Supplementary Data for the Theory of the Seepage Through an Earth Dam (II): with special reference to the results of the hodograph theory of P. Ya. Polubarinova-Kochina

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Supplementary Data for the Theory of the Seepage Through an Earth Dam (II)

with special reference to the results of the hodograph theory of P. Ya. Polubarinova-Kochina

By
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1. Following the foregoing note [4], where the experimental data for the seepage through an earth dam with a trapezoidal cross-section have been particularly referred, we shall examine here to some extent the behaviour of the exact solution established for a rectangular dam and compare them with our simplified theory which one of the present authors has described in his previous paper [3].

2. After P. Ya. Polubarinova-Kochina [2] we have the following expressions for an earth dam with vertical faces :[2)]

$$X = L - \int_0^{\varphi} \frac{K[\sin^2\varphi]\ \sin\varphi\, d\varphi}{\sqrt{(1-\lambda\sin^2\varphi)(1-\mu\sin^2\varphi)}}, \tag{2.1}$$

$$Y = \int_0^{\varphi} \frac{K[\cos^2\varphi]\ \sin\varphi\, d\varphi}{\sqrt{(1-\lambda\sin^2\varphi)(1-\mu\sin^2\varphi)}}, \tag{2.2}$$

$$L = \int_0^{\pi/2} \frac{K[\sin^2\varphi]\ \sin\varphi\, d\varphi}{\sqrt{(1-\lambda\sin^2\varphi)(1-\mu\sin^2\varphi)}} = \int_0^{\pi/2} \frac{K[\mu+(\mu-\lambda)\sin^2\varphi]\, d\varphi}{\sqrt{\lambda'-(\mu-\lambda)\sin^2\varphi}}, \tag{2.3}$$

$$H_0 = \int_0^{\pi/2} \frac{K[\mu+\mu'\sin^2\varphi]\, d\varphi}{\sqrt{(\mu-\lambda)+\mu'\sin^2\varphi)}} = \frac{1}{\sqrt{\lambda'}} \int_0^{\pi/2} \frac{K[1-\mu'\sin^2\varphi]\, d\varphi}{\sqrt{1-\gamma'\sin^2\varphi}}, \tag{2.4}$$

$$H_1 = \sqrt{\gamma} \int_0^{\pi/2} \frac{K[\lambda\sin^2\varphi]\ \sin\varphi\, d\varphi}{\sqrt{(1-\lambda\sin^2\varphi)(1-\gamma\sin^2\varphi)}}, \tag{2.5}$$

$$H_S = \int_0^{\pi/2} \frac{K[\cos^2\varphi]\ \sin\varphi\ \cos\varphi\, d\varphi}{\sqrt{(1-\lambda'\sin^2\varphi)(1-\mu'\sin^2\varphi)}}, \tag{2.6}$$

$$Q' = \frac{Q}{k} = \frac{1}{\sqrt{\lambda'}} \int_0^{\pi/2} \frac{K[\mu'\sin^2\varphi]\, d\varphi}{\sqrt{1-\gamma'\sin^2\varphi}}, \tag{2.7}$$

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1) Ministry of Construction, Ôita Construction Office.

2) A common multiplicative factor is omitted in these expressions; it may be noticed that if all of the quantities X, Y, L, H_0, H_1, H_S and Q' are multiplied by a constant, they are still the solutions which correspond to the same parameters λ and μ.

where X and Y are the current coordinates of the free surface measured in the sense of Fig. 1, λ and μ disposable parameters such that $0 \leq \lambda \leq \mu \leq 1$,[3] with the abbreviations

$$\lambda' = 1 - \lambda, \quad \mu' = 1 - \mu, \quad \gamma = \lambda/\mu, \quad \gamma' = \mu'/\lambda', \tag{2.8}$$

and
$$K[x] = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x \sin^2 \theta}}, \tag{2.9}$$

or the complete elliptic integral of the first kind, all of the other notations being the same as before (cf. [3], p. 4).

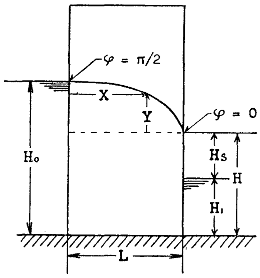

FIG. 1.

On the other hand it has been independently proved [3] that

$$Q' = \frac{H_0^2 - H_1^2}{2L}, \quad \text{or} \quad q = \frac{1}{2}(h_0^{\ 2} - h_1^{\ 2}). \tag{2.10}$$

This relation, as is called the Dupuit-Forchheimer's formula, makes us possible to proceed to some further analysis of the above-cited integrals.

3. In the first place, the quantities L, H_0, H_1 and H_S are computed from (2.3)-(2.6) for the various values of λ and μ by applying the Simpson rule of numerical integrations.[4] Since, however, the integrand for H_0 becomes infinitely large (in a logarithmic way) at $\varphi = 0$, the expansion in series should be used near $\varphi = 0$, which gives

$$\int_0^{\varepsilon} \frac{K[1 - \mu' \sin^2 \varphi]\, d\varphi}{\sqrt{1 - \gamma' \sin^2 \varphi}} = -(\log \varepsilon)\Big[\varepsilon + \frac{p}{3}\varepsilon^3 + \cdots\Big] + \Big[(1 + C)\,\varepsilon + \Big(\frac{p}{9} + \frac{q}{3}\Big)\varepsilon^3 + \cdots\Big] \tag{3.1}$$

for small ε, where

$$C = \log \frac{4}{\sqrt{\mu'}}, \quad p = \frac{\mu'}{4} + \frac{1}{2}\gamma', \quad q = Cp - \frac{\mu'}{4}. \tag{3.2}$$

The results thus obtained are shown in Table 1. Although once the values of L, H_0 and H_1 were known then we could readily find the value of Q' from (2.10), it is calculated directly through the integral expression (2.7) too, in order to see the accuracy of our numerical computations. It will be seen from the last two columns of Table 1 that the accuracy may be satisfactory for our present purpose.

For the sake of later convenience these data are retabulated in Table 2 together with those given by M. Muskat [1]. They are also plotted in Fig's 2 and 3.

4. Next let us consider the special cases for which the asymptotic behaviour of the solution can be expressed in an analytical form. Making use of the series expansion and with the aid of the relation (2.10), we come to the following conclusions:[5]

[3] λ and μ are the reciprocals of the parameters a and b respectively, which have been used in Muskat's book [1].

[4] The values of $K[x]$ are read from the tables of K. Hayashi, "*Kòtôkansûhyô*" (in Japanese) 3rd ed. (1947) Table 29,

[5] The details of calculation are omitted here; they are a little tedious but straightforward in view of the relations shown in Appendices A and B.

TABLE 1. Numerical results of hodograph theory for some typical values of λ and μ.

λ	μ	L	H_0	H_1	H_S	(Q')	Q'
0	0.05	2.515	5.705	0	3.846	6.447	6.471
	0.20	2.683	4.805	0	2.843	4.306	4.303
	0.40	2.974	4.516	0	2.383	3.427	3.429
	0.60	3.427	4.516	0	2.129	2.974	2.975
	0.80	4.306	4.805	0	1.957	2.683	2.681
	0.95	6.447	5.705	0	1.859	2 515	2.524
0.20	0.40	3.239	5.481	1.553	1.640	4.261	4.263
	0.60	3.738	5.226	1.158	1.502	3.474	3.474
	0.80	4.711	5.444	0.965	1.405	3.045	3.047
0.40	0.60	4.171	6.497	2.348	1.342	4.403	4.399
	0.80	5.278	6.430	1.793	1.260	3.615	3.612
	0.95	8 019	7.422	1.579	1.210	3.273	3.279
0.60	0.80	6.165	8.336	3.253	1.168	4.792	4.778
0.80	0.95	12.44	13.41	5.597	1.060	5.964	5.969

(Q') : obtained from the integral expression (2.7),
Q' : obtained from the D.-F. formula (2.10).

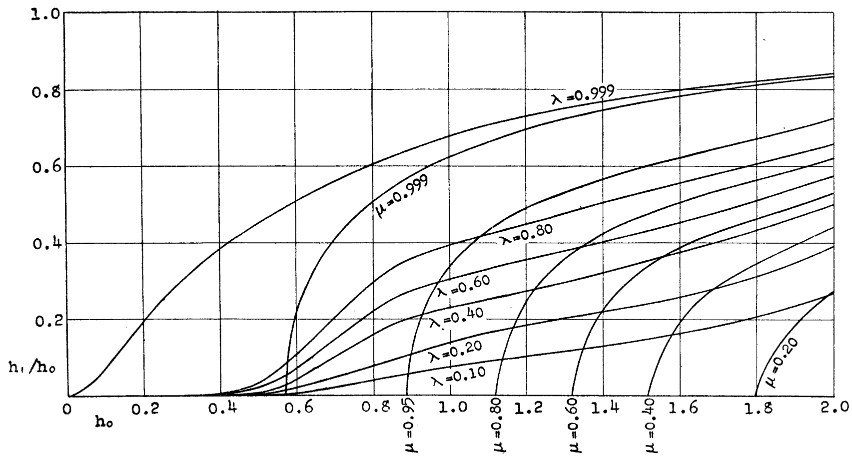

FIG. 2. λ-μ diagram, showing the relationship between the shape parameters of a dam (h_0, h_1) and the hodograph parameters (λ, μ) for $h_0 \leq 2.0$.

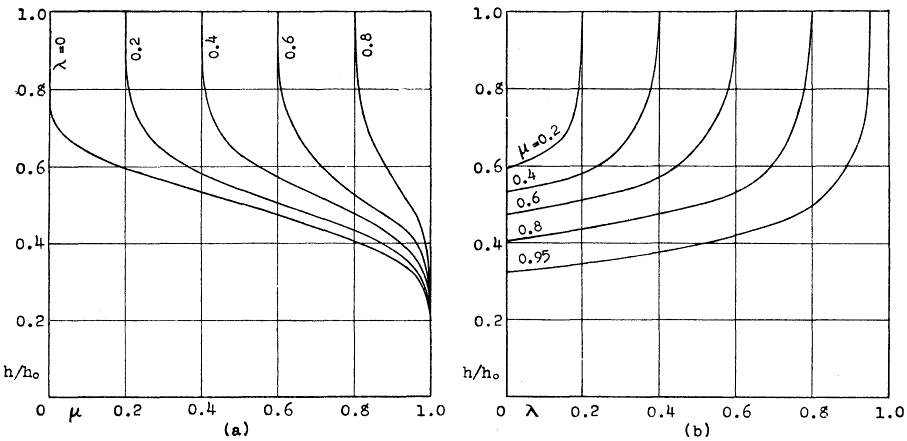

FIG. 3. Change of $h = h_1 + h_S$ with λ and μ.

Case 1. $0 \fallingdotseq \lambda \ll \mu \ll 1$, and accordingly $\gamma = \lambda/\mu \ll 1$.

$$L = \frac{\pi^2}{4}[1 + O(\mu, \gamma)],$$

$$H_0 = \left(\frac{\pi}{4} \log \frac{64}{\mu}\right)[1 + O(\mu, \gamma)],$$

$$H_1 = \frac{\pi}{2} \gamma^{1/2}[1 + O(\mu, \gamma)],$$

$$H_S = \left(\frac{\pi}{4} \log \frac{64}{\mu} - M - \frac{\pi}{2} \gamma^{1/2}\right)[1 + O(\mu, \gamma)],$$

and
$$Q' = \frac{1}{8}\left(\log \frac{64}{\mu}\right)^2 [1 + O(\mu, \gamma)],$$

where
$$M = 2G = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1.8319\cdots, \quad G: \text{Catalan's constant},$$

or
$$h_0 = \left(\frac{1}{\pi} \log \frac{64}{\mu}\right)[1 + O(\mu, \gamma)],$$

$$h_1 = \frac{2}{\pi} \gamma^{1/2}[1 + O(\mu, \gamma)],$$

and
$$h_S = \frac{1}{\pi}\left(\log \frac{64}{\mu} - \frac{4}{\pi} M - 2\gamma^{1/2}\right)[1 + O(\mu, \gamma)] \fallingdotseq h_0 - h_1 - \frac{4M}{\pi^2}.$$

Case 2. $\lambda \fallingdotseq \mu \ll 1$, and accordingly $\gamma = \lambda/\mu = 0(1)$.

$$L = \frac{\pi^2}{4}[1 + O(\lambda, \delta)],$$

$$H_0 = \left(\frac{\pi}{4} \log \frac{64}{\mu\delta}\right)[1 + O(\lambda, \delta)],$$

$$H_1 = \left(\frac{\pi}{4}\ \log\frac{4}{\delta}\right)[1 + O(\lambda,\ \delta)]\,,$$

$$H_S = \left(\frac{\pi}{4}\ \log\frac{16}{\mu} - M\right)[1 + O(\lambda,\ \delta)]\,,$$

and $$Q' = \frac{\pi}{8}\left[\left(\log\frac{64}{\mu\,\delta}\right)^2 - \left(\log\frac{4}{\delta}\right)^2\right][1 + O(\lambda,\ \delta)]\,,$$

where $\delta = 1 - \gamma$,

or $$h_0 = \left(\frac{1}{\pi}\ \log\frac{64}{\mu\,\delta}\right)[1 + O(\lambda,\ \delta)]\,,$$

$$h_1 = \left(\frac{1}{\pi}\ \log\frac{4}{\delta}\right)[1 + O(\lambda,\ \delta)]\,,$$

and $$h_S = \left(\frac{1}{\pi}\ \log\frac{16}{\mu} - \frac{4M}{\pi^2}\right)[1 + O(\lambda,\ \delta)] \doteqdot h_0 - h_1 - \frac{4M}{\pi^2}\,.$$

Case 3. $\lambda \ll 1$ and $\mu \doteqdot 1$ (i.e. $\mu' \ll 1$).

$$L = \frac{1}{8}\left(\log\frac{64}{\mu'}\right)^2 [1 + O(\lambda,\ \mu')]\,,$$

$$H_0 = \left(\frac{\pi}{4}\ \log\frac{64}{\mu'}\right)[1 + O(\lambda,\ \mu')]\,,$$

$$H_1 = \frac{\pi}{2}\ \lambda^{1/2}[1 + O(\lambda,\ \mu')]\,,$$

$$H_S = \left(M - \frac{\pi}{2}\ \lambda^{1/2}\right)[1 + O(\lambda,\ \mu')]\,,$$

and $$Q' = \frac{\pi^2}{4}[1 + O(\lambda,\ \mu')]\,,$$

or $$h_0 = \left(\frac{2\pi}{\log\frac{64}{\mu'}}\right)[1 + O(\lambda,\ \mu')]\,,$$

$$h_1 = \frac{4\pi\,\lambda^{1/2}}{\left(\log\frac{64}{\mu'}\right)^2}[1 + O(\lambda,\ \mu')]\,,$$

and $$h_S = \frac{8M - 4\pi\,\lambda^{1/2}}{\left(\log\frac{64}{\mu'}\right)^2}[1 + O(\lambda,\ \mu')] \doteqdot \frac{2M}{\pi^2}\ h_0^2 - h_1\ .$$

Case 4. $\lambda \doteqdot \mu \doteqdot 1$ ($\lambda' \doteqdot \mu' \ll 1$) and accordingly $\gamma' = \mu'/\lambda' = 0(1)$.

$$L = \left(\frac{\pi}{4\sqrt{\lambda'}}\ \log\frac{16}{\lambda'}\right)[1 + O(\lambda',\ \delta')]\,,$$

$$H_0 = \frac{1}{4\sqrt{\lambda'}}\left[\log\frac{16}{\lambda'}\cdot\log\frac{32}{\delta'} + \frac{\pi^2}{2}\right][1 + O(\lambda',\ \delta')]\,,$$

$$H_1 = \frac{1}{4\sqrt{\lambda'}}\left[\log\frac{16}{\lambda'}\cdot\log\frac{32}{\delta'} - \frac{\pi^2}{2}\right][1 + O(\lambda',\ \delta')]\,,$$

$$H_S = 1 + O(\lambda', \delta'),$$

and
$$Q' = \left(\frac{\pi}{4\sqrt{\lambda'}} \log \frac{32}{\delta'}\right)[1 + O(\lambda', \delta')],$$

where $\delta' = 1 - \gamma'$,

or
$$h_0 = \frac{1}{\pi}\left(\log\frac{32}{\delta'} + \frac{\pi^2}{2\log\frac{16}{\lambda'}}\right)[1 + O(\lambda', \delta')],$$

$$h_1 = \frac{1}{\pi}\left(\log\frac{32}{\delta'} - \frac{\pi^2}{2\log\frac{16}{\lambda'}}\right)[1 + O(\lambda', \delta')],$$

and
$$h_S = \frac{4\sqrt{\lambda'}}{\pi\log\frac{16}{\lambda'}}[1 + O(\lambda', \delta')].$$

Case 5. $\lambda \doteqdot 1$, $\mu \doteqdot 1$, $1 \gg \lambda' \gg \mu' \doteqdot 0$ and accordingly $\gamma' \ll 1$.

$$L = \frac{1}{8\sqrt{\lambda'}}\left(\log\frac{256}{\lambda'\mu'}\right)\left(\log\frac{32}{\gamma'}\right)[1 + O(\lambda', \gamma')],$$

$$H_0 = \left(\frac{\pi}{4\sqrt{\lambda'}}\log\frac{64}{\mu'}\right)[1 + O(\lambda', \gamma')],$$

$$H_1 = \frac{\pi}{4\sqrt{\lambda'}}\left[\left(\log\frac{2}{\lambda'}\right)^2 - \log 2\cdot\log\frac{32}{\gamma'}\right]^{1/2}[1 + O(\lambda', \gamma')],$$

$$H_S = 1 + O(\lambda', \gamma'),$$

and
$$Q' = \frac{\pi^2}{4\sqrt{\lambda'}}[1 + O(\lambda', \gamma')],$$

or
$$h_0 = \frac{2\pi\log\frac{64}{\mu'}}{\Delta}[1 + O(\lambda', \gamma')],$$

$$h_1 = \frac{2\pi\left[\left(\log\frac{2}{\lambda'}\right)^2 - \log 2\cdot\log\frac{32}{\gamma'}\right]^{1/2}}{\Delta}[1 + O(\lambda', \gamma')],$$

and
$$h_S = \frac{8\sqrt{\lambda'}}{\Delta}[1 + O(\lambda', \gamma')],$$

where
$$\Delta = \left(\log\frac{256}{\lambda'\mu'}\right)\cdot\left(\log\frac{32}{\gamma'}\right).$$

5. Now our approximate formula which is to be compared with these results is

$$h_S = \frac{h_0^2 - h_1^2}{(h_0^2 - h_1^2) + 2}\times(h_0 - h_1), \quad \text{or} \quad h = \frac{(h_0^2 - h_1^2)h_0 + 2h_1}{(h_0^2 - h_1^2) + 2} \tag{5.1}$$

for the seepage level at the outflow face ([3], p. 6).

The values of h(approx.) obtained in this way are shown in Table 2. As is indicated by the values of $\eta = h/[h(\text{approx.})]$ in the last column of that table, the height of the terminal of the seepage surface are predicted in the correct order of magnitude in these cases.

TABLE 2. Numerical comparison between the hodograph-theoretical data and the approximate ones.

λ	μ	h_0	h_1/h_0	q	I : h/h_0	II : h/h_0 (approx.)	η=I/II
0	0.05	2.27	0	2.57	0.674	0.720	0.94
	*0.10	2.04	0	2.09	0.640	0.676	0.95
	0.20	1.791	0	1.604	0.592	0.616	0.96
	0.40	1.518	0	1.153	0.528	0.536	0.98
	*0 50	1.419	0	1.007	0.502	0.502	1.00
	0.60	1.318	0	0.868	0.471	0.465	1.01
	0.80	1.116	0	0.623	0.407	0.384	1.06
	*0.833…	1.077	0	0.580	0.395	0.367	1.08
	0.95	0.885	0	0.391	0.325	0.281	1.16
0.10	*0.20	1.988	0.261	1.842	0.640	0.740	0.86
0.20	0.40	1.692	0.283	1.316	0.583	0.690	0.84
	*0.50	1.509	0.236	1.075	0.537	0.632	0.85
	0.60	1.398	0.222	0.929	0.509	0.597	0.85
	0.80	1.156	0.177	0.647	0.435	0.500	0.87
0.40	0.60	1.558	0.361	1.055	0.568	0.689	0.82
	0.80	1.218	0.279	0.684	0.475	0.572	0.83
	0.95	0.926	0.213	0.409	0.376	0.441	0.85
0.60	0.80	1.352	0.390	0.775	0.530	0.656	0.81
0.80	0.95	1.078	0.417	0.480	0.496	0.606	0.82

* These cases are taken from Muskat's book [1].

It is also possible to evaluate the error for any of the special cases considered above. Roughly speaking, the approximation becomes better and better as h_0 increases (*Cases* 1, 2 and 4), whereas for small inflow head (*Cases* 3 and 5) an appreciable error will be associated. Excluding this latter case ($h_0 < 0.5$, say), we may suppose that the formula (5.1) is applicable within the errors of about 20 percent or less.

APPENDIX A

We shall note here two integral relations which can be proved in connection with the present considerations.

(A. 1)

$$\int_0^{\pi/2} \frac{K[1-\mu\sin^2\varphi]\,d\varphi}{\sqrt{1-\mu\sin^2\varphi}} = \int_0^{\pi/2} \frac{K[1-\mu'\sin^2\varphi]\,d\varphi}{\sqrt{1-\mu'\sin^2\varphi}} \qquad (= J, \text{ say}),$$

provided that $\mu+\mu'=1$ and $0 \leq (\mu, \mu') \leq 1$.

Proof: If $\lambda=0$ in (2.3)~(2.7), it is readily seen that

$$L(0, \mu) = Q'(0, \mu'), \quad Q'(0, \mu) = L(0, \mu') \quad \text{and} \quad H_1(0, \mu) \equiv 0.$$

Then upon using the relation (2,10) we have

$$H_0(0, \mu) = \{2\, L(0, \mu) \cdot Q'(0, \mu)\}^{1/2} = \{2\, Q'(0, \mu') \cdot L(0, \mu')\}^{1/2} = H_0(0, \mu').$$

This is nothing but what is to be proved.

(A. 2)

$$\int_0^{\pi/2} \frac{K[\cos^2\varphi] \sin\varphi\, d\varphi}{\sqrt{1-\mu\sin^2\varphi}} + \int_0^{\pi/2} \frac{K[\cos^2\varphi] \sin\varphi\, d\varphi}{\sqrt{1-\mu'\sin^2\varphi}} = J,$$

the conditions for μ and μ' being the same as above.

Proof: This is equivalent to say that

$$H_S(0, \mu) + H_S(0, \mu') = H_0(0, \mu).$$

Now putting $\varphi=\pi/2$ in (2.2) we have in general

$$Y\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{K[\cos^2\varphi]\, \sin\varphi\, d\varphi}{\sqrt{(1-\lambda\, \sin^2\varphi)(1-\mu\, \sin^2\varphi)}} = H_0 - (H_1 + H_S),$$

(cf. Fig. 1). Further assuming here that $\lambda=0$, it becomes

$$H_0(0, \mu) - H_S(0, \mu) = \int_0^{\pi/2} \frac{K[\cos^2\varphi]\, \sin\varphi\, d\varphi}{\sqrt{1-\mu\, \sin^2\varphi}} = H_S(0, \mu').$$

Q. E. D.

APPENDIX B

Short list of mathematical formulas for the discussion of the asymptotic behaviours of the integrals

(B. 1) $$K[x] = \frac{\pi}{2}\left(1+\frac{x}{4}+\frac{9x^2}{64}+\frac{25x^3}{256}+\cdots\right), \quad (0 \leq x \ll 1).$$

(B. 2) $$K[1-x] = \Lambda + \frac{\Lambda-1}{4}x + \frac{9}{64}\left(\Lambda-\frac{7}{6}\right)x^2 + \frac{25}{256}\left(\Lambda-\frac{37}{30}\right)x^3+\cdots, \quad (0 \leq x \ll 1).$$

where $\Lambda = \frac{1}{2}\log\frac{16}{x}$.

(B. 3) $$\int_0^{\pi/2} \frac{\theta\, d\theta}{\sin\theta} = \int_0^1 K[x^2]\, dx = M.$$

(B. 4) $$\int_0^{\pi/2} K[\cos^2\varphi]\, \sin\varphi\, \cos\varphi\, d\varphi = \frac{1}{2}\int_0^1 K[x]\, dx = 1.$$

(B. 5) $$\int_0^{\pi/2} \frac{\sin\varphi\, d\varphi}{\sqrt{1-p^2\sin^2\varphi}} = \frac{1}{2p}\log\frac{1+p}{1-p} = \left(1+\frac{1}{3}p^2+\frac{1}{5}p^4+\frac{1}{7}p^6+\cdots\right), \quad (p \ll 1).$$

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