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HIGHEST WAVES OF PERMANENT TYPE ON THE SURFACE OF DEEP WATER

By Hikoji YAMADA

Résumé.— Under the Lévi-Civita's formulation the highest waves on the surface of deep water was calculated. For $\Omega = \theta + i\tau$ we assume the form $\frac{i}{3}\log(1-\zeta) + \sum a_n \zeta^n$; a's are 12 in number. ζ is the unit circle corresponding to water region, and $\theta(\sigma)$, $\tau(\sigma)$ on the circumference $\zeta = e^{i\sigma}$ $(-\pi < \sigma \leq \pi)$ solve the problem. To obtain $\theta(\sigma)$, $\tau(\sigma)$ we use an iterative procedure, one cycle of it consisting of two stages: to get $\tau(\sigma)$ from $\theta(\sigma)$ through the surface condition and to get $\theta(\sigma)$, the harmonic conjugate of $\tau(\sigma)$. Calculations were all numerical and almost converged after 8 cycles. Obtained results fulfil the susface condition with %-error. One refinement of approximation is proposed and error is reduced to one third.

§1. Introduction. The problem of permanent wave train which travels on the surface of deep water was placed on a sound basis by T. Lévi-Civita,[1] and the case of relatively small amplitude was solved exactly by himself, but the case of higher amplitude or the case of extreme height brings some difficulties in the numerical determination of exact solution. With regard to the waves of extreme height the fact that the crests are angular points of angle $2\pi/3$ *radian* is known since the time of Sir G. G. Stokes, and J. H. Michell[2] and T. H. Havelock[3] tried to find this extreme wave train, making use of this fact directly. Their analysis is complicated and in spite of the correctness of the mathematical procedure their numerical results are shadowed by some doubts.[4] Recently T. V. Davies has developed an ingenious treatment of surface waves and mentioned the extreme waves as a special case of his general theory,[5] but to push his results for small amplitude waves to this extreme case seems to introduce, we think, some errors in the estimation of numerical values. Up to now, therefore, the problem of highest waves on deep water surface has not only exact solution, but any approximate one with sufficient accuracy, and this is the reason why we have attempted here an approximate treatment of the problem with sufficient simplicity and clearness.

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1) T. Lévi-Civita, Math. Annalen, vol. 93, p. 264 (1925).
2) J. H. Michell, Phil. Mag., ser. 5, vol. 36, p. 430 (1893).
3) T. H. Havelock, Proc. Roy. Soc. London (A), vol. 95, p. 38 (1919).
4) H. Jeffreys, Quat. Jour. Mech. Appl. Math., vol. 4, p. 385 (1951).
5) T. V. Davies, Proc. Roy. Soc. London (A), vol. 208, p. 475 (1951).

We use here the formulation of problem due to Lévi-Civita. Assumed form of solution is identical with that of Michell-Havelock, for by the introduction of angular crests due to Stokes any natural reasoning leads the solution into that form. Our manipulating plane is, however, a unit circle which corresponds to the water region, in contrast to their potential plane. In this formulation not only calculations are simplified a good deal, but an iterative procedure for the determination of solution could be used with success. Obtained results are reasonable and the degree of approximation is indicated by the defect in the fulfilment of surface condition.

At the numerical computations of solution we had to cut a Fourier series short to a finite terms, and when we desire a more accurate solution increased number of terms are to be included. But this procedure makes calculations laborious, and so we have tried another method which has brought satisfactory precisions.

§2. Lévi-Civita's formulation of wave problem. As we follow here the Lévi-Civita's formulation, it will be convenient to describe in this section its outlines briefly, as far as we require in the followings. The problem is two-dimensional. A permanent wave train (of infinite length) which travels on the calm water of infinite depth, being its wave-length L and proceeding to left with velocity U, is observed from the rectangular axes O-xy which follows the waves with the same velocity. The wave form on the surface stops, and instead water flows from left to right steadily, the velocity at the infinite depth where the influence of waves disappears being horizontal and magnitude U. Origin of the coordinates is at a wave crest, x-axis is horizontal and directed to right, y-axis vertical and upwards.

Our waves are irrotational, and to any point (x, y) in water there are velocity potential $\varphi(x, y)$ and stream function $\psi(x, y)$; the complex potential $W = \varphi + i\psi$ is a holomorphic function of the position $z = x + iy$ everywhere in water. We take the origin of W-plane at a point which corresponds to a wave crest and then W-plane and z-plane correspond to each other as is sketched qualitatively in Fig. 1. With this potential W

$$\frac{1}{U}\frac{dW}{dz} = q e^{-i\theta}, \tag{1}$$

where q is the magnitude, measured with U as unit, and θ is the angle, measured from Ox direction upwards, of the flow velocity. q never vanishes in water and then, when we introduce a function $\Omega(z)$ by the definition

$$\Omega(z) = i \log\left(\frac{1}{U}\frac{dW}{dz}\right) = \theta + i\tau, \tag{2}$$

$$\tau = \log q, \tag{3}$$

Ω is everywhere holomorphic in water, being its real part the flow direction and the imaginary part logarithm of the magnitude of velocity. To know this function is the essential part of the solution of our problem and

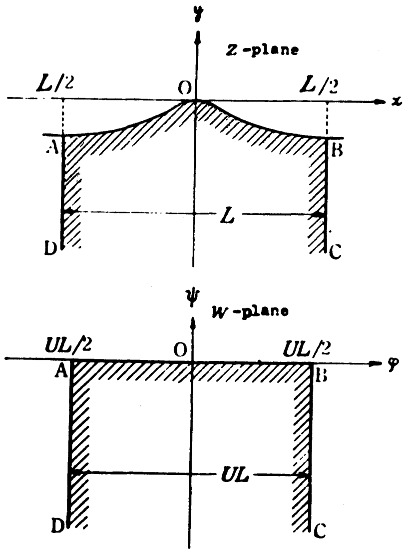

Fig. 1.

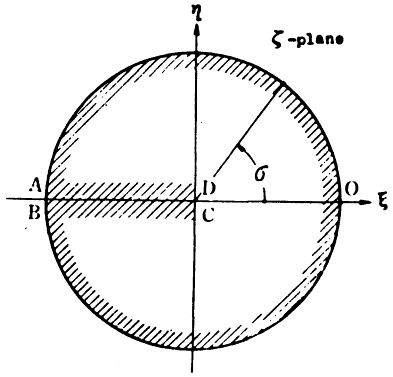

Fig. 2.

we can find the potential from this function by the relation

$$\frac{dW}{dz} = Ue^{-i\Omega(z)}. \tag{4}$$

Here we transform a part of W-plane, which corresponds to a wave length $(-UL/2 \leq \varphi < UL/2,\ \psi \leq 0)$ and hatched in Fig. 1, to a unit circle $|\zeta| \leq 1$ of $\zeta = \xi + i\eta$ plane by the equation

$$\zeta = e^{-2\pi i(W/UL)}, \quad W = i\frac{UL}{2\pi}\log\zeta\,; \tag{5}$$

the correspondence is one to one as is shown in Fig. 2. Then through W a wave length domain of z, which is also hatched in Fig. 1, corresponds also to unit circle $|\zeta| \leq 1$ one to one, and any function of z becomes a function of ζ, specially $\Omega(z)$ to $\Omega(\zeta)$. $\Omega(z)$ is a periodic function of a wave length peried: the values of $\Omega(z)$ on the line AD is wholly identical with those on the line BC, and this results in a wholly holomorphic $\Omega(\zeta)$ in the domain $|\zeta| < 1$. Moreover, as we are dealing with symmetrical waves, we find $\theta = 0$ on the real axis $\zeta = \xi$ $(-1 \leq \xi < 1)$, and then that $\Omega(\zeta)$ is pure imaginary on this axis.

If we can find $\Omega(\zeta)$ by one way or another, (4) can be written in the form

$$\frac{dW}{d\zeta}\cdot\frac{d\zeta}{dz} = Ue^{-i\Omega(\zeta)}, \quad i.e. \quad \frac{dz}{d\zeta} = \frac{1}{U}\frac{dW}{d\zeta}e^{i\Omega(\zeta)},$$

and this relation, by virtue of (5), results in

$$dz = i\frac{L}{2\pi\zeta}e^{i\Omega(\zeta)}d\zeta\,, \tag{6}$$

which gives by integration $z=z(\zeta)$ or $\zeta=\zeta(z)$. $\Omega(\zeta)$ can then be rewritten into $\Omega(z)$, which is our aimed function. Our problem has thus been transformed into the determination of $\Omega(\zeta)$.

When y tends to $-\infty$ *i.e.* ψ to $-\infty$, θ and τ and accordingly Ω vanish, and then $\Omega(\zeta)$ has to vanish at $\zeta=0$. $\Omega(\zeta)$ is thus a holomorphic function which vanishes at the origin, and fulfils on the circumference of unit circle $\zeta=e^{i\sigma}(-\pi<\sigma\leq\pi)$ the surface condition. By the surface condition $\Omega(\zeta)$ is defined. Of cousse there remains one parameter free, if limited, which may be identified *e.g.* with the wave steepness.

The surface condition means that along the stream line which constitutes the water surface the statical pressure must be that of atmosphere, or in other word constant, and by Bernoulli's theorem we have

$$\frac{1}{2}U^2q^2+gy=\text{const.} \qquad i.e. \qquad \frac{2gy}{U^2}+q^2=\text{const.} \tag{7}$$

As the introduction of this form of surface condition directly into the theory is difficult we use instead of that that differentiated along surface stream line, *i.e.*

$$\frac{2g}{U^2}\frac{dy}{ds}=-2q\frac{dq}{ds},$$

where arc element ds and depth element dy are combined by the relation $dy=ds\sin\theta$, and we have

$$-q\frac{dq}{ds}=\frac{g}{U^2}\sin\theta, \tag{7'}$$

which is our required form. q and θ are functions of arc length s.

Now we put $\zeta=e^{i\sigma}$, $d\zeta=ie^{i\sigma}d\sigma$ $(-\pi<\sigma\leq\pi)$ in the expression (6) of dz. As dz is evidently equal to $ds\,e^{i\theta}$ we have

$$-ds=\frac{L}{2\pi q}d\sigma, \tag{8}$$

and when we look θ and q as defined on $|\zeta|=1$, *i.e.* as functions of σ, the surface condition (7′) assigned to these functions becomes finally by virtue of (8):

$$q^2\frac{dq}{d\sigma}=p\sin\theta \qquad (q=e^{\tau(\sigma)}), \tag{9}$$

where p is a parameter defined by

$$p=\frac{gL}{2\pi U^2}. \tag{10}$$

When we find a $\Omega(\zeta)$ whose real and imaginary parts $\theta(\sigma)$, $\tau(\sigma)$ on the unit circle satisfy the condition (9) our problem is settled, and this formulation is nothing but the one due to Lévi-Civita. To each p, the eigenvalue, which is less than 1 but sufficiently close to 1 belongs a certain unique solution; to smaller p belongs steeper waves; there is the smallest p which corresponds to highest waves with angular crests; all these facts are well-known, but the value of the smallest p and the corresponding wave form have not been fully determined.[6]

§3. Calculation scheme for the highest waves. Here we take up the problem of numerical determination of the highest waves. As long as the angular crests do not appear $\Omega(z)$ is analytic on the free surface and then $\Omega(\zeta)$ holomorphic in the domain $|\zeta|\leq 1$. Accordingly the power series which represents $\Omega(\zeta)$:

$$\Omega(\zeta)=\sum_{n=1}^{\infty} b_n \zeta^n \tag{11}$$

can be cut short to an appropriate polynomial with good approximation up to as far as $|\zeta|=1$. But when angular points make their appearance in z-plane $\zeta=1$ becomes a singular point of $\Omega(\zeta)$, and any polynomial does not approximate the real behavior of function in the neighborhood of singular point. We have then to eliminate the singularity before we can approximate the behavior with a polynomial of a few number of terms, which is the necessary device in numerical treatment.

The nature of singularity at the wave crest is well-known and in this neighborhood

$$W=\text{const.}\,z^{3/2}, \tag{12}$$

and by means of (5) this is equivalent to

$$\log(1-\overline{1-\zeta})=\text{const.}\,z^{3/2}, \quad i.e. \quad 1-\zeta=\text{const.}\,z^{3/2}, \tag{12'}$$

which, introduced into (2), results in

$$\Omega=i\log(\text{const.}\,z^{1/2})=\frac{i}{3}\log(1-\zeta)+\text{const.}$$

This approximate expression in the neighborhood of $\zeta=1$ has no other singularity in the entire domain $|\zeta|\leq 1$ and then real Ω can be split into two parts:

$$\Omega(\zeta)=\Omega_0(\zeta)+\Omega_r(\zeta), \tag{13}$$

where[7]

$$\Omega_0(\zeta)=\frac{i}{3}\log(1-\zeta). \tag{14}$$

As Ω_r is holomorphic in $|\zeta|\leq 1$ except one point $\zeta=1$, where Ω_r is continuous, our requiremant is realized and we can cut the expansion

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6) *c.f.* T. von Kármán's J. W. Gibbs lecture, Proc. Amer. Math. Soc. (August, 1940), p. 615-, esp. p. 649.

7) $\Omega_r(\zeta)$ itself has the main properties which $\Omega(\zeta)$ must possess, as easily examined. Then it will be near to generalise this expression to $\Omega_0=i/3\log(1-\lambda\zeta)$ $(0<\lambda\leq 1)$ and regard as an approximate solution for general waves. This is just identical to Davies' first approximation. *c.f.* footnote 5).

$$\Omega_r(\zeta) = \sum_{n=1}^{\infty} b_n \zeta^n \tag{15}$$

short to finite number of terms with sufficient accuracy.

(13), (14) and (15) are identical with the form of solution which Michell-Havelock have used. Their solution was manipulated in the W-plane and assumed solution had in our axes and notation the form[8)]

$$\frac{dW}{dz} = \left(i 2 U^{-3} \sin \frac{\pi W}{UL}\right)^{-1/3} \cdot e^{i\pi W/3UL} \cdot \left(1 + c_1 e^{-i2\pi W/UL} + c_2 e^{-i4\pi W/UL} + \cdots\right),$$

which by virtue of (2) and (3) transforms into above expression. Davies' expression[5)] on the other hand is for a permanent waves of arbitrary steepness

$$e^{3(\tau - i\theta)} = 1 - A e^{-i 2\pi W/UL} + B e^{-i 4\pi W/UL} + \cdots,$$

and this transforms into

$$\Omega(\zeta) = \frac{i}{3} \log(1 - A\zeta + B\zeta^2 + \cdots);$$

we can bring this easily into our form if desired, for for the highest waves the argument of logarithm has to vanish at $\zeta = 1$ and we have

$$\Omega(\zeta) = \frac{i}{3} \log\{(1-\zeta)(1 + \overline{1-A}\zeta + \cdots)\}.$$

In all these methods of solution the accuracy seems to depend on the number of expansion coefficients determined numerically.

As is mentioned above $\Omega(\zeta)$ has pure imaginary values on the real axis $\zeta = \xi$, and as $\Omega_0(\zeta)$ do so $\Omega_r(\zeta)$ has also to do so. Then the expansion coefficients in (15) must all be pure imaginary and we write

$$\Omega_r(\zeta) = i \sum_{n=1}^{\infty} a_n \zeta^n \tag{15'}$$

with real coefficients a_n. Therefore on the circumference of unit circle $\zeta = e^{i\sigma}$ its real and imaginary parts are

$$\theta_r(\sigma) = -\sum_{n=1}^{\infty} a_n \sin n\sigma, \quad \tau_r(\sigma) = \sum_{n=1}^{\infty} a_n \cos n\sigma. \tag{16}$$

On the same circumference $(-\pi < \sigma \leq \pi)$ $\Omega_0(\zeta)$ has the two parts:

$$\theta_0(\sigma) = \begin{cases} \dfrac{\pi - \sigma}{6} \ (\pi \geq \sigma > 0), \\ \dfrac{-\pi - \sigma}{6} \ (-\pi < \sigma < 0); \end{cases} \qquad \tau_0(\sigma) = \frac{1}{3} \log\left(2\left|\sin\frac{\sigma}{2}\right|\right); \tag{16'}$$

and these functions contitute with (16) the total Ω:

$$\Omega(e^{i\sigma}) = \theta(\sigma) + i\tau(\sigma); \quad \begin{aligned} \theta(\sigma) &= \theta_0(\sigma) + \theta_r(\sigma), \\ \tau(\sigma) &= \tau_0(\sigma) + \tau_r(\sigma). \end{aligned} \tag{16''}$$

8) *c.f.* the equation (5) of Havelock's paper cited in footnote 3).

The coefficients a_n have to be so determined as to satisfy the surface condition. If we substitute (16), (16′) and (16″) directly into the surface condition (9) the resulting equation is so much complicated that we can at most determine first several coefficients only. Therefore we left such an analytical method and adopt a numerical treatment: iterative determination of numerical constants.[9] We consider a certain number of the coefficients $a_1, a_2, \cdots, a_N$ and abandon all the others, *i.e.* we replace the infinite power series by a polynomial of certain degree, and our problem is to determine the numerical values of a_n $(n=1, \cdots, N)$ to satisfy the surface condition as good as possible. The intial values are arbitrarily but adequately set. One cycle of the iterative procedure consists of two stages: first the determination of θ_r (or τ_r) from the other τ_r (or θ_r) through the surface condition (*abr.* S), and second the determination of τ_r (or θ_r), the harmonic conjugate function of θ_r (or τ_r) (*abr.* H); the iterative procedure is then as shown in the following scheme:

$$\text{———}\overset{\text{(S)}}{}\tau_r\text{———}\overset{\text{(H)}}{}\underbrace{\theta_r\text{———}\overset{\text{(S)}}{}\tau_r\text{———}\overset{\text{(H)}}{}\theta_r}_{\text{one cycle}}\text{———}\overset{\text{(S)}}{}\tau_r\text{———}\overset{\text{(H)}}{}$$

The sense of this procedure is, however, at first unknown, and is to be determined *a posteriori*. If iteration in one sense diverges, that in the reversed sense is hoped for convergence. After a few trials we know that the procedure converges for the sense from left to right.

Resorting to the expressions (16) harmonic coniugate functions can be constructed at once, and the main part of calculation is the derivation of $\tau_r(\sigma)$ from $\theta_r(\sigma)$ throngh the condition (9), which we use in the form:

$$q^3(\sigma) = 3p \int_0^\sigma \sin\theta(\sigma')\, d\sigma', \qquad \tau(\sigma) = \frac{1}{3} \log q^3(\sigma)\,; \tag{17}$$

here of course $\tau = \tau_0 + \tau_r$, $\theta = \theta_0 + \theta_r$.

One remark is necessary: that which is determined by (17) is not $q(\sigma)$, but $q(\sigma)/q(\pi)$. To determine p or $q(\pi)$ one more equation is necessary, which is one of the expressions of the characteristic natures of $q(\sigma)$. Here we take up the relation[10]

$$\int_0^L q^2 dx = L, \quad i.e. \quad \int_0^{-L/2} q^2 dx = -\frac{L}{2} \tag{18}$$

as the most convenient one, q being of course the surface value. Of the line-elements of the water surface:

$$\left.\begin{aligned} dx &= ds \cos\theta(\sigma) = -\frac{L}{2\pi q(\sigma)} \cos\theta(\sigma)\, d\sigma \\ dy &= ds \sin\theta(\sigma) = -\frac{L}{2\pi q(\sigma)} \sin\theta(\sigma)\, d\sigma \end{aligned}\right\}, \tag{19}$$

[9] *c.f.* G. Birkhoff *et al.*, Proc. Symp. App. Math., vol. 4, p. 117 (1953).
[10] *c.f.* H. Lamb: Hydrodynamics, 6th ed. (1932), p. 420.

which follow from (8), we use the former into (18) and get to the required equation:

$$\int_0^{\pi} q(\sigma)\cos\theta(\sigma)\,d\sigma = \pi\,. \tag{18'}$$

Combining this equation with (17) we obtain a formula:

$$(3p)^{1/3} = \pi\Big/\int_0^{\pi}\cos\theta(\sigma)\,d\sigma\left(\int_0^{\sigma}\sin\theta(\sigma')\,d\sigma'\right)^{1/3}, \tag{20}$$

which fixes the eigenvalue p.

When p is known the values of $q(\sigma)$ and then $\tau(\sigma)$ follow from (17), and particularly the speed $q(\pi)$ at the centre of trough is

$$q(\pi) = (3p)^{1/3}\left(\int_0^{\pi}\sin\theta(\sigma')\,d\sigma'\right)^{1/3}. \tag{21}$$

The values of $\tau_r(\sigma)$ are to be computed from $\tau(\sigma)$ thus obtained by subtraction of $\tau_0(\sigma)$, and its limiting value $\tau_r(0+)$ is

$$\tau_r(0+) = \lim_{\sigma\to 0+}\left\{\frac{1}{3}\log q^3(\sigma) - \frac{1}{3}\log\left(2\sin\frac{\sigma}{2}\right)\right\},$$

which by virtue of $\theta(0+) = \pi/6$ reduces to

$$\tau_r(0+) = \frac{1}{3}\log\left(\frac{3}{2}p\right). \tag{22}$$

The Fourier decomposition of $\tau_r(\sigma)$ is as usual: a number of coefficients are determined making use of the values of $\tau_r(\sigma)$ at the same number of points with equal intervals.[11]

§ 4. **Numerical computations and results.** We adopted the initial values $a_1 = 1/6$ and others zero which realize a flat trough, but such a supposition proved no benifit and it would have been rather simple to start

[11] The iterative computation above stated is also applicable to waves with general amplitude: As the calculation of waves for given amplitude is originally difficult, we assume here the ratio of water velocity at crest to that at trough *i.e.* $q(0)/q(\pi) = \rho$ as given. Then the analogue of (17) is now

$$q^3(\sigma) = q^3(0) + 3p\int_0^{\sigma}\sin\theta(\sigma')\,d\sigma',$$

and we obtain from this

$$p = \frac{1}{3}q^3(\pi)(1-\rho^3)\Big/\int_0^{\pi}\sin\theta(\sigma')\,d\sigma';$$

$$\{q(\sigma)/q(\pi)\}^3 = \rho^3 + (1-\rho^3)\int_0^{\sigma}\sin\theta(\sigma')\,d\sigma'\Big/\int_0^{\pi}\sin\theta(\sigma')\,d\sigma'.$$

Using the latter into (18') $q(\pi)$ can be fixed, and with $q(\pi)$ determines the latter the flow velocity and the former the eigenvalue p. The constant λ in our present analogue of (14) *i.e.* $\Omega_0 = (i/3)\log(1-\lambda\zeta)$ can be fixed *e.g.* by the relation $\tau_0(0) = \log q(0)$. With these remarks the iteration procedure above stated determines $\theta_r(\sigma)$, $\tau_r(\sigma)$, or their expansion coefficients a_n. Ω_0 can be fixed once for all, and is not necessary to renew it at every cycle of iteration.

with all zeros. Of the Fourier series representing τ_r first 12 harmonic components have been considered. As this is a cosine series 13 dividing points $\sigma=\pi n/12$ ($n=0,1,2,\cdots,12$) have been defined and from the values of $\tau_r(\sigma)$ at these points c_n's are determined. Almost all integrals are evaluated by the Simpson rule, and, except a somewhat rough estimation in first few cycles, precisions of numerical integrations were all the same.

One step of numerical integration is $\pi/12$, *i.e.* the values of $\sin\theta=\sin(\theta_0+\theta_r)$ at every $\pi/24$-point are used, but in the neighborhood of $\sigma=0$ the integral for $\sigma=\pi/24$ is also computed. The reason why is that the integrand of the denominator of (20) has a vertical tangent at $\sigma=0$ such that

$$\cos\theta(\sigma)\left(\int_0^\sigma \sin\theta(\sigma')\,d\sigma'\right)^{1/3}=\cos 30^\circ\cdot 2^{-1/3}\sigma^{1/3}(1+a\sigma+b\sigma^2) \tag{23}$$

is its approximate expression. We fix the constants a and b through the

Table 1.

a_1	0.0412
a_2	0.0120
a_3	0.0061
a_4	0.0039
a_5	0.0029
a_6	0.0023
a_7	0.0019
a_8	0.0017
a_9	0.0016
a_{10}	0.0014
a_{11}	0.0014
a_{12}	0.0007

Table 2.

σ (deg.)	$\theta_0(\sigma)$	$\tau_0(\sigma)$	$\theta_r(\sigma)$	$\tau_r(\sigma)$
0	0.5236	$-\infty$	0.0000	0.0771
3.75	0.5127	−0.9089	−0.0131	0.0748
7.5	0.5018	−0.6780	−0.0241	0.0686
15	0.4800	−0.4477	−0.0350	0.0515
22.5	0.4582	−0.3137	−0.0353	0.0398
30	0.4363	−0.2195	−0.0360	0.0351
37.5	0.4145	−0.1473	−0.0406	0.0299
45	0.3927	−0.0891	−0.0433	0.0213
52.5	0.3709	−0.0409	−0.0422	0.0139
60	0.3491	0.0000	−0.0409	0.0095
67.5	0.3273	0.0351	−0.0414	0.0052
75	0.3054	0.0656	−0.0412	−0.0008
82.5	0.2836	0.0922	−0.0387	−0.0060
90	0.2618	0.1155	−0.0362	−0.0094
97.5	0.2400	0.1360	−0.0348	−0.0126
105	0.2182	0.1539	−0.0329	−0.0168
112.5	0.1964	0.1695	−0.0296	−0.0203
120	0.1745	0.1831	−0.0264	−0.0226
127.5	0.1527	0.1948	−0.0239	−0.0247
135	0.1309	0.2047	−0.0211	−0.0272
142.5	0.1091	0.2129	−0.0174	−0.0293
150	0.0873	0.2195	−0.0138	−0.0305
157.5	0.0655	0.2246	−0.0107	−0.0313
165	0.0436	0.2282	−0.0073	−0.0323
172.5	0.0218	0.2303	−0.0037	−0.0330
180	0.0000	0.2311	0.0000	−0.0332

values of the left-hand side at $\sigma=\pi/24$ and $\sigma=\pi/12$, and its integral over $(0, \pi/12)$ is easily obtainable.

After 7 or 8 cycles of the iteration procedure almost stationary values were reached, and we took for final $\tau_r(\sigma)$ the values after 8 cycles. From these values final a_n's were calculated and given in Table 1. $\theta_r(\sigma)$, $\tau_r(\sigma)$ computed with these values of a_n's by the formulas (16) are in Table 2 columns 4, 5, and $\theta(\sigma)=\theta_0(\sigma)+\theta_r(\sigma)$, $q(\sigma)=\exp\{\tau_0(\sigma)+\tau_r(\sigma)\}$ in Table 3 columns 2, 3. When (20) is calculated with $\theta(\sigma)$ thus given, according to the manner above explained and used in each cycle of iteration, the eigenvalue p is determined:

$$p=0.8402\,, \tag{24}$$

which is also a final result. From this value, by virtue of the definition (10), the wave velocity is found:

Table 3.

σ (deg.)	θ	q	$-x/L$	$-y/L$	$\dfrac{q^2+4\pi p\frac{y}{L}}{q^2}\times 100$
0	0.5236	0.0000	0.0000	0.0000	—
3.75	0.4996	0.4343			
7.5	0.4777	0.5437	0.0500	0.0276	1.3
15	0.4450	0.6729		0.0428	0.1
22.5	0.4229	0.7604	0.1069	0.0550	−0.5
30	0.4003	0.8316		0.0655	0.0
37.5	0.3740	0.8893	0.1533	0.0747	0.3
45	0.3494	0.9344		0.0827	0.0
52.5	0.3287	0.9733	0.1952	0.0900	−0.3
60	0.3082	1.0096		0.0966	0.0
67.5	0.2858	1.0412	0.2346	0 1025	0.1
75	0.2643	1.0670		0.1079	0.0
82.5	0.2449	1.0900	0.2723	0.1127	−0.2
90	0.2256	1.1120		0.1172	0.0
97.5	0.2052	1.1313	0.3089	0.1211	0.1
105	0.1853	1.1469		0.1247	−0.1
112.5	0.1667	1.1609	0.3446	0.1278	−0.1
120	0.1481	1.1741		0.1306	0.0
127.5	0.1288	1.1854	0.3796	0.1331	0.0
135	0.1098	1.1942		0.1351	−0.1
142.5	0.0917	1.2015	0.4143	0.1369	−0.1
150	0.0735	1.2081		0.1383	−0.1
157.5	0.0548	1.2132	0.4487	0.1394	0.0
165	0.0363	1.2164		0.1402	0.0
172.5	0.0181	1.2182	0.4830	0.1407	−0.1
180	0.0000	1.2188	0.5001	0.1408	−0.1

$$U=1.0909\times\sqrt{\frac{gL}{2\pi}}. \tag{25}$$

This is about 9% greater than the case of infinitesimal amplitude.

Water surface condition (7) becomes, by virtue of (10) and the condition $C=0$ now existing, to

$$q^2+4\pi p\frac{y}{L}=0, \tag{26}$$

and this equation applied to a particular point $\sigma=\pi$ (centre of trough) gives following expression of the wave steepness:

$$\frac{H}{L}=\frac{q^2(\pi)}{4\pi p}, \tag{26'}$$

H being the vertical distance between crest and trough. Then making use of (24) and the value of $q(\pi)$ in Table 3 we have the result:

$$\frac{H}{L}=0.1407; \tag{26''}$$

this is almost equal to the corresponding value 0.1418 of Michell-Havelock.[3] The value 0.1443 given by Davies[5] is doubtful, as stated above.

Wave form can be calculated without difficulty. Integration of (19):

$$-\frac{x}{L}=\frac{1}{2\pi}\int_0^\sigma\frac{\cos\theta(\sigma')}{q(\sigma')}d\sigma', \qquad -\frac{y}{L}=\frac{1}{2\pi}\int_0^\sigma\frac{\sin\theta(\sigma')}{q(\sigma)}d\sigma' \tag{19'}$$

are accomplished by Simpson rule, except the neighborhood of $\sigma=0$, where integrands become infinite as $\sigma^{-1/3}$. In this neighborhood we have

$$\left.\begin{aligned}\frac{\cos\theta}{q}&=0.8017\sigma^{-1/3}(1+a\sigma+b\sigma^2)\\ \frac{\sin\theta}{q}&=0.4628\sigma^{-1/3}(1+a'\sigma+b'\sigma^2)\end{aligned}\right\}, \tag{27}$$

and the constants a, b, a', b' in the right-hand sides are to be fixed by the function values at $\sigma=\pi/48$ and $\pi/24$. Integrations of (27) between $(0, \pi/24)$ are easy. Obtained results are given in Table 3 columns 4, 5, and graphed in Fig. 3. The values of $-x/L$ and $-y/L$ at the point $\sigma=\pi$ (trough), *i.e.* 0.5001 and 0.1408, are very reasonable (the latter compared with (26'')), and assure the accuracies of our numerical calculations.

At the end we examine the degree of approximation. As θ and τ obtained are accurately conjugate to each other and as the surface condition is almost fulfilled within the accuracy of numerical calculations, the only source of error is the short cut harmonic analysis of $\tau_r(\sigma)$. Leaving some discussions of it in next section we here see only the degree of fulfilment of the original surface condition (26), which is the figures given in Table 3 column 6. Errors are less than 1% at all points but the neighborhood of $\sigma=0$, where both terms of the condition vanish.

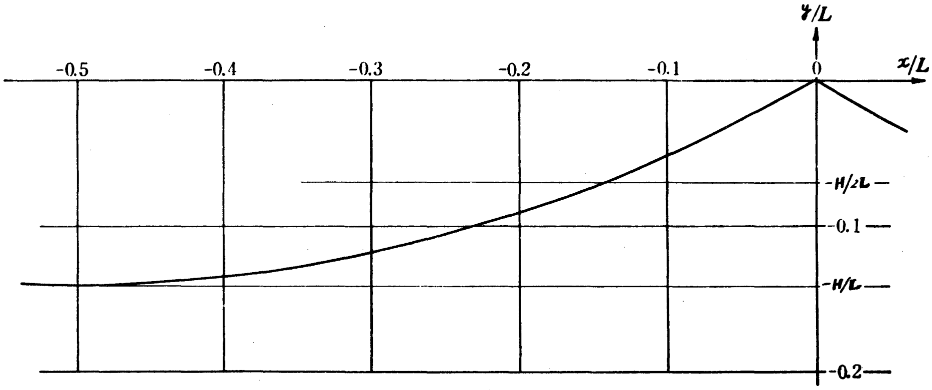

Fig. 3.

§ 5. **Improvement of approximation.** To refine the approximation further we have to increase the number of components of the Fourier analysis of $\tau_r(\sigma)$, or to replace the Fourier analysis by a direct numerical determination of the harmonic conjugate $\theta_r(\sigma)$. But in both of these methods necessary numerical computations are so much tedious that we try in the following another method of improvement.

The final values of $\tau_r(\sigma)$ which are tabulated in Table 2 column 5, are the interpolations by means of the 12-terms Fourier cosine series, determined by the values of it at the 13 points $\sigma=\pi n/12$ $(n=0,1,\cdots,12)$. As it is cosine series of finite terms it has necessarily a horizontal tangent at $\sigma=0$, and does not reproduce the slope of original $\tau_r(\sigma)$ at $\sigma=0+$. By reason of this the interpolated curve undulates about the original one (Fig. 4), and this disagreement is the source of errors. We can then hope to use the technique by which in the section 3 above eliminated the discontinuity of $\theta(\sigma)$, and try to eliminate the slope $\tau_r'(0+)$ by a certain known function with same slope. The difference of them has a horizontal tangent at $\sigma=0$, and can be approximated minutely by a cosine series with a few terms.

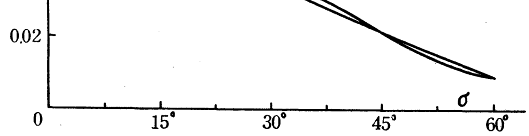

Fig. 4.

One of the simple functions which has known slope ($\neq 0$) at $\sigma=0+$ and horizontal tangent at $\sigma=\pi$ is $\sin(\sigma/2)$ and we define by

$$\tau_1(\sigma)=1-\sin\frac{\sigma}{2}\ (0\leq\sigma\leq\pi),\quad \tau_1(\sigma)=\tau_1(-\sigma)\ (0\geq\sigma\geq-\pi) \tag{28}$$

an even function on the circumference of unit circle $\zeta=e^{i\sigma}$; $\theta_1(\sigma)$ which is conjugate to $\tau_1(\sigma)$ is determined as usual:

$$\theta_1(\sigma)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\tau_1(\sigma')-\tau_1(\sigma)\}\times\cot\frac{\sigma'-\sigma}{2}d\sigma', \tag{29}$$

and is

$$\theta_1(\sigma)=\frac{2}{\pi}\sin\frac{\sigma}{2}\log\left|\tan\frac{\sigma}{4}\right|. \tag{29'}$$

The functional form (28) selected for $\tau_1(\sigma)$ is the one which enables to accomplish the quadrature (29), this being a condition for simple manipulation. The holomorphic function $\Omega_1(\zeta)$ in the unit circle $|\zeta|<1$, which becomes to $\theta_1(\sigma)+i\tau_1(\sigma)$ on that circle $|\zeta|=1$, is easily found by Fourier expansion of $\theta_1(\sigma)$ or $\tau_1(\sigma)$ and is

$$\Omega_1(\zeta)=i(1-2/\pi)+i\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2-1/4}\zeta^n. \tag{30}$$

The values $\theta_1(\sigma)$ and $\tau_1(\sigma)$ are as seen in Table 4 columns 2, 3, and are graphed in Fig. 5. Differential quotient $\tau_1'(0+)=-1/2$; that of $\theta_1(\sigma)$ is

$$\theta_1'(\sigma)=\frac{1}{\pi}\left(\cos\frac{\sigma}{2}\log\left|\tan\frac{\sigma}{4}\right|+1\right), \tag{31}$$

Table 4.

σ (deg.)	$\theta_1(\sigma)$	$\tau_1(\sigma)$	$\Delta\tau(\sigma)$
0	0.0000	1.0000	−0.1384
3.75	−0.0857	0.9673	
7.5	−0.1424	0.9346	(−0.1381)
15	−0.2265	0.8695	−0.1358
22.5	−0.2879	0.8049	
30	−0.3341	0.7412	−0.1245
37.5	−0.3686	0.6786	
45	−0.3934	0.6173	−0.1114
52.5	−0.4100	0.5577	
60	−0.4192	0.5000	−0.0979
67 5	−0.4219	0.4444	
75	−0.4187	0.3912	−0.0848
82.5	−0.4102	0.3407	
90	−0.3968	0.2929	−0.0723
97.5	−0.3789	0.2482	
105	−0.3571	0 2067	−0.0610
112.5	−0.3316	0.1685	
120	−0.3029	0.1340	−0.0512
127.5	−0.2713	0.1031	
135	−0.2372	0.0761	−0.0434
142 5	−0.2009	0.0531	
150	−0.1629	0.0341	−0.0374
157.5	−0.1234	0.0192	
165	−0.0829	0.0086	−0.0339
172.5	−0.0416	0.0021	
180	0.0000	0 0000	−0.0327
	$\tau_1'(0)=-1/2$, $\theta_1'(0)=-\infty$,	$\tau_1'(\pi)=0$; $\theta_1'(\pi)=1/\pi$	

and has a weak infinity (vertical tangent of $\theta_1(\sigma)$) at $\sigma=0$; $\theta_1'(\pi)=\pi^{-1}$.

Now we express $\Omega(\zeta)$ in the following manner:

$$\Omega(\zeta)=\Omega_0(\zeta)+\mu\Omega_1(\zeta)+\Delta\Omega(\zeta), \tag{32}$$

$$\Delta\Omega(\zeta)=i\sum_{n=0}^{\infty}a_n\zeta^n; \tag{32'}$$

and determine, as is said above, the constant μ so that the gradient of

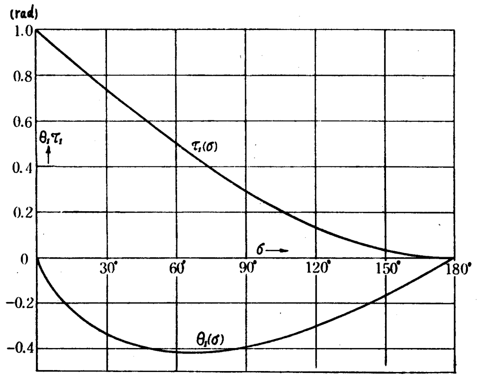

Fig. 5.

Table 5.

a_0	-0.0783
a_1	-0.0507
a_2	-0.0067
a_3	-0.0022
a_4	-0.0009
a_5	-0.0004
a_6	-0.0001
a_7	$+0.0001$
a_8	$+0.0001$
a_9	$+0.0002$
a_{10}	$+0.0002$
a_{11}	$+0.0002$
a_{12}	$+0.0001$

$\Delta\tau = \tau_r - \mu\tau_1$ along the circumference of unit circle vanishes at $\sigma = 0$. $\tau_r(\sigma)$ has been calculated from p and $\theta_r(\sigma)$ of the preceding cycle of iteration through the surface condition, and then by means of (9) there exists a relation between them:

$$2\sin\frac{\sigma}{2}e^{3\tau_r}\left(\frac{1}{6}\cot\frac{\sigma}{2} + \tau_r'\right) = p\sin\theta\,,$$

from which, referring to (22), we have

$$\tau_r'(0+) = \frac{1}{2\sqrt{3}}\theta'(0)\,. \tag{33}$$

Now $\theta'(0) = -0.3732$ and consequently $\tau_r'(0+) = -0.10774$; with this value μ is easily determined to 0.2155. $\Delta\tau(\sigma)$ thus determined are the figures in Table 4 column 4, and has really horizontal tangent at $\sigma = 0$ (and at $\sigma = \pi$). Fourier expansion coefficients a_n ($n = 0, 1, 2, \cdots, 12$) of this $\Delta\tau(\sigma)$ are in Table 5. The solution has thus been settled and among other things

$$\left.\begin{aligned} \theta(\sigma) &= \theta_0(\sigma) + 0.2155\,\theta_1(\sigma) - \sum_{n=1}^{12} a_n \sin n\sigma\,, \\ \tau(\sigma) &= \tau_0(\sigma) + 0.2155\,\tau_1(\sigma) + \sum_{n=0}^{12} a_n \cos n\sigma\,; \end{aligned}\right\} \tag{34}$$

these two functions being tabulated in Table 6 columns 2 and 3.[12]

Eigenvalue p is evaluated, as above, by means of (20) and θ-values of (34) and in Table 6. On the occasion of integral in the denominator, how-

[12] $\theta(\sigma)$ includes $\theta_1(\sigma)$ and therefore $\theta'(0+)$ is $-\infty$. This fact does, however, not mean that the wave profile has a large curvature near the crest, for

$$\frac{d\theta}{ds} = \frac{d\theta}{d\sigma}\cdot\frac{d\sigma}{ds} = -\frac{2\pi q}{L}\frac{d\theta}{d\sigma} \to \text{const}\cdot\sigma^{1/3}\cdot\log\sigma \to 0 \quad (\sigma \to 0+)\,,$$

which means a straight line segment at $\sigma \to 0+$.

ever, we must take care of the term $\sigma\log\sigma$ near $\sigma=0$. In the neighborhood of $\sigma=0$

$$\left.\begin{aligned}\cos\theta(\sigma) &\doteqdot 0.8660-0.03429\sigma\log\sigma+a\sigma+b\sigma^2,\\ \sin\theta(\sigma) &\doteqdot 0.5000+0.05940\sigma\log\sigma+a'\sigma+b'\sigma^2,\end{aligned}\right\}\tag{35}$$

and from these follows the approximate expansion near $\sigma=0$:

$$\cos\theta(\sigma)\left(\int_0^\sigma\sin\theta(\sigma')d\sigma'\right)^{1/3}=0.6873\sigma^{1/3}(1-0.01980\sigma\log\sigma+\alpha\sigma+\beta\sigma^2).\tag{35'}$$

Constants α, β are determined, as usual, by virtue of function values of the left-hand side at $\sigma=\pi/24$ and $\sigma=\pi/12$, and integration easily done in the interval $(0,\pi/12)$; in the remaining interval Simpson rule being adopted. Result is

Table 6.

σ (deg.)	$\theta(\sigma)$	$q(\sigma)$	$-x/L$	$-y/L$	$\dfrac{q^2+4\pi p\frac{y}{L}}{q^2}\times 1{,}000$
0	0.5236	0.0000	0.0000	0.00000	—
3.75	0.4988	0.4322	0.0314	0.01765	4.8
7.5	0.4803	0.5409	0.0502	0.02767	4.1
15	0.4500	0.6729	0.0808	0.04305	−1.3
22.5	0.4240	0.7624	0.1070	0.05531	−2.1
30	0.3992	0.8317	0.1310	0.06578	−1.6
37.5	0.3748	0.8885	0.1534	0.07490	0.8
45	0.3514	0.9347	0.1747	0.08303	−0.9
52.5	0.3292	0.9748	0.1953	0.09029	−0.7
60	0.3075	1.0099	0.2152	0.09688	−0.4
67.5	0.2861	1.0405	0.2347	0.10280	−0.1
75	0.2653	1.0673	0.2536	0.10819	−0.4
82.5	0.2451	1.0910	0.2724	0.11305	−0.3
90	0.2252	1.1122	0.2907	0.11747	−0.2
97.5	0.2054	1.1309	0.3089	0.12142	0.1
105	0.1860	1.1473	0.3268	0.12500	−0.2
112.5	0.1669	1.1617	0.3446	0.12814	0.0
120	0.1478	1.1744	0.3622	0.13097	−0.2
127.5	0.1289	1.1853	0.3797	0.13337	0.1
135	0.1103	1.1944	0.3970	0.13549	−0.3
142.5	0.0919	1.2021	0.4144	0.13722	−0.1
150	0.0734	1.2085	0.4316	0.13867	0.0
157.5	0.0549	1.2134	0.4488	0.13975	0.3
165	0.0366	1.2167	0.4659	0.14055	0.1
172.5	0.0183	1.2187	0.4830	0.14101	0.1
180	0.0000	1.2194	0.5001	0.14117	0.1

$$p = 0.8381, \tag{36}$$

and from this follow the wave velocity:

$$U = 1.0923 \times \sqrt{\frac{gL}{2\pi}}, \tag{37}$$

and, making use of $q(\pi)$ in Table 6, the steepness:

$$\frac{H}{L} = 0.1412; \tag{38}$$

the latter value being very near to that of Michell-Havelock.

In the calculation of wave profile by (19′) there appears also the term $\sigma \log \sigma$, and assuming the forms:

$$\left.\begin{aligned} \frac{\cos\theta}{q} &\doteqdot \sigma^{1/3}(0.8018 - 0.03175\sigma \log\sigma + a\sigma + b\sigma^2 + c\sigma^3), \\ \frac{\sin\theta}{q} &\doteqdot \sigma^{1/3}(0.4629 + 0.05499\sigma \log\sigma + a'\sigma + b'\sigma^2 + c'\sigma^3), \end{aligned}\right\} \tag{39}$$

we fix the constants on one sides by virtue of the function values on the other sides at the points $\sigma = \pi/48$, $\pi/24$ and $\pi/12$. These expansions are used for the integral values at $\sigma = \pi/48$ and $\pi/24$; Outside of this interval Simpson rule, and rarely trapezoidal method, being used. Resulting wave profile is given in Table 6 columns 4, 5, the profile coinciding very perfectly with that of Fig. 3.

The degree of failure in the fulfilment of surface condition is likewise calculated and tabulated in Table 6 column 6, and is the order of several per mille at most. In this case of calculation the Fourier expansion into finite components is expected to reproduce the original function $\Delta\tau(\sigma)$ well in all interval $(-\pi, \pi)$, and the harmonic conjugate nature between θ and τ is as perfect as above. The inaccuracy of the solution seems then to arise mainly from that of the fulfilment of surface condition, for our θ and τ is not bound directly by that condition.

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