p)$. Second step is the derivation of $\theta_r(\sigma)$ which is the harmonic conjugate function of $\tau_r(\sigma)$ just obtained. For this purpose we expand $\tau_r(\sigma)$ into Fourier cosine series with coefficients $a_0, a_1, \cdots, a_N$, by virtue of $N+1$ values of it at the points $\sigma = n\pi/N$ $(n = 0, 1, \cdots, N)$. With these coefficients a_n's the required $\theta_r(\sigma)$ can be written down at once according to the expression (15).

$\theta_r(\sigma)$ thus obtained is again substituted into (17) and next cycle begins. Starting from an arbitrarily chosen intial function $\theta_r(\sigma)$ the iterative procedure converges practically after a number of cycles, and by any further cycles θ_r, τ_r and p are not changed in practice.

These final functions and eigenvalue are the ones required and any quantity of importance can be deduced from them. The velocity of wave is calculated by virtue of (11′) as

$$U = \frac{1}{\sqrt{\pi p}} \sqrt{gH}, \tag{19}$$

and surface condition (9) applied to the point $\sigma = 0$ and π brings for the height of wave the expression:

$$\frac{A}{H} = \frac{1}{2\pi p}, \tag{20}$$

these quantities being computed at once. For the wave profile (x, y) the equation (8) with $\zeta = e^{i\sigma}$ is useful and gives:

$$\frac{dx + i\,dy}{H} = -\frac{1}{\pi} \cdot \frac{e^{i\theta(\sigma)}}{q(\sigma)} \sec\frac{\sigma}{2} d\sigma, \tag{21}$$

x and y being obtained by mere quadrature.

Our solution θ_r, τ_r and p satisfy the surface condition and the conjugate relation simultaneously, the former, however, approximately because the satisfaction of the condition is exact only at a number of discrete points $\sigma = n\pi/N$ $(n = 0, 1, \cdots, N)$. Or, so to speak, the surface condition is exact but the Fourier analysis is not exact. The Fourier cosine series with finite terms $\sum a_n \cos n\sigma$ coincides with original $\tau_r(\sigma)$, which is the direct consequence of the integral expression (17), at these points exactly and in the intervals between them with errors. As an indication of the degree of approximation we can take *e.g.* the error in the satifaction of original surface condition (9), which with a new ordinate $y_1 = y - (A+H)$——*i.e.* through the parallel displacement of origin from O to O_1——is written in the form:

$$q^2 + 2\pi p \frac{y_1}{H} = 0. \tag{22}$$

§4. Numerical treatment and results. We took $N=12$ and $\theta_r(\sigma)$ of start was zero *i.e.* $a_n=0$ ($n=0, 1, \cdots, 12$). Quadratures are done by the Simpson rule with 15° steps (*i.e.* 7.5° divisions) between $\sigma=0°$ and 180°, the points to be mentioned for notice are $\sigma=0°$ and 180°, where $\tau_r(0+)=(1/3)\log(3p)$ and $(\sin\theta\sec(\sigma/2))_{\sigma=\pi}=-2\theta'(\pi)$. The values of a_n's and p after 10 cycles of iteration procedure have been chosen as final ones,[4] which are given in Table 1 (above all $p=0.1925$) and $\theta_r(\sigma)$, $\tau_r(\sigma)$ calculated with them

Table 1.

a_0	−0.1584
a_1	−0.0659
a_2	+0.0366
a_3	−0.0117
a_4	+0.0124
a_5	−0.0054
a_6	+0.0071
a_7	−0.0034
a_8	+0.0051
a_9	−0.0026
a_{10}	+0.0042
a_{11}	−0.0022
a_{12}	+0.0020
p	0.1925

Table 2.

(deg.) σ	θ_0	τ_0	θ_r	τ_r	
0	0.5236	$-\infty$	0.0000	−0.1825*	(−0.1829)
3.75	0.5127	−1.1399	−0.0044	−0.1839	
7.5	0.5018	−0.9091	−0.0074	−0.1876	
15	0.4800	−0.6787	−0.0063	−0.1972*	(−0.1977)
22.5	0.4582	−0.5448	0.0011	−0.2025	
30	0.4363	−0.4505	0.0077	−0.2032*	(−0.2036)
37.5	0.4145	−0.3783	0.0125	−0.2038	
45	0.3927	−0.3202	0.0183	−0.2049*	(−0.2053)
52.5	0.3709	−0.2719	0.0252	−0.2044	
60	0.3491	−0.2311	0.0317	−0.2027*	(−0.2032)
67.5	0.3273	−0.1959	0.0385	−0.2006	
75	0.3054	−0.1654	0.0457	−0.1967*	(−0.1972)
82.5	0.2836	−0.1388	0.0515	−0.1915	
90	0.2618	−0.1155	0.0566	−0.1869*	(−0.1874)
97.5	0.2400	−0.0951	0.0634	−0.1817	
105	0.2182	−0.0772	0.0704	−0.1728*	(−0.1733)
112.5	0.1964	−0.0615	0.0734	−0.1622	
120	0.1745	−0.0480	0.0751	−0.1543*	(−0.1548)
127.5	0.1527	−0.0363	0.0805	−0.1459	
135	0.1309	−0.0264	0.0857	−0.1305*	(−0.1310)
142.5	0.1091	−0.0182	0.0828	−0.1124	
150	0.08727	−0.0116	0.0764	−0.1005*	(−0.1010)
157.5	0.06545	−0.0065	0.07761	−0.0887	
165	0.04363	−0.0029	0.07877	−0.0602*	(−0.0606)
172.5	0.02182	−0.0007	0.05467	−0.0200	
176.25	0.01091	−0.0002			
180	0.00000	0.0000	0.00000	0.0000*	(0.0000)
			$\left(\begin{array}{l}\tau_r'(0+)=-0.06875\\ \tau_r'(\pi-)=0.2453\end{array}\right)$		

[4] As we found irregularities in the values of $\tau_r(\sigma)$ after a few cycles which seemed to due to mistakes in numerical computation we smoothed them arbitrarily; if it were not convergency would have been more rapid.

by (15) are in Table 2 columns 4, 5. Numbers in the parentheses in column 5 are the eleventh approximation, and the differences from the tenth approximation are within 0.7 %, which indicates the degree of convergence.

Making use of the value $p = 0.1925$ in (19) we have for the wave velocity

$$U = 1.286\sqrt{gH}\,, \tag{23}$$

and for the wave height

$$\frac{A}{H} = 0.8266\,, \tag{24}$$

Table 3.

(deg.) σ	θ	q	$-x/H$	$-y/H$	$\frac{\left(q^2+2\pi p\frac{y_1}{H}\right)\times 100}{q^2}$
0	0.5236	0.0000	0.0000	0.0000	--
3.75	0.5083	0.2661			
7.5	0.4944	0.3340	0.1626	0.0913	1.1
15	0.4731	0.4165			
22.5	0.4592	0.4736	0.3458	0.1857	−0.2
30	0.4440	0.5201			
37.5	0.4270	0.5587	0.4964	0.2573	0.3
45	0.4110	0.5916			
52.5	0.3961	0.6211	0.6365	0.3184	0.2
60	0.3808	0.6480			
67.5	0.3658	0.6727	0.7746	0.3737	0.1
75	0.3511	0.6962			
82.5	0.3351	0.7187	0.9165	0.4256	0.3
90	0.3184	0.7391			
97.5	0.3034	0.7582	1.0683	0.4757	−0.1
105	0.2885	0.7788			
112.5	0.2698	0.7996	1.2373	0.5257	0.5
120	0.2496	0.8169			
127.5	0.2332	0.8335	1.4359	0.5763	−0.4
135	0.2166	0.8548			
142.5	0.1919	0.8776	1.6868	0.6309	0.9
150	0.1636	0.8940			
157.5	0.1431	0.9092	2.0495	0.6905	−1.1
165	0.1224	0.9389			
172.5	0.0765	0.9795	2.7931	0.7737	2.4
180	0.0000	0.0000	∞	0.8250	0.2
	$\theta'(0+) = -0.2382$				
	$\theta'(\pi) = -0.6369$				
	$\left(\sin\theta\sec\frac{\sigma}{2}\right)_{\sigma=\pi-} = 1.2738$				

which differ from the values of McCowan, $1.25\sqrt{gH}$ and 0.780, a few percents respectively; it is interesting to note that in the $U/\sqrt{gH}$ *vs* A/H diagram our point (0.8266, 1.286) is just on the extrapolation of the experimental curve of the Hydrodynamical Laboratory of M.I.T.,[5] and McCowan's point (0.780, 1.25) is decidedly not.

Wave profile is calculated by (21), which is rewritten making use of the ordinate y_1 into

$$\frac{x+iy_1}{H} = -\frac{1}{\pi}\int_0^{\sigma} \frac{e^{i\theta(\sigma')}}{q(\sigma')} \sec\frac{\sigma'}{2}\, d\sigma'. \tag{25}$$

Quadratures are done by Simpson rule with the values $\theta(\sigma)$, $q(\sigma)$ given in Table 3 columns 2 and 3, which are computed by means of Table 2. Near $\sigma=0$ the expansions

$$\left.\begin{aligned} \frac{\cos\theta}{q}\sec\frac{\sigma}{2} &= 1.3096\sigma^{-1/3}(1+a\sigma+b\sigma^2) \\ \frac{\sin\theta}{q}\sec\frac{\sigma}{2} &= 0.7561\sigma^{-1/3}(1+a'\sigma+b'\sigma^2) \end{aligned}\right\} \tag{26}$$

are used. Such quadratures have been used frequently in (D) and we use here without description. The results are given in Table 3 columns 4 and 5, and in Fig. 3. $-y_1/H$ is 0.8250 for $\sigma=\pi$, and the practical agreement of this with A/H of (24) tells the accuracy of our calculation.

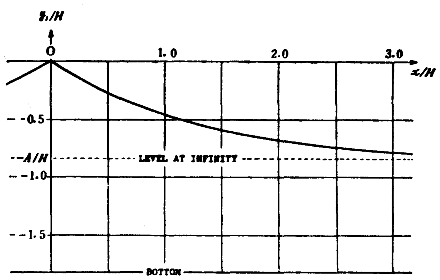

Fig. 3.

To know the values of quantities in water it is necessary to integrate (8) and obtain the relation $z=z(\zeta)$, which is prettly tedious and we do not take up here. For the bottom ζ is equal to $e^{-i\pi}\xi$ and $e^{i\pi}\xi$, and calculations are easy; specially at the point O which is just under the crest O_1 $\zeta=0$, and then $\tau=-3^{-1}\log 2+a_0=-0.3895$, $q=0.6774$ and the pressure head $H_0=H(1+\overline{1-q^2}/2\pi p)=1.4473H$.

Finally the degree of approximation is shown in Table 3 column 6, by virtue of errors in the surface condition (22) above stated. Errors are generally less than 1% except crest and distant bases. Specially at the latters errors mount to 2.4% and can scarcely be overlooked.

§5. Improvement of approximation. Errors in the satisfaction of surface condition are due to the short cut of the Fourier expansion, and for finer agreement more terms are to be included, or $\theta_r(\sigma)$, the harmonic con-

[5] *c.f.* Coastal Engineering, vol. 3 (1952), div. 1, chap. 2 by J. W. Daily and S. C. Stephan Jr.

jugate function of $\tau_r(\sigma)$, has to be calculated all numerically. But in these processes necessary numerical computations are so much that we try in the following another device of improvement, which we used in (D) successfully.

In the present case $\tau_r(\sigma)$ which is the direct consequence of the integral expression (17) is the numbers asterisked in Table 2 column 5, and interpolations by the 12-conponents Fourier series are the numbers between them. The relation between these two $\tau_r(\sigma)$ is visualised in Fig. 4 and we can observe at once the reason why the discrepancies are larger near the ends. If we can obtain an analytical expression of $\tau_r(\sigma)$ which represents $\tau_r'(0+)$ and $\tau_r'(\pi-)$ exactly, and if we can obtain an expression of $\theta_r(\sigma)$ which is harmonic conjugate to $\tau_r(\sigma)$, these τ_r and θ_r will solve the problem with far better approximation. $\tau_r(\sigma)$ is an even function of σ and then Fourier cosine series is to be consulted. As this series with short cut terms, however, has necessarily horizontal tangents at $\sigma=0$ and π, we attempt to realize horizontal end-tangents of the function, which is to be expanded into cosine series. For this purpose we take up two functions; the one which has been used in (D) is for $\sigma=0$, and the other $\Omega_2(\zeta)$ which is derivable from $\Omega_1(\zeta)$ is for $\sigma=\pi$. These functions are

$$\Omega_1(\zeta)=i\left(1-\frac{2}{\pi}\right)+\frac{i}{\pi}\sum_{n=}^{\infty}\frac{1}{n^2-1/4}\zeta^n, \tag{27}$$

$$\left.\begin{aligned}&\Omega_1(e^{i\sigma})=\theta_1(\sigma)+i\tau_1(\sigma),\\ &\theta_1(\sigma)=\frac{2}{\pi}\sin\frac{\sigma}{2}\log\left|\tan\frac{\sigma}{4}\right|,\quad \tau_1(\sigma)=\begin{cases}1-\sin\dfrac{\sigma}{2}\,(\sigma>0),\\ 1+\sin\dfrac{\sigma}{2}\,(\sigma<0),\end{cases}\end{aligned}\right\} \tag{27'}$$

$$\tau_1'(0+)=-1/2,\quad \tau_1'(\pi)=0; \tag{27''}$$

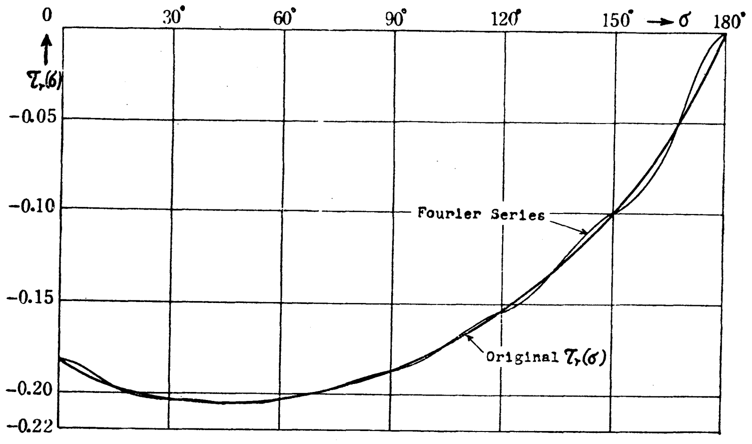

Fig. 4.

and

$$\Omega_2(\zeta) = i - \Omega_1(-\zeta), \tag{28}$$

$$\left.\begin{aligned} &\Omega_2(e^{i\sigma}) = \theta_2(\sigma) + i\tau_2(\sigma), \\ &\theta_2(\sigma) = \frac{2}{\pi}\cos\frac{\sigma}{2}\log\left|\tan\frac{\pi-\sigma}{4}\right|, \quad \tau_2(\sigma) = \cos\frac{\sigma}{2}, \end{aligned}\right\} \tag{28'}$$

$$\tau_2'(0) = 0, \quad \tau_2'(\pi-) = -1/2; \tag{28''}$$

$\theta(\sigma)$'s and $\tau(\sigma)$'s being tabulated in Table 4 and represented graphically in Fig. 5. $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ are not only holomorphic in the unit circle ($|\zeta|<1$) and continuous on it ($|\zeta|=1$), but have the same symmetry character as $\Omega_r(\zeta)$ itself.

Table 4.

(deg.) σ	θ_1	τ_1	θ_2	τ_2	$\varDelta\tau$
0	0.0000	1.0000	0.0000	1.0000	0.1701*
3.75	−0.0857	0.9673	−0.0208	0.9995	0.1698
7.5	−0.1424	0.9346	−0.0416	0.9979	0.1697
15	−0.2265	0.8695	−0.0829	0.9914	0.1691*
22.5	−0.2879	0.8049	−0.1234	0.9808	0.1689
30	−0.3341	0.7412	−0.1629	0.9659	0.1682*
37.5	−0.3686	0.6786	−0.2009	0.9464	0.1661
45	−0.3934	0.6173	−0.2372	0.9239	0.1630*
52.5	−0.4100	0.5577	−0.2713	0.8969	0.1587
60	−0.4192	0.5000	−0.3029	0.8660	0.1528*
67.5	−0.4219	0.4444	−0.3316	0.8315	0.1459
75	−0.4187	0.3912	−0 3571	0.7934	0.1380*
82.5	−0.4102	0.3407	−0.3789	0.7518	0.1293
90	−0.3968	0.2929	−0.3968	0.7071	0.1192*
97.5	−0.3789	0.2482	−0.4102	0.6594	0.1083
105	−0.3571	0.2067	−0.4187	0.6088	0.0968*
112.5	−0.3316	0.1685	−0.4219	0.5556	0.0848
120	−0.3029	0.1340	−0.4192	0.5000	0.0720*
127.5	−0.2713	0.1031	−0.4100	0.4423	0.0590
135	−0.2372	0.0761	−0.3934	0.3827	0.0462*
142.5	−0.2009	0.0531	−0.3686	0.3214	0.0338
150	−0.1629	0.0341	−0.3341	0.2588	0.0213*
157.5	−0.12340	0.0192	−0.28786	0.1951	0.0097
165	−0.08285	0.0086	−0.22645	0.1305	0.0023*
172.5	−0.04162	0.0021	−0.14237	0.0654	−0.0001
176.25	−0.02082	0.0005	−0.08567	0.0327	−0.0002
180	0.00000	0,0000	0.00000	0.0000	0.0000*

As we adopt for $\tau_r(\sigma)$ the numbers in parentheses in Table 2 column 5 $\theta(\sigma)$ in the integral expression of (17) must be the final one of preceding section. By means of this equation (17) or rather its differential form (11) we obtain

$$e^{3\tau_r(\sigma)}\left(\frac{1}{6}\cos^2\frac{\sigma}{2}+\frac{1}{2}\sin\sigma\,\tau_r'(\sigma)\right)=p\sin\theta(\sigma)\,, \quad (29)$$

from which it results, at one end $\sigma=0$

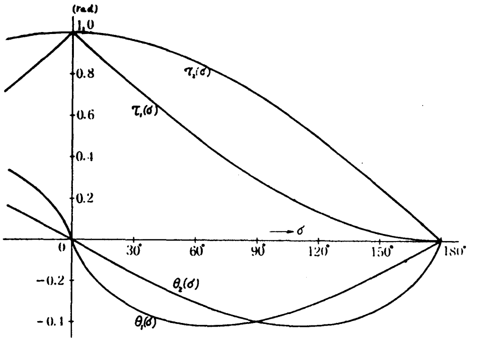

Fig. 5.

$$\tau_r'(0+)=\frac{1}{2\sqrt{3}}\theta'(0)\,, \qquad (29')$$

and at the other end $\sigma=\pi$

$$\tau_r'(\pi-)=-2p\theta'(\pi)\,, \qquad (29'')$$

the relation $\tau_r(0+)=3^{-1}\log(3p)$ being used in the derivation of (29'). Values of $\theta'(0)$ and $\theta'(\pi)$ are given in Table 3, and inserting these values we obtain

$$\tau_r'(0+)=-0.06875\,, \quad \tau_r'(\pi-)=0.2453\,. \qquad (30)$$

After these preparations we construct the function $\varDelta\Omega(\zeta)$ by the definition:

$$\Omega_r(\zeta)=0.1375\,\Omega_1(\zeta)-0.4906\,\Omega_2(\zeta)+\varDelta\Omega(\zeta)\,, \quad (31)$$

where $\tau_r(\sigma)$ is the function defined above precisely. $\varDelta\Omega(\zeta)$ is holomorphic in the unit circle, continuous on it and has the same symmetry character as $\Omega_r(\zeta)$. Above all its imaginary part $\varDelta\tau(\sigma)$ on the unit circle has horizontal tangents along the circular arc at $\sigma=0$ and π. Therefore $\varDelta\tau(\sigma)$ can be expanded pretty accurately into a Fourier cosine series with a few terms, this being our required property. We take in first 12 components as before, thus

$$\varDelta\Omega(\zeta)=i\sum_{n=0}^{12}a_n\zeta^n\,, \qquad (32)$$

and consequently

$$\varDelta\tau(\sigma)=\sum_{n=0}^{12}a_n\cos n\sigma\,, \quad \varDelta\theta(\sigma)=-\sum_{n=1}^{12}a_n\sin n\sigma\,. \qquad (32')$$

Table 5.

a_0	+0.1028
a_1	+0.0845
a_2	−0.0178
a_3	+0.0018
a_4	−0.0015
a_5	−0.0002
a_6	+0.0002
a_7	−0.0004
a_8	+0.0005
a_9	−0.0003
a_{10}	+0.0005
a_{11}	−0.0003
a_{12}	+0.0003
p	0.1921

Numerical values of $\Delta\tau(\sigma)$ at 13 points $n\pi/12$ ($n=0, 1, 2, \cdots, 12$) are the asterisked ones in Table 4 column 6, the Fourier coefficients deduced from those are given in Table 5, and interpolations by this series are the numbers between the asterisked ones in Table 4 column 6. We can see pretty good elimination of the undulation of interpolated values.

Now our final solution of the problem is

$$\Omega(\zeta) = \Omega_0(\zeta) + 0.1375\,\Omega_1(\zeta) - 0.4906\,\Omega_2(\zeta) + \Delta\Omega(\zeta)\,, \tag{33}$$

and specially on the circumference of the unit circle

Table 6.

(deg.) σ	θ	q	$-x/H$	$-y/H$	$\dfrac{\left(q^2+2\pi p\frac{y_1}{H}\right)\times 1{,}000}{q^2}$
0	0.5236	0.0000	0.0000	0.0000	—
3.75	0.5077	0.2652			2.8
7.5	0.4959	0.3328	0.1631	0.0915	1.8
15	0.4766	0.4164			1.2
22.5	0.4597	0.4742	0.3462	0.1864	−0.9
30	0.4433	0.5199			0.0
37.5	0.4271	0.5582	0.4968	0.2580	0.3
45	0.4115	0.5913			0.9
52.5	0.3961	0.6210	0.6370	0.3192	0.8
60	0.3809	0.6478			1.0
67.5	0.3656	0.6725	0.7751	0.3745	0.2
75	0.3503	0.6957			0.8
82.5	0.3348	0.7179	0.9172	0.4264	1.2
90	0.3190	0.7387			1.3
97.5	0.3030	0.7587	1.0690	0.4764	0.7
105	0.2867	0.7783			1.2
112.5	0.2693	0.7977	1.2381	0.5262	1.6
120	0.2510	0.8165			1.8
127.5	0.2324	0.8352	1.4368	0.5770	1.4
135	0.2128	0.8543			2.2
142.5	0.1907	0.8739	1.6880	0.6309	2.6
150	0.1665	0.8936			2.5
157.5	0.1410	0.9141	2.0505	0.6911	1.6
165	0.11180	0.9386			5.0
172.5	0.07126	0.9680	2.7971	0.7682	10.3
176.25	0.04294	0.9837			
180	0.00000	1.0000	∞	0.8278	0.7

$$
\left.\begin{aligned}
\theta(\sigma) &= \frac{\pi-\sigma}{6} + 0.1375 \times \frac{2}{\pi} \sin\frac{\sigma}{2} \log\tan\frac{\sigma}{4} \\
&\qquad - 0.4906 \times \frac{2}{\pi} \cos\frac{\sigma}{2} \log\tan\frac{\pi-\sigma}{4} - \sum_{n=1}^{12} a_n \sin n\sigma\,, \\
\tau(\sigma) &= \frac{1}{3} \log\sin\frac{\sigma}{2} + 0.1375\left(1 - \sin\frac{\sigma}{2}\right) \\
&\qquad - 0.4906 \cos\frac{\sigma}{2} + \sum_{n=0}^{12} a_n \cos n\sigma\,; \qquad (\sigma > 0)
\end{aligned}\right\} \tag{33'}
$$

the values of $\theta(\sigma)$ and $q(\sigma) = \exp\tau(\sigma)$ being computed at equal interval points and tabulated in Table 6 columns 2 and 3. With these values we determine numerically the wave profile according to (25), the neighborhoods of $\sigma = 0$ and $\sigma = \pi$ being treated specially by the approximate expansions:

$$
\sigma = 0: \quad \left\{\begin{aligned}
\frac{\cos\theta}{q} \sec\frac{\sigma}{2} &= 1.3101\sigma^{-1/3}(1 + a\sigma + b\sigma^2 + c\sigma^3)\,, \\
\frac{\sin\theta}{q} \sec\frac{\sigma}{2} &= 0.7564\sigma^{-1/3}(1 + a'\sigma + b'\sigma^2 + c'\sigma^3)\,,
\end{aligned}\right\} \tag{34}
$$

and

$$
\sigma = \pi: \quad \frac{\sin\theta}{q} \sec\frac{\sigma}{2} = -0.3123 \log(\pi-\sigma) + a'' + b''(\pi-\sigma)\,. \tag{34'}
$$

The result is as shown in Table 6 columns 4 and 5, and differs so little from previous result (Table 3 columns 4 and 5), that graphical representation is perfectly identical with Fig. 3. Wave height $(-y_1/H)_{\sigma=\pi} = 0.8278$.

Eigenvalue p is given by numerical integration of (18). Here also $\sigma = 0$ and π require approximate expansions:

$$
\sigma = 0: \quad \sin\theta \sec\frac{\sigma}{2} = 0.5000 + 0.03791\sigma\log\sigma + a\sigma + b\sigma^3\,, \tag{35}
$$

$$
\sigma = \pi: \quad \sin\theta \sec\frac{\sigma}{2} = -0.3123 \log(\pi-\sigma) + a' + b'(\pi-\sigma)\,; \tag{35'}
$$

the result is $p = 0.1921$. Making use of this value

$$
U = 1.287\sqrt{gH}\,, \qquad \frac{A}{H} = 0.8284\,, \tag{36}
$$

the second of these being very near to the value of $(-y_1/H)_{\sigma=\pi}$ above mentioned, which proves the exactitude of our numerical method. The velocity on the bottom just under the crest is $q = 0.6767$ and the pressure head there is $H_0 = 1.4491H$. All these values are almost identical with our previous results. In conclusion errors of surface condition are calculated and inserted in Table 6 column 6, and we see the errors reduced notably. Along almost all part of wave errors are a few per mille and mount only to 1 percent atmost at far remote bases, and, we think, our solution is sufficiently accurate.

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