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SUHARA, Tashiro
Research Institute for Applied Mechanics, Kyushu University

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On the Stresses in Laterally Loaded Plates

By Toshiro SUHARA

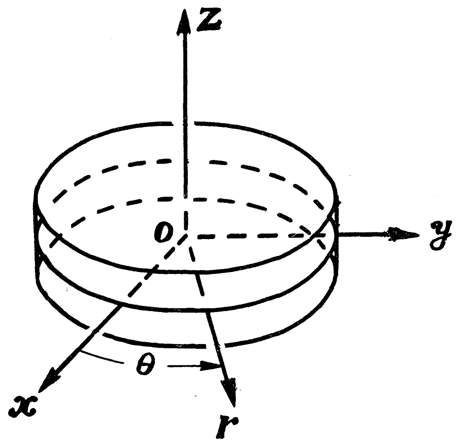
1. Three dimensional stress distribution is claculated in the case of the plate bent by lateral loads. Considering an infinite plate loaded on one face by the pressure $P(r, \theta)$ which is expressible by Neumann's integral form(1)

$$P(r, \theta) = \frac{1}{2\pi}\int_0^\infty \kappa\, d\kappa \int_0^\infty \int_{-\pi}^{\pi} P(R, \Phi)\, J_0\{\kappa\sqrt{R^2 + r^2 - 2Rr\cos(\Phi - \theta)}\}\, R(d\Phi\, dR) \quad (1),$$

or, if the change of order of the integrations and the summations is permitted, expanded in series,

$$\left.\begin{aligned} P(r, \theta) &= \sum_{n=0}^{\infty} \epsilon_n \{\cos n\theta \int_0^\infty J_n(\kappa r)\, G_{cn}(\kappa)\, \kappa\, d\kappa \\ &\quad + \sin n\theta \int_0^\infty J_n(\kappa r)\, G_{sn}(\kappa)\, \kappa\, d\kappa\} \\ \text{where} \quad G_{cn}(\kappa) &= \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} P(R, \Phi)\, \mathrm{con}\, n\Phi\, J_n(\kappa R)\, R(d\Phi\, dR) \\ G_{sn}(\kappa) &= \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} P(R, \Phi)\, \sin n\Phi\, J_n(\kappa R)\, R(d\Phi\, dR) \end{aligned}\right\} \quad (2),$$

we have to obtain the expressions for stress components in cylindrical polar coordinates.



2. For the sake of convenience we proceed to find particular solutions of the fundamental equations of the present problem in rectangular coordinate.

i) Equations of equilibrium.

$$(\partial/\partial x_j)\, \sigma_{ij} = 0 \quad (3).$$

ii) Conditions of compatibility in terms of stress components in the absence of body forces.

$$\Delta\sigma_{ij} = -\frac{1}{1+\nu}\frac{\partial^2}{\partial x_i\, \partial x_j}\Theta \quad (4),$$

(1) The stresses in plates bent by the rotationally symmetrical loads have been calculated by H. Jung; Z. angew. Math. Mech. Bd. 32, S. 46, 1952.

where ν is poisson's ratio, $\Theta = \sigma_x + \sigma_y + \sigma_z$, and $\varDelta\Theta = 0$

Hence
$$\varDelta\varDelta\,\sigma_{ij} = 0 \tag{5}$$

iii) Boundary conditions.

$$\left.\begin{aligned} &\sigma_z = P(x, y), \quad \tau_{yz} = \tau_{zx} = 0 \quad \text{at} \quad z = h \\ &\sigma_z = 0, \quad \tau_{yz} = \tau_{zx} = 0 \quad \text{at} \quad z = -h \end{aligned}\right\} \tag{6}$$

where h is half thickness of the plate.

In the beginning we define the functions φ_0, $\varphi_0{}^*$, φ_1 and $\varphi_1{}^*$, as follows,

$$\left.\begin{aligned} &\left.\begin{matrix}\varphi_0\\ \varphi_0{}^*\end{matrix}\right\} = \frac{K\cosh K + \sinh K}{2K + \sinh 2K}\cdot\left.\begin{matrix}\cosh K\frac{z}{h}\\ \sinh K\frac{z}{h}\end{matrix}\right\} - \frac{K\sinh K + \cosh K}{2K - \sinh 2K}\cdot\left.\begin{matrix}\sinh K\frac{z}{h}\\ \cosh K\frac{z}{h}\end{matrix}\right\} \\ &\left.\begin{matrix}h\varphi_1\\ h\varphi_1{}^*\end{matrix}\right\} = -\frac{K\sinh K}{2K + \sinh 2K}\cdot\left.\begin{matrix}\sinh K\frac{z}{h}\\ \cosh K\frac{z}{h}\end{matrix}\right\} + \frac{K\cosh K}{2K - \sinh 2K}\cdot\left.\begin{matrix}\cosh K\frac{z}{h}\\ \sinh K\frac{z}{h}\end{matrix}\right\} \end{aligned}\right\} \tag{7}$$

where $K = \kappa h$.

A particular solution σ_z^p of the equation (5) which contents the condition,

$$\left.\begin{aligned} &\sigma_z = 1, \quad \tau_{yz} = \tau_{zx} = 0 \quad \text{at} \quad z = h \\ &\sigma_z = 0, \quad \tau_{yz} = \tau_{zx} = 0 \quad \text{at} \quad z = -h \end{aligned}\right\} \tag{6'}$$

is
$$\sigma_z^p = (\varphi_0 + z\,\varphi_1)\,e^{i\lambda x + i\mu y}, \quad (\lambda^2 + \mu^2 = \tau^2) \tag{8}$$

From this equation and the equation (4)

$$\Theta^p = -(1+\nu)\frac{2}{\kappa}\,\varphi_1{}^*\,e^{i\lambda x + i\mu y} \tag{9}$$

Utilizing equations (3), (4), (8) and (9), we can get at first τ_{yz}^p and τ_{zx}^p satisfying the conditions (6′) and then have remaining components σ_x^p, σ_y^p, τ_{xy}^p; that is,

$$\left.\begin{aligned} \sigma_x{}^p &= -\frac{1}{\kappa^3}\{\lambda^2\kappa(\varphi_0 + z\,\varphi_1) + 2(\lambda^2 + \nu\,\mu^2)\,\varphi_1{}^*\}e^{i\lambda x + i\mu y} \\ \sigma_y{}^p &= -\frac{1}{\kappa^3}\{\mu^2\kappa(\varphi_0 + z\,\varphi_1) + 2(\mu^2 + \nu\,\lambda^2)\,\varphi_1{}^*\}e^{i\lambda x + i\mu y} \\ \sigma_z{}^p &= (\varphi_0 + z\,\varphi_1)\,e^{i\lambda x + i\mu y} \\ \tau_{yz}{}^p &= \frac{i\,\mu}{\kappa^2}\{\kappa(\varphi_0{}^* + z\,\varphi_1{}^*) + \varphi_1\}\,e^{i\lambda x + i\mu y} \\ \tau_{zx}{}^p &= \frac{i\,\lambda}{\kappa^2}\{\kappa(\varphi_0{}^* + z\,\varphi_1{}^*) + \varphi_1\{\,e^{i\lambda x + i\mu y} \\ \tau_{xy}{}^p &= -\frac{1}{\kappa^3}\{\lambda\,\mu\,\kappa(\varphi_0 + z\,\varphi_1) + 2(1-\nu)\,\lambda\,\mu\,\varphi_1{}^*\}\,e^{i\lambda x + i\mu y} \end{aligned}\right\} \tag{10}$$

The desired expressions of the each of stress-components σ_x, σ_y, $\cdots$ are, therefore, obtained making use of Fourier integral form,

$$\left.\begin{aligned}\sigma_x &= \frac{1}{(2\pi)^2}\int\int\int_0^{\infty}\int P(\alpha,\beta)\sigma_x{}^p\, e^{-i\lambda\alpha-i\mu\beta}\,d\lambda\,d\mu\,d\alpha\,d\beta \\ &= -\frac{1}{(2\pi)^2}\int_0^{\infty}\int_0^{\infty}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(R,\Phi)\{(\varphi_0+z\varphi_1)\cos^2\psi \\ &\qquad +\frac{2}{\kappa}\varphi_1{}^*(\cos^2\psi+\nu\sin^2\psi)\}\,e^{i\rho\kappa\cos(\varphi-\psi)}\,d\psi\,R\,(d\Phi\,dR)\,\kappa\,d\kappa \\ &\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\} \quad (11).$$

in which the following notations are used.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases},\quad \begin{cases}\lambda=\kappa\cos\psi \\ \mu=\kappa\sin\psi\end{cases},\quad \begin{cases}\alpha = R\cos\Phi \\ \beta = R\sin\Phi\end{cases},\quad \begin{cases} x-\alpha=\rho\cos\varphi \\ y-\beta=\rho\sin\varphi\end{cases} \quad (12).$$

Utilizing (10) and the formula

$$\int_{-\pi}^{\pi} e^{i\kappa\rho\cos(\varphi-\psi)}\begin{Bmatrix}\cos m\psi \\ \sin m\psi\end{Bmatrix}d\psi = 2\pi\, i^m\begin{Bmatrix}\cos m\varphi \\ \sin m\varphi\end{Bmatrix}J_m(\kappa\rho) \quad (13)^{(2)}$$

and transforming (11) into cylindrical-polar coordinates, we have,

$$\left.\begin{aligned}\sigma_r &= -\frac{1}{4\pi}\int_0^{\infty}(f_1+f_2)\,\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\,J_0(\kappa\rho)\,R(d\Phi\,dR) \\ &\quad +\frac{1}{4\pi}\int_0^{\infty}(f_1-f_2)\,\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\cos 2(\varphi-\theta)\,J_2(\kappa\rho)\,R(d\Phi\,dR), \\ \sigma_\theta &= -\frac{1}{4\pi}\int_0^{\infty}(f_1+f_2)\,\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\,J_0(\kappa\rho)\,R(d\Phi\,dR) \\ &\quad -\frac{1}{4\pi}\int_0^{\infty}(f_1-f_2)\,\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\cos 2(\varphi-\theta)\,J_2(\kappa\rho)\,R(d\Phi\,dR), \\ \sigma_z &= \frac{1}{2\pi}\int_0^{\infty}\left(f_1-\frac{1}{\nu}f_2\right)\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\,J_0(\kappa\rho)\,R\,(d\Phi\,dR), \\ \tau_{rz} &= -\frac{1}{2\pi}\int_0^{\infty}g_1\,\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\cos(\varphi-\theta)\,J_1(\kappa\rho)\,R(d\Phi\,dR), \\ \tau_{z\theta} &= -\frac{1}{2\pi}\int_0^{\infty}g_1\,\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\sin(\varphi-\theta)\,J_1(\kappa\rho)\,R(d\Phi\,dR), \\ \tau_{r\theta} &= \frac{1}{4\pi}\int_0^{\infty}(f_1-f_2)\,\kappa\,d\kappa\int_0^{\infty}\int_{-\pi}^{\pi}P(R,\Phi)\sin 2(\varphi-\theta)\,J_2(\kappa\rho)\,R(d\Phi\,dR),\end{aligned}\right\} \quad (14),$$

(2) Watson; Theory of Bessel Function p. 124.

where

$$\left.\begin{aligned} f_1 &= \varphi_0 + z\,\varphi_1 + \frac{2}{\kappa}\varphi_1{}^* \\ f_2 &= \frac{2}{\kappa}\nu\,\varphi_1{}^* \\ g_1 &= \varphi_0{}^* + z\,\varphi_0{}^* + \frac{1}{\kappa}\varphi_1\,, \qquad \rho = \sqrt{\{R^2 + r^2 - 2\,R\,r\,\cos(\Phi - \theta)\}} \\ &\varphi \text{ is the function of } r,\ \theta,\ R,\ \Phi \text{ as shown in (12).} \end{aligned}\right\}$$

These are the solutions corresponding to the loads given by (1).

3. To evaluate the above integrals we expand $J_\mu(\kappa\rho)$ in the integrands as

$$\frac{J_\mu(\kappa\rho)}{(\kappa\rho)^\mu} = \sum_{m=0}^{\infty} \epsilon_{m+\mu}\frac{J_{m+\mu}(\kappa R)}{(\kappa R)^\mu}\cdot\frac{J_{m+\mu}(\kappa r)}{(\kappa r)^\mu}\cdot\frac{d^\mu\cos\{(m+\mu)(\Phi-\theta)\}}{d\{\cos(\Phi-\theta)\}^\mu} \qquad (15)^{(3)},$$

where ϵ_n is defined to be equal to 2 when n is not zero and to be equal to 1 when n is zero, and integrate term by term about the variable Φ, expressing $P(R, \Phi)$ in series given by (2). The n th terms of the series calculated are, then,

$$\left.\begin{aligned}
\sigma_{rn} &= -\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\int_0^\infty f_1\,J_n(\kappa r)\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\kappa\,d\kappa \\
&\quad + \begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\int_0^\infty (f_1 - f_2)\Big\{n(n-1)\frac{J_n(\kappa r)}{\kappa r} + J_{n+1}(\kappa r)\Big\}\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\frac{\kappa\,d\kappa}{\kappa r}, \\
\sigma_{\theta n} &= -\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\int_0^\infty f_2\,J_n(\kappa r)\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\kappa\,d\kappa \\
&\quad - \begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\int_0^\infty (f_1 - f_2)\Big\{n(n-1)\frac{J_n(\kappa r)}{\kappa r} + J_{n+1}(\kappa r)\Big\}\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\frac{\kappa\,d\kappa}{\kappa r}, \\
\sigma_{zn} &= \begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\int_0^\infty \Big(f_1 - \frac{1}{\nu}f_2\Big)J_n(\kappa r)\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\kappa\,d\kappa, \\
\tau_{rzn} &= \begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\int_0^\infty g_1\Big\{n\frac{J_n(\kappa r)}{\kappa r} - J_{n+1}(\kappa r)\Big\}\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\kappa\,d\kappa, \\
\tau_{z\theta n} &= -n\begin{Bmatrix}\sin n\theta\\ -\cos n\theta\end{Bmatrix}\int_0^\infty g_1\frac{J_n(\kappa r)}{\kappa r}\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\kappa\,d\kappa, \\
\tau_{r\theta n} &= n\begin{Bmatrix}\sin n\theta\\ -\cos n\theta\end{Bmatrix}\int_0^\infty (f_1 - f_2)\Big\{(n-1)\frac{J_n(\kappa r)}{\kappa r} + J_{n+1}(\kappa r)\Big\}\begin{Bmatrix}G_{cn}(\kappa)\\ G_{sn}(\kappa)\end{Bmatrix}\frac{\kappa\,d\kappa}{\kappa r},
\end{aligned}\right\} \qquad (16),$$

where G_{cn} and G_{sn} are given by (2). Remaining integrals about κ can be easily evaluated by the usual method of summing up the residues of functions f_1, f_2 and g_1 in the upper or lower half of complex κ-plane.

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(3) Watson; *loc. cit.*, p. 362.