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ON THE ELASTIC FAILURE AND THE BUCKLING OF A COLUMN UNDER ECCENTRIC LOADS

(2nd Report)

By Shosaburo NEGORO

In accordance with the definition of the buckling stated in the 1st report,(*) we think in the present report that a column receives an elastic failure, that is, the buckling when the greatest stress on compression side of the column under eccendtric loads reaches the stress at the yield point of its material. Now, in the actual compression of a column, we always have unavoidable eccentric quantities and these ones appear in the buckling of the column chiefly. Hence, for the actual design of the column, we are in need of knowing what loading states within the range of the eccentricities assumed properly lead the column to the buckling most easily. Then, assuming that the column, a prismatic bar of uniform circular cross-section, is hinged at the both ends and its material is mild steel with the yield stress $\sigma_t = 2.23 \times 10^3 \mathrm{kg/cm^2}$ and Young's modulus $E = 2.08 \times 10^6 \mathrm{kg/cm^2}$, we find actually the bucklings of the columns loaded at several positions in the domains within the eccentric quantities stated above by using the buckling formulae stated in the 1st report.(*) From these calculating results, we know that as to the effects of the eccentricities of loading on the both end surfaces of the column on the buckling, it is greatest when these eccentricities are on the plane of bending caused by the eccentric loads and are on the same side of the column-axis as well as these quantities are equal to each other and, on the contrary, it is smallest when these eccentricities are on the plane of bending and are on the opposite side of the column-axis as well as these quantities are nearly equal to each other and, moreover, that in the latter cases stated above the relative curves of buckling loads and slenderness ratios are nearly expressed by the curves consisted of e-constant lines corresponding to the constants (k_0, k_0'), these being defined in the proviso of the buckling formulae, and partial curve of Euler's one. Furthermore, expressing the foregoing descriptions in other words, for leading the column to the buckling the former is the easiest state and the latter is the most difficult one. Then we intend to notice that we generally have some plastic demain in the column when we apply Euler's formula to the buckling of the column and, accordingly, that the formula is already inapplicable as it is.

After all, we know that in actual design of a column under eccentric loads, first assuming the eccentric quantities from degree of accuracy required actaully in the problems and determing dimensions of the column by utilizing the method described in the present report on the basis of the loading conditions in the former case stated above, we have always the column without danger of the buckling, provided that the column is loaded normally with the eccentricities within the range of the quantity assumed above.

I. Introduction. In the present report, as stated in the 1st report,(*) we think that a column receives an elastic failure, that is, the buckling, when the greatest stress caused in the column by eccentric loads at the both ends reaches the stress at the yield point of the material and already know that the effect of eccentricities which generally take place in loading at the both ends on the buckling of the column are considerable. Now, in the actual compression of a column, we always have unavoidable eccentric quantities and these ones appear in the buckling of the column chiefly. Therefore, first assuming the range of the eccentric quantities from actual compressive conditions and then finding the easiest loading states, for leading the column to the buckling, of all the ones which take place in the compressions within the range of its eccentric quantities, we think it is most proper to design the column by using the well-known buckling fromulae in the 1st report(*) on the basis of the loading states described above.

Thereupon, in the present report, using the buckling formulae, we find the bucklings of columns loaded at several positions in the domains within the range of the eccentric quantities assumed properly and draw the relative curves between buckling loads and slenderness ratios. From these results, we intend to point out what loading states within the range of the eccentricities lead the column to the buckling most easily.

II. The buckling formulae of a column. As well-known already in the 1st report,(*) the buckling formulae of columns which are assumed to be prismatic bars of uniform cross-sections, to be loaded freely at the ends and to be hinged at the both ends are given as follows:—

(A) On such a case as one of the principal axes of the cross-section exists on the plane of bending.

In this case, we have the buckling formulae as follows:—

(i) In the range of $p \leqq p_0 (= \pi/2 - \nu)$,

$$(\sigma, e) = (\sigma_l, e_l)/\{(\pm 1) + k_0 k_1^2 \Theta_0\} \qquad (1_1).$$

(ii) In the range of $p > p_0$,

$$(\sigma, e) = (\sigma_l, e_l) \bigg/ \left\{(\pm 1) + \frac{k_0 k_1^2}{\sin p} \Theta_0 \sqrt{4(1-\kappa)\sin^2\frac{p}{2} + \kappa^2}\right\} \qquad (1_2),$$

in which $(e, e_l) = (\sigma, \sigma_l)/E$, $\sigma_l =$ yield stress, $\sigma = P/A$, $P = W\cos\gamma$,

$$\Theta_0 \doteqdot 1 + (a')^2 \kappa(\kappa + p\eta), \quad \Theta_1 = \cos\theta_l + c\sin\theta_l, \quad c = \tan\gamma,$$
$$\eta = (\sin\nu - \cos p)/\sin p, \quad \kappa = 1 - \sin\nu, \quad \sin\nu = \bar{a}_0/a_0, \quad a_0 = a\Theta_0,$$
$$\bar{a}_0 = \bar{a}\Theta_1, \quad p = \lambda\sqrt{e(1-e)}, \quad a', h' = (a, h)/l, \quad \lambda = l/i,$$

$\bar{a}$, $a =$ eccentric quantities of loading at the upper and the lower end surfaces, $h =$ the furthest distance from the axis of the column to its boundary, $\gamma =$ the inclination of the line of the loads to the initial axis of the column, $k_0 = a'/h'$ ($=$eccentricity), $k_1 = h'\lambda =$ the number determined by the form of the cross-section ($=2$ for circular cross-section, or $=1.75$ for rectangular cross-section etc.) i = radius of gyration of the cross-section, $I = Ai^2$, A = cross-sectional area, l = length of the column, W = compressive load and as to the double signs, we take the positive sign when the greatest stress on compression side of the column reaches first the yield point of the material and negative sign when the greatest stress on tension side reaches first the yield point and, furthermore, as to the eccentric quantity $\bar{a}$, we take $\bar{a}$ so as to be positive when the two eccentric quantities a and $\bar{a}$ at the ends are on the same side of the axis and, moreover, take a so as not to be smaller then $\bar{a}$.

(B) On such a case as the principal axes of the cross-section are outside of the plane of bending.

In this case, assuming a column to be one with circular cross-section for convenience of the treatments in addition to the assumptions mentioned above, we have the buckling formulae as follows:—

$$\left.\begin{aligned} (\sigma, e) &= 2(\sigma_l, e_l)/\{F_1 + \sqrt{F_1^2 + 4F_2^2}\} \\ \sigma &= 2\tau_l/\sqrt{F_1^2 + 4F_2^2} \end{aligned}\right\} \qquad (2),$$

in which $F_1 = (\pm 1) + k_{01} k_1^2 \Theta_0 F_3$, $F_3 = \sqrt{1 + \eta_{01} \tan\varphi}$, $\varphi = p\xi_m$,

$$F_2 = 2k_0 k_0' \frac{k_1}{\lambda}(\kappa_0' - \kappa_0), \quad \kappa_0 = 1 - \sin\nu_0, \quad \kappa_0' = 1 - \sin\nu_0',$$

$$k_0 = a/h, \quad k_0' = b/h, \quad p = \lambda\sqrt{e(1-e)}, \quad \sin\nu_0 = \bar{a}_0/a_0,$$

$$\sin\nu_0' = \bar{b}_0/b_0, \quad \eta_{01} = (\sin\nu_{01} - \cos p)/\sin p, \quad k_{01} = \sqrt{k_0^2 + k_0'^2},$$

$$\sin\nu_{01} \doteqdot \frac{1}{k_{01}^2}(k_0^2 \sin\nu_0 + k_0'^2 \sin\nu_0'), \quad \kappa = 1 - k_{01}^2 (k_0^2 \sin^2\nu_0 + k_0'^2 \sin^2\nu_0'),$$

ξ_m = the position of the cross-section, on which the column receives the maximum bending moment, σ_l, τ_l, e_l = normal stress (compressive or tensile stress), shear stress and normal strain at the yield point, and as to φ,

$$\left.\begin{aligned} &p \leqq p_0\left(= \frac{\pi}{2} - \nu_{01}\right): \quad && \varphi = 0, \quad (\xi_m = 0), \\ &p > p_0: && \tan 2\varphi = 2\eta_{10} \sin^2 p/(\kappa + \eta_{01} \sin 2p) \end{aligned}\right\}$$

and, furthermore, we take the positive one of double signs in the definition of the function F_1 in such a case as the greatest stress on the compression side of the column becomes equal to the yield point of the material and the negative sign in such a case as the greatest stress on the tension side of the column becomes equal to the yield point of the material and, moreover $\bar{a}$, $\bar{b}$ denote the components of the eccentric quantity $\bar{r}$ of loading on the upper end surface in the directions of y- and z-axes on the cross-section of the column respectively and its signs are determined by their positions referring to the axis of the column in similar manner as the rules stated in the part (A). (See the 1st report.(*))

III. Numerical Calculations. In the present calculations, assuming that a columns is a prismatic bar of circular cross-section and its material is mild steel with $\sigma_l = 2.23 \times 10^3$ kg/cm^2 and $E = 2.08 \times 10^6$ kg/cm^2 in addition to the assumptions described previously, we treat the buckling in such a case as the greatest stress on the compression side of the column becomes equal to the yield point of the material.

(A) On such a case as one of the principal axes of the cross-section exists on the plane of bending.

In this case, we are able to use the buckling formulae (1) as stated above. Namely, from the above assumptions, first we have $k_1 = 2$ in Eqs. (1). Here, substituting suitable valuse for k_0 and ν_0, from Eqs. (1) we find all of the undetermined quantities required in the present problems. In this calculation, we find the buckling loads and the slenderness ratios for the twenty cases determined by the combinations of $\nu = (0.5\pi, 0.25\pi, 0, -0.25\pi \,\&\, -0.4\pi)$ or $(\bar{a}_0/a_0 = 1, 0.707, 0, -0.707 \,\&\, -0.951)$ and $k_0 = (0.5, 0.1, 0.01 \,\&\, 0.001)$ respectively. Fig. (1) shows the, results, that is, the relative curves between buckling loads and slenderness ratios. In this figure, the curve marked with □ denotes the Euler's one. From this figure we know that the curves obtained above approach to Euler's one gradually as the eccntricities are smaller and slenderness ratios are larger and, moreover, that the effects of eccentricities on the buckling of the column are smaller as $p_0 (= \pi/2 - \nu)$ is larger from zero and that when p_0 is the extreme value, that is $p_0 \doteqdot \pi$ or $a_0 \doteqdot -\bar{a}_0$, the curves $(e \sim \lambda)$ are always expressed by e-constant lines corresponding to the values of k_0 and partial curve of Euler's one nearly; furthermore, in other words, in this case we are able to apply Euler's formula to the buckling of the columns beyond the range of the slenderness ratio expressed by the intersecting point of the two lines of the e-constant line and Euler's curve and, on the contrary, are unable to apply the formula to the buckling of the columns within the range of the slenderness ratio stated above.

In conclusion, from the above descriptions we intend to notice especially that in the present case the column receives the buckling most easily when

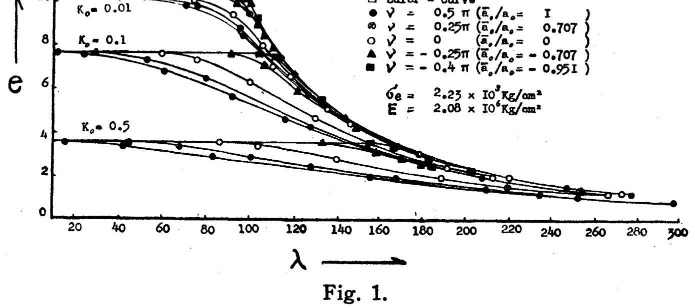


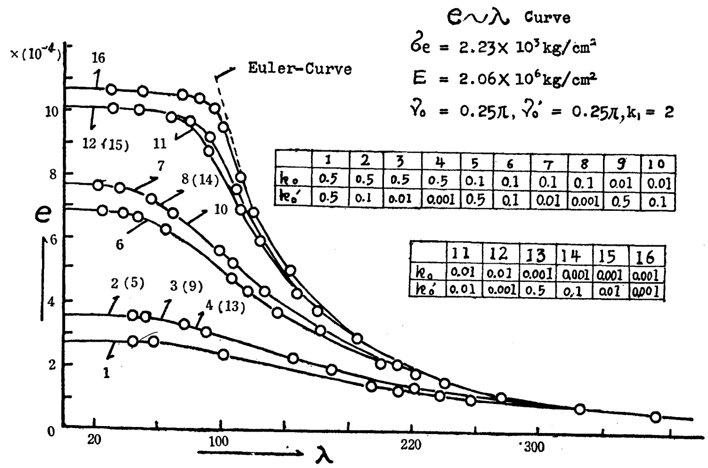
Fig. 1.

the eccentricities on the both ends are on the plane of bending caused by the compressive loads and are on the same side of the column-axis as well as their quantities are equal to each other, that is $a = \bar{a}$ or $\nu = 0.5\pi$.

Therefore, the column, the dimensions of which are determined by utilizing the formulae (1) on the basis of the loading states described above, is always out of danger of the buckling, provided that the column is loaded normally with the eccentricities within the range of the quantity a asuumed above.

(B) On such a case as the principal axes of the cross-section are outside of the plane of bending.

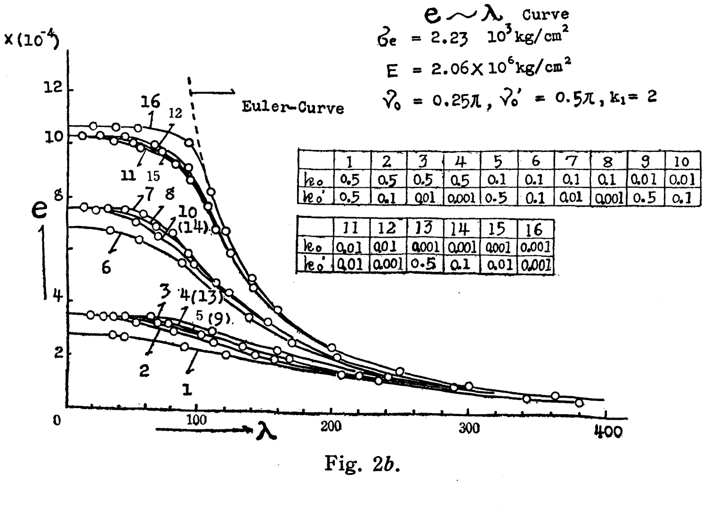
In this case, as stated previously, we are able to use the buckling formulae (2). Namely, first assuming that the cross-section of the column is



	1	2	3	4	5	6	7	8	9	10
k_0	0.5	0.5	0.5	0.5	0.1	0.1	0.1	0.1	0.01	0.01
k_0'	0.5	0.1	0.01	0.001	0.5	0.1	0.01	0.001	0.5	0.1

	11	12	13	14	15	16
k_0	0.01	0.01	0.001	0.001	0.001	0.001
k_0'	0.01	0.001	0.5	0.1	0.01	0.001

Fig. 2*a*.



	1	2	3	4	5	6	7	8	9	10
k_0	0.5	0.5	0.5	0.5	0.1	0.1	0.1	0.1	0.01	0.01
k_0'	0.5	0.1	0.01	0.001	0.5	0.1	0.01	0.001	0.5	0.1

	11	12	13	14	15	16
k_0	0.01	0.01	0.001	0.001	0.001	0.001
k_0'	0.01	0.001	0.5	0.1	0.01	0.001

Fig. 2*b*.

circular as stated in previous part, we have $k_1 = 2$ in Eqs. (2). Here, substituting suitable values for (ν_0, ν_0') and (k_0, k_0'), from Eqs. (2) we find $(\kappa_0, \kappa_0', k_{01}, \kappa, \nu_{01}, p_0(=\pi/2 - \nu_{01}), \eta_{01}, \varphi, F_3 \& F_1)$ in succession and, accordingly, all of undetermined quantities required in the present problems are always found.

In the present calculation, we find the buckling loads and the slenderness ratios for such cases as we take proper combinations of (0.5, 0.1, 0.01 & 0.001) for (k_0, k_0') and $(0.5\pi, 0.25\pi, 0, -0.25\pi \& -0.4\pi)$ for (ν_0, ν_0') respectively and draw these relative curves $(e \sim \lambda)$. Figs. 2 and 3 show the results; namely, Figs. 2_a and 2_b show the relations in such cases as we take constants for (ν_0, ν_0') and take these values (0.5, 0.1, 0.01 & 0.001) successively for (k_0, k_0') respectively and Figs. 3_a and 3_b show the relations in such cases as we take constants for (k_0, k_0') and take these values $(0.5\pi, 0.25\pi, 0, -0.25\pi \& -0.4\pi)$ successively for (ν_0, ν_0') respectively and in these Figs. 2 and 3 the curves marked with (→) denote the Euler's one.

From these figures, in the present case we have also the results quite similar as the ones described in previous part (A); namely the curves obtained above approach to Euler's one gradually as the eccentricities are smaller and slenderness ratios are larger and, moreover, the effects of eccentricities on the buckling of the column are smaller as $p_0(=\pi/2 - \nu_{01})$ is larger and larger from zero and, accordingly, when p_0 is the extreme value, that is $p_0 \overset{\cdot}{\underset{\cdot}{\approx}} \pi$ or $r_0 \overset{\cdot}{\underset{\cdot}{\approx}} -\bar{r}_0$ $(a \overset{\cdot}{\underset{\cdot}{\approx}} -\bar{a}, b \overset{\cdot}{\underset{\cdot}{\approx}} -\bar{b})$, their effects are the smallest and in these cases the curves $(e \sim \lambda)$ are always expressed by e-constant lines corresponding to the values of (k_0, k_0') and partial curve of Euler's one nearly.

In conclusion, as stated in the previous part, from the above descriptions we intend to notice especially that in the present case the column receives also the buckling most easily when the column is loaded in such states as

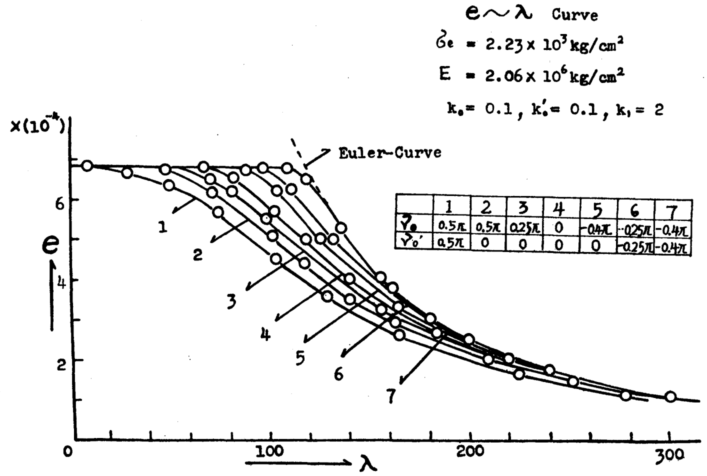
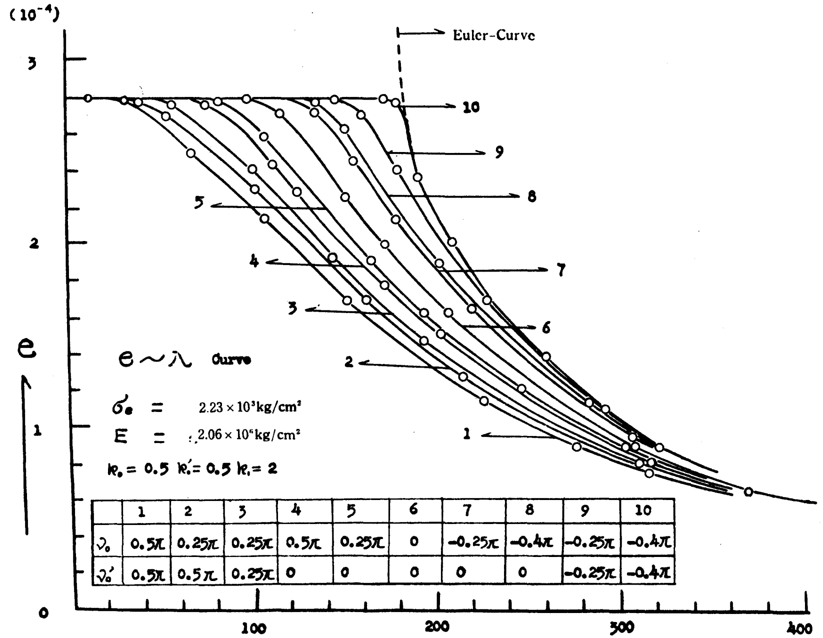


Fig. 3*a*.



	1	2	3	4	5	6	7	8	9	10
ν_0	0.5π	0.25π	0.25π	0.5π	0.25π	0	-0.25π	-0.4π	-0.25π	-0.4π
ν_0'	0.5π	0.5π	0.25π	0	0	0	0	0	-0.25π	-0.4π

Fig. 3*b*.

the eccentricity on the lower end surface r_0 coincides with the one on the upper end surface $\bar{r}$, that is $(a=\bar{a}$ & $b=\bar{b}$ or $\nu=0.5\pi$ & $\nu'=0.5\pi)$ and, on the contrary, that it is most difficult to lead the column to the buckling in such cases as the relative curves are nearly expressed by the combinations of e-constant lines corresponding to the values (k_0, k_0') and partial curve of Euler's one, that is in such cases as $r_0 \fallingdotseq -\bar{r}_0$ $(a \fallingdotseq -\bar{a}, b \fallingdotseq -\bar{b})$ or the eccentricities tend to zero approximately, say $r_0 \to 0$, furthermore, in other words, in such cases as Euler's formula is held. Then, in the actual design, determining the dimensions of the columns on the basis of the loading conditions in the former case stated above by using the method described in the present report, we have always the columns without danger of the buckling, provided that the columns are loaded hormally with the eccentricities within the range of the quantity r_0 assumed from degree of accuracy required actually in the problem.

IV. Conclusion. In the present report, as stated in previous parts, first we assume that the column, a prismatic bar of uniform circular cross-section, is hinged at the both ends and, moreover, receives the buckling when the greatest stress on compression side of the column reaches the yield stress of its material. Then from the calculating results treated in the present report we know that as to the effect of the eccentricities of loading on the both end surfaces of the column on the buckling, it is greatest when these eccentricities are on the plane of bending caused by the eccentric loads and are on the same side of the column-axis as well as these quantities are equal to each other, namely $r_0 = \bar{r}_0$ $(a = \bar{a},\ b = \bar{b})$ and, on the contrary, it is smallest when these eccentricities are on the plane of bending and are on the opposite side of the column-axis as well as these quantities are nearly equal to each other, say $r_0 \overset{\cdot}{\underset{\cdot}{\approx}} -\bar{r}_0$ $(a \overset{\cdot}{\underset{\cdot}{\approx}} -\bar{a},\ b \overset{\cdot}{\underset{\cdot}{\approx}} -\bar{b})$ and, moreover, that in the latter cases stated above the relative curves of buckling loads and slenderness ratios ($e \sim \lambda$ curves) are nearly expressed by the curves consisted of e-constant lines corresponding to $(k_0,\ k_0')$ and partial curve of Euler's one. Furthermore, expressing the foregoing descriptions in other words, for leading the column to the buckling, the former is the easiest state and the latter is the most difficult one. Therefore, in actual design of a column under eccentric loads, first assuming the eccentric quantities from degree of accuracy required actually in the problems and determining dimensions of the column by utilizing the method described in the present report on the basis of the loading conditions in the former case stated above, namely $r_0 = \bar{r}_0$ $(a = \bar{a},\ b = \bar{b})$, we have always the column without danger of the buckling, provided that the column is loaded normally with the eccentricities within the range of the quantity r_0 assumed above.

In the previous descriptions, however, assuming that the eccentric quantities are small, we have no regard for the influences of twisting moment caused in the column by eccentric loads on the buckling and, moreover, assume that the buckling treated in the present calculations denotes the elastic one complying with the definition mentioned in the introduction. Therefore, we are unable to utilize the buckling formulae described above not only for the plastic buckling but also for the elastic buckling of higer order which gernerally takes place in the range of $p > \pi$ stated in the proviso of the formulae, because we have already plastic domain in the column under the loads $p > \pi$ and the formulae are derived on the basis of the elastic bending formulae as stated in the 1st report(*) fully. Then we have still necessity to research these buckling of the column.

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