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ON THE ELASTIC FAILURE AND THE BUCKLING OF A COLUMN UNDER ECCENTRIC LOADS

By Shosaburo NEGORO

In this report, we treat the problem for the purpose of finding the effect of eccentricities at the both ends of loading in compression on the buckling of a column on the basis of the principle of Sunatani's[1] "buckling theory":— namely, we think that the column receives an elastic failure, that is, the buckling, when the greatest skin stress in the column reaches the stress at the yield point of the material. First, columns treated in the report are assumed to be prismatic bars of uniform cross-section and to be hinged at both ends. Next, utilizing author's "bending formulae of a strut under an axial load" already proposed, we find the greatest bending moments occurring in the columns and, from this moment and the bending formulae, the buckling formulae of columns loaded as follows:— namely, columns, the cross-sections of which are arbitrary forms, are loaded in such states as one of the principal axes of its cross-section exists in the plane of bending caused in the column by eccentric loads and columns, the cross-sections of which are circular forms, are loaded in such more general states as the principal axes of its cross-section are outside of the plane of bending stated above. In these formulae, needless to say, the eccentricities at the both ends are quite arbitrary with each other. Next, we find the relations between stains and slenderness ratios from the former buckling formulae stated above, in which we assume that the material of the column is mild steel with the yield stress $\sigma_e = 2.23 \times 10^3$ kg/cm^2 and Young's modulus $E = 2.08 \times 10^6$ kg/cm^2 and compare these curves with Euler's one. From the results we know that these curves approach to Euler's one gradually as the eccentricities are smaller and slenderness ratios are larger and, further, we see clearly that in the range of slenderness ratios of usual struts the effect of the eccentricities which generally take place in loading on the buckling of the column are still considerable.

After all, from the above descriptions we notice especially that it seems to be no connection indeed between the buckling stated in the report and usual one defined by stability theory well known as Euler's buckling, but there are never discrepancy between them, provided that we compare with each other in the same loading conditions without eccentricities at both ends, because the curves obtained above coincide with Euler's one as the eccentricities at the both ends approach to zero infinitely as shown in Fig. 3 and, moreover, that in the range of slenderness ratios of usual struts the effect of the eccentricities which generally take place in loading on the buckling of column are still considerable and, consequently, in actual designs of struts we need to consider these effects sufficiently.

Introduction. On the buckling of a column, many papers[(2)] have already been published, since Euler first pointed out a theory for it.

The author also treats the problem for the purpose of finding the effect of eccentricity of loading in compression on the buckling of a column on the basis of the principle of Sunatani's "buckling theory":—namely, we think that the column receives an elastic failure when the greatest skin stress in the column reaches the stress at the yield point of the material and regard these facts in this report as the definition of the buckling of the column.

Now, in the actual compression of a column, we always have unavoidable eccentricities of loading and we think that the influences of these eccentricities on the buckling of the column are considerable. But these eccentric quantities can not be measured generally. Therefore, we are necessary to find the effects of all the eccentricities on the buckling which we consider to have in actual compression of a column.

In the present report, first, utilizing author's "bending formulae of a strut under axial loads" already proposed, we find the greatest bending moments occurring in the columns loaded as follows:—the columns, the cross-sections of which are arbitrary forms, are loaded in such state as one of the principal axes of its cross-section exists in the plane of bending caused in the column by eccentric loads and the columns, the cross-sections of which are circular forms, are loaded in such more general state as the principal axes of the cross-section are outside of the plane of bending stated above and accordingly, from these moments and the bending formulae we find the buckling formulae of columns complying with the definition mentioned above.

I. Bending Formulae of a strut. From the curved beam theory, we find the bending formulae of a strut under axial loads as follows:—

$$\left.\begin{aligned} &H = EA\varepsilon, \quad \frac{1}{\rho} = M/(1+\varepsilon)EI, \quad \theta = \theta_0 + \int_0 \frac{M}{EI}ds, \\ &u = u_0 - \int_0 \{1-(1+\varepsilon)\cos\theta\}\,ds, \quad v = v_0 + \int_0 (1+\varepsilon)\sin\theta\,ds, \\ &x_1 = x+u, \quad y_1 = y+v, \quad f = \frac{H}{A} - \frac{M}{I}\zeta \end{aligned}\right\} \quad (1.1),$$

where H, M, f denote the axial force, the bending moment and the fibre stress respectively and ρ, θ, u, v, x_1, y_1 the radius of curvature, the inclination, the displacement-components in the directions of the x- and y-axes, and the positions, of the deflected axis of the loaded strut respectively and E, A, I, ε, Young's modulus, the area of the cross-section, moment of inertia of the cross-section and the normal strain of the axis of the bar and ζ represents the distance of a given point on the cross-section from the axis of the bar and takes positive or negative sign when the point exists in compression or tension side of the column.

II. Fundamental Principle of the Theory. In this report, we treat the buckling of a column which is hinged at the both ends.

(A) On such a case as one of the principal axes of the cross-section exists in the plane of bending.

(2.1) The Bending of a Prismatic Column.

Columns treated in this report are assumed to be prismatic bars of uniform cross-section and to be loaded freely at the ends in a plane containing the column axis and one of the principal axes of the cross-section as shown in Fig's 1 and 2. In these Fig's ACB and $AC'B$ represent the axes of a column before and after the loads are added and a, $\bar{a}$, and l, the eccentric quantities of loading at the lower and upper ends, the length of the column, and γ, θ_0, θ_l the inclination of the line of the loads to the initial axis ACB and the inclinations of the deflected axis at the both ends to the initial axis.

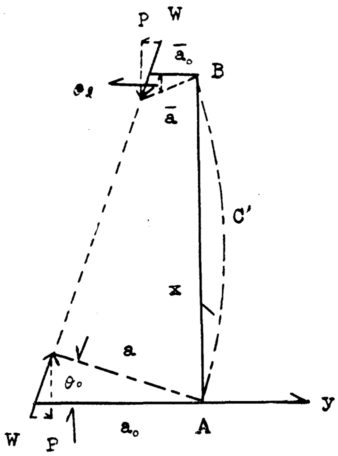

FIG. 1

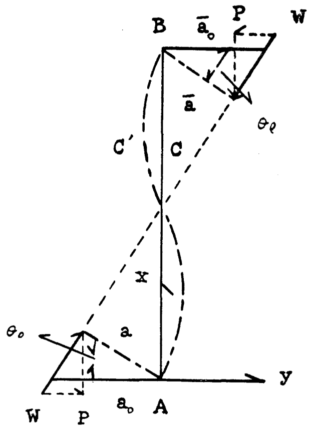

FIG. 2

Then, the bending moment induced at the cross-section c, its position being (x_1, y_1) apart from the lower end A, is given by the equations

$$M_{\underset{x}{\leftarrow}} = P(y_1 + a \cos\theta_0) - W \sin\gamma\,(x_1 - a \sin\theta_0)$$
$$= P(y_1 - c\,x_1) + M_0 ,$$

in which $M_{\underset{x}{\leftarrow}}$ denotes the bending moment at the negative side of the cross-section c, it being equal to $(-M_{\underset{x}{\rightarrow}})$, also

$$P = W \cos\gamma , \quad M_0 = P a_0 = P a\,\Theta_0 , \quad \Theta_0 = \cos\theta_0 + c \sin\theta_0 \doteqdot 1 + c\,\theta_0 ,$$
$$c = \tan\gamma$$

and M_0 stated above is the bending moment itself at the netative side of the lower end surface A of the column.

Further utilizing the 4th, the 5th, and the 6th equations of Eqs. (1.1), the above equation can be rewritten as follows:

$$M_{\underset{x}{\leftarrow}} = M_0 + P P_1 \left(\int_0 \theta_1 \alpha s - c s\right) \qquad (2.1),$$

in which $P_1 = (1 - P/E A)$.

As to the inclination of the deflected axis, from Eqs. (1.1) we have

$$\theta = \theta_0 + \int_0 \frac{M}{EI} ds \qquad (2.2).$$

Here, solving Eqs. (2.1) and (2.2) for θ and M by using the iteration method, we have the solutions

$$\left.\begin{aligned} \theta &= (\theta_0 - c)\cos\alpha s - \frac{M_0}{\alpha EI}\sin\alpha s + c \\ M &= \alpha EI(c - \theta_0)\sin\alpha s - M_0\cos\alpha s \end{aligned}\right\} \qquad (2.3),$$

in which $\alpha = \sqrt{P_1 P_2}$, $P_2 = P/EI$.

Hence, the displacement-components u and v are derived from the above equations and the 4th, the 5th, the 6th and the 7th equations of Eqs. (1.1) as follows:

$$\left.\begin{aligned} u &= x_1 - s = -Ps/EA \\ v &= y_1 = P_0(\theta_0 - c)\sin\alpha s - a_0(1 - \cos\alpha s) + cs \end{aligned}\right\} \qquad (2.4),$$

in which $P_0 = \sqrt{P_1/P_2}$.

After all, from the above results we know that the general solutions of the inclination θ, the bending moment M, and displacements u, v, required in the present problem are given by Eqs. (2.3) and (2.4) and the undetermined constants contained in these equations can be determined from the binding conditions at the both ends of the column.

(2.2) The Greatest Bending Moment within the Elastic Limit of a Column.

In this report, as stated previously we treat such a case as the both ends of a column are hinged.

In this case, as to the binding conditions at the both ends, we have

$$\left.\begin{aligned} s = 0 &: v_0 = 0, \quad M = -M_0 \\ s = l &: v_l = 0, \quad M = M_1. \end{aligned}\right\}.$$

First, it is clear that Eqs. (2.2) and (2.3) satisfy the above conditions at the lower end as they are and from the condition of displacement at the upper end and the 2nd equation of Eqs. (2.4), we have

$$\theta_0 = \left\{\left(P_0 c + a\tan\frac{p}{2}\right)\sin p - cl\right\}\Big/\left(P_0 - ac\tan\frac{p}{2}\right)\sin p \qquad (2.5_1),$$

in which $p = \alpha l = l\sqrt{P_1 P_2}$.

Then, from Eqs. (2.5_1) and the 1st equation of Eqs. (2.3), the inclination of the deflected axis of the loaded column at the upper end θ_l is given by the equation

$$\theta_l = \frac{1}{P_0 \sin p}\{(a_0 - cl)\cos p - a_0\} + c \qquad (2.5_2)$$

and applying Eqs. (2.5_l) also to the 2nd equation of Eqs. (2.3), bending moments can be expressed by the equation

$$M=\frac{P}{\sin p}[c\,l\ \sin p\,\xi-a_0\{\sin p\,\xi+\sin p\,(1-\xi)\}] \qquad (2.6),$$

it being $\xi=s/l$.

Hence, finding the bending moment (M_l) at the upper end from the above equation and the external force respectively, we have the following relation

$$\left.\begin{aligned} M_l &= P(c\,l-a_0)=-P\bar{a}_0 \\ \therefore\quad c\,l &= a_0-\bar{a}_0 \end{aligned}\right\} \qquad (2.7),$$

in which $\bar{a}_0=\bar{a}\,\Theta_l$, $\Theta_l=\cos\theta_l+c\,\sin\theta_l$ and we take the eccentric quantity $\bar{a}$ so as to be positive sign when the two eccentric quantities a and $\bar{a}$ at the ends are on the same side of the axis as shown in Fig. 1 and to be negative sign when a and $\bar{a}$ are in opposite sides of the axis as shown in Fig. 2. By the way, let us take a so as to be no smaller then $\bar{a}$ for the convenience of the following explanations and even if it does so, we have no restriction in the present problem.

Here, substituting Eqs. (2.7) for cl in Eqs. (2.6), as to the bending moment in the present problem, we have

$$M=-M_0\ \sin p\,\xi\,(\cot p\,\xi+\eta) \qquad (2.8),$$

in which $\eta=(\sin\nu-\cos p)/\sin p$ and $\sin\nu=\bar{a}_0/a_0$.

Further, let us find the maximum moment M_m and the position ξ_m of the cross-section received its moment. First, in consequence of the usual methods, differentiating Eqs. (2.8) as to ξ, we have

$$\left.\begin{aligned} \frac{\partial M}{\partial\xi} &= p\,M_0\,(\tan p\,\xi-\eta)\ \cos p\,\xi \\ \frac{\partial^2 M}{\partial\xi^2} &= p^2 M_0\,(1+\eta\ \tan p\,\xi)\ \cos p\,\xi \end{aligned}\right\} \qquad (2.9).$$

From these equations, we know the following facts.

(i) In such a case as p is not smaller than $p_0(=\pi/2-\nu)$:—

In this case, first, from Eqs. (2.8), we know that η is zero for $p=p_0$ and positive quantity for $p>p_0$. From these facts $\eta\geqq 0$ and the 1st equation of Eqs. (2.9), the position ξ_m of the cross-section, at which the bending moment is M_m, is given by the equations

$$\tan\varphi=\eta=(\sin\nu-\cos p)/\sin p\geqq 0 \qquad (2.10_1),$$

in which $\varphi=p\,\xi_m$.

And, moreover, utilizing the position ξ_m and Eqs. (2.10_1), the 2nd equation of Eqs. (2.9), becomes

$$\frac{\partial^2 M}{\partial\xi^2}=p^2M_0\,(1+\eta^2)\ \cos\varphi \qquad (2.10_2).$$

The Eqs. (2.10_2) tells us that the sign of $\partial^2 M/\partial\xi^2$ is determined by the sign of M_0 under the restriction $p<\pi$:— namely, as M_0 shown in the Fig's (1) and (2) is positive, the sign of $\partial^2 M/\partial\xi^2$ is positive. From this fact, we know that the moment M_m at the cross-section situated at the position ξ_m is the algebraic minimum value :— that is to say, the absolute value of M_m is maximum, because its moment M_m is negative.

After all, the above descriptions tell us that the position ξ_m of the cross-section, at which the absolute bending moment is maximum at the real part of the column, is given by Eq. (2.10_1) and its maximum moment M_m in absolute value is found from Eqs. (2.8) and (2.10_1) as follows :—

$$M_m = -M_0/\cos\varphi \qquad (2.11),$$

in which $\cos\varphi = \sin p\sqrt{4(1-\kappa)\sin^2\frac{p}{2}+\kappa^2}$, $\kappa = 1-\sin\nu$ for $p>p_0$ and $\cos\varphi=1$ for $p=p_0$.

(ii) In such a case as p is smaller than p_0.

In this case, first, from Eqs. (2.8) and the restriction $p<p_0$ stated above, η is negative quantity. Let us separate the present case again into the following two cases :— that is, $\nu\geqq 0$ and $\nu<0$.

(a) In such a case as $\nu\geqq 0$:—

In this case, it is clear that $p<p_0\leqq\pi/2$ from the definitions ($p_0=\pi/2-\nu$) and the restriction $\nu\geqq 0$. Here, adopting the above results to the 1st equation of Eqs. (2.9) and considering the fact that M_0 is positive, we have

$$\frac{\partial M}{\partial\xi}>0.$$

The above equation tells us that the absolute value of the bending moments occurring in the column go on decreasing from the end A towards the end B :— namely, the moment M_0 is the greatest bending moment.

(b) In such a case as $\nu<0$:—

First, adopting the position ξ_0 of the cross-section, at which the bending moment is zero, to Eqs. (2.8), we have

$$\eta = -\cot p\,\xi_0 = -\cos p\,\xi_0/\sin p\xi_0 < 0.$$

From the above equation, we know that $p\,\xi_0<\pi/2$ in the range of $p<\pi$. Moreover, from the above equation and Eqs. (2.8), we have

$$\sin\nu = -\sin p\,(1-\xi_0)/\sin p\,\xi_0.$$

Here, considering the above two equations with $\nu<0$,

$$\sin p\,\xi_0 \geqq \sin p\,(1-\xi_0)$$

and, moreover, considering the restriction $a_0 \not< \bar{a}_0$ together with the above equation, we have

$$\xi_0\geqq(1-\xi_0) \qquad \therefore\ \xi_0\geqq 1/2.$$

Further, adopting the above results to the 1st equation of Eqs. (2.9), we have

$$\frac{\partial M}{\partial\xi} = p\,M_0\frac{\cos p\,(\xi_0-\xi)}{\sin p\,\xi_0}>0.$$

From the above equation, in this case too we have the same results as the descriptions in the case (a) :— namely, the absolute values of the bending moments occurring in the column go on decreasing from the end A towards the end B.

After all, from the results stated in the both cases (a) and (b), we know that within the range of $p < p_0$, the moment M_0 at the end A is the greatest bending moment M_g at the real part of the column.

In conclusion, the above descriptions tell us that, within the range of $p < p_0(=\pi/2-\nu)$, M_0 is M_g and in the range of $p \geqq p_0$, M_m is M_g at the same time, so that the amount of M_m and its position ξ_m may be found from Eqs. (2.11) and Eqs. (2.10$_1$). Summarizing the above statements, the greatest bending moment can be found by the equation

$$M_g = -M_0/\cos\varphi \qquad (2.12),$$

where, within the range of $p \leqq p_0(=\pi/2-\nu)$, $\cos\varphi = 1$ and in the range $p > p_0$,

$$\cos\varphi = \sin p \Big/ \sqrt{4(1-\kappa)\sin^2\frac{p}{2} + \kappa^2}.$$

(2.3) The Elastic Failure and the Buckling of a Column.

As stated in the introduction, we think that the column receives an elastic failure when the greatest skin stress in the column becomes equal to the yield point of the material and regard the facts stated above as the definition of the buckling of the column.

Then, first the greatest skin stress (f_g) in the column can be easily found by substituting Eqs. (2.12) for M in the 8th equation of Eqs. (1.1) as follows :—

$$f_g = \sigma\{(\pm 1) + k_0 k_1^2 \Theta_0/\cos\varphi\} \qquad (2.13),$$

in which $\sigma = P/A$, $k_0 = a'/h'$ (= eccentricity), $k_1 = h'\lambda$ = the number determined by the form of the cross-section of the column (= 2 for circulur cross-section, or = 1.75 for rectangular cross-section), $a', h' = (a, h)/l$, $\lambda = l/i$ (= slenderness ratio), i = radius of gyration of the cross-section, $1 = A i^2$, h = the furthest distance from the axis of the column to its boundary and we take the positive sign of double ones in the equation when f_g denotes the greatest skin stress in the compression side of the column and the negative sign when f_g shows the stress in the tension side of the column.

From the above descriptions, we think that the column receives the buckling when the greatest skin stress expressed by Eqs. (2.13) becomes equal to the yield point of the material. Therefore, in the present case the buckling formulae are given as follows :—

(i) In the range of $p \leqq p_0(=\pi/2-\nu)$:—

$$(\sigma, e) = (\sigma_l, e_l)/\{(\pm 1) + k_0 k_1^2 \Theta_0\} \qquad (2.14_1).$$

(ii) In the range of $p > p_0$ —

$$(\sigma, e) = (\sigma_l, e_l) \Big/ \left\{ (\pm 1) + \frac{k_0 k_1^2}{\sin p} \Theta_0 \sqrt{4(1-\kappa)\sin^2\frac{p}{2} + \kappa^2} \right\} \quad (2.14_2),$$

in which $(e, e_l) = (\sigma, \sigma_l)/E$, σ_l = yield stress,

$$\Theta_0 \doteqdot 1 + (a')^2 \kappa(\kappa + p\eta), \quad p = \lambda\sqrt{e(1-e)},$$

$$\eta = (\sin\nu - \cos p)/\sin p, \quad \kappa = 1 - \sin\nu.$$

and as to the double signs, as well known already in Eqs. (2.13), we take the positive sign when the greatest skin stress in compression side of the column reaches first the yield point of the material and negative sign when the stress in tension side reaches first the yield point.

Here, putting $\nu = \pi/2$, that is to say $\bar{a}_0/a_0 = 1$, and assuming that the greatest skin stress in compression side of the column reaches first to yield point of the material, the 2nd equations of the formulae can be rewritten as follows:

$$(\sigma, e) = (\sigma_l, e_l) \Big/ \left(1 + k_0 k_1^2 \sec\frac{p}{2}\right),$$

and these equations are the well-known buckling formulae already proposed by Smith.[3]

Further, from the above equations, we find the following relations approximately

$$\left. \begin{aligned} & e \doteqdot (p/\lambda)^2 \\ & \lambda \doteqdot \frac{p}{\sqrt{e_l}} \sqrt{(\pm 1) + \frac{k_0 k_1^2}{\sin p} \left\{ 4(1-\kappa)\sin^2\frac{p}{2} + \kappa^2 \right\}^{1/2}} \end{aligned} \right\} (2.14').$$

In conclusion, from the above results, we know that from Eqs. (2.14) the buckling load for any slenderness ratio can be found by the trial method, and from Eqs. (2.14') the relations are found directly with some approximations, if only the quantities of any one of these pairs (a', c), $(a, \bar{a}')$, $(a'_0, \bar{a}'_0)$, and $(a', \bar{a}_0/a_0)$ are given and accordingly, the present problems are solved.

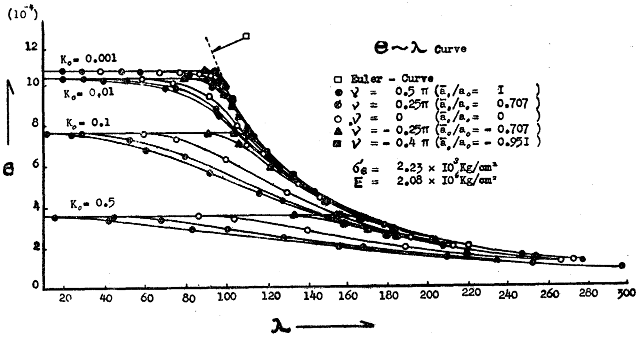

FIG. 3

Fig. 3 shows us the $e \sim \lambda$ curves found from Fqs. (2.14), in which assuming that the material of the column is mild steel with $\sigma_l = 2.23 \times 10^3$ kg/cm² and $E = 2.08 \times 10^6$ kg/cm², we calculate for the twenty cases $\nu = 0.5\pi$, 0.25π, 0, -0.25π and -0.4π, that is to say $\bar{a}_0/a_0=1$, 0.707, 0, -0.707 & -0.955 at the values $k_0=0.5$, 0.1, 0.01 and 0.001 respectively. The curve marked with ◎ denotes the Euler's one. From this figure we know that the curves obtained above approach to Euler's one gradually as the eccentricities are smaller and slenderness ratios are larger and, moreover, we see clearly that the effect of unavoidable eccentricities in actual compressions on the buckling are considerable. After all, from the results we notice especially that it seems to be no continuation indeed between the buckling stated in this report and usual one defined by stability theory well known as Euler's buckling, but there are never discrepancy between them, provided that we compare with each other in the same loading conditions without eccentricities at both ends, because the curves obtained above coincide with Euler's one as the eccentricities at the both ends approach to zero infinitely as shown in Fig. 3 and, moreover, that in the range of slenderness ratios of usual struts the effect of the eccentricities which generally take place in loading on the buckling of the column are still considerable and consequently, in actual designs of struts we need to consider these effects sufficiently.

(B) On such a case as the principal axes of the cross-section are outside of the plane of bending.

(2.4) The moments of a prismatic column.

Columns treated in the present report are assumed to be prismatic bars of uniform cross-section and to be loaded freely at the ends in such a state

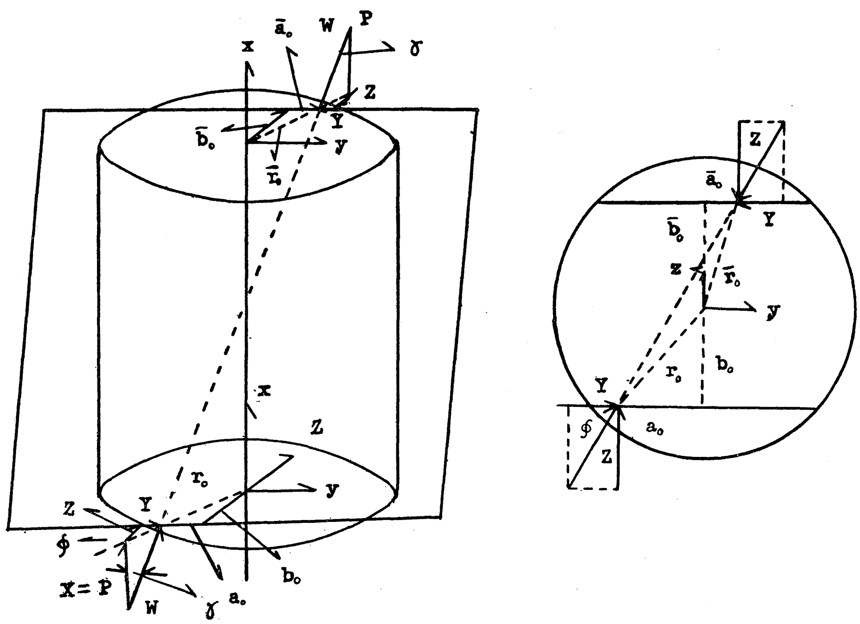

Fig. 4

as the principal axes of the cross-section of the column are out-side of the plane of bending caused in the column by eccentric loads.

Now, let us consider a plane which is parallel to one of the principal axes of the cross-section and contains the line of the loads and take x-, y- and z-axes in the directions of the axis of the column and the principal axes of the cross-section respectively. Then, the components ($X = P$, Y, Z) of the external force W in the directions of x-, y- and z-axes are given by the equations

$$P = X = W \cos\gamma\,, \quad Y = W \sin\gamma \cos\phi\,, \quad Z = W \sin\gamma \sin\phi \qquad (2.15)\,,$$

in which γ denotes the inclination of the line of the loads to the x-axis, and ϕ the inclination of the projecting line of the loading line to the lower end surface to the y-axis.

Here, first, from the Fig. (4), the twisting moment $T_{\underset{x}{\rightarrow}}$ about the x-axis is given by the equation

$$T_{\underset{x}{\rightarrow}} = Z \cdot a_0 - Y \cdot b_0 = P(a_0 c' - b_0 c) \qquad (2.16)\,,$$

in which $c = \tan\gamma \cos\phi\,, \quad c' = \tan\gamma \sin\phi\,,$

$$a_0 = a(\cos\theta_{z0} + c \sin\theta_{z0}) = a\,\Theta_{z0}\,, \quad b_0 = b(\cos\theta'_{y0} + c' \sin\theta'_{y0}) = b\,\Theta'_{y0},$$

$$\theta'_{y0} = -\theta_{y0}$$

and a, b denote the components of the eccentric quantity (r) of loading at the lower end surface in the directions of y- and z-axes respectively and θ_{y0}, θ_{z0} the components of the inclinations of the deflected axis at the lower end to the initial axis in the directions of y- and z-axes respectively.

Moreover, from the condition of the equilibrium of twisting moment about x-axis

$$T_{\underset{0}{\leftarrow}} + T_{\underset{1}{\rightarrow}} = Y(b_0 - \bar{b}_0) - Z(a_0 - \bar{a}_0) = 0$$

$$\therefore \quad Z/Y = (b_0 - \bar{b}_0)/(a_0 - \bar{a}_0) = \tan\phi \qquad (2.17)\,,$$

in which
$$\bar{a}_0 = \bar{a}(\cos\theta_{z1} + c \sin\theta_{z1}) = \bar{a}\,\Theta_{z1}\,,$$

$$\bar{b}_0 = \bar{b}(\cos\theta'_{y1} + c' \sin\theta'_{y1})\,\bar{b}\,\Theta'_{y1},$$

and $\bar{a}$, $\bar{b}$ denote the components of the eccentric quantity ($\bar{r}$) of loading at the upper end surface in the directions of y- and z-axes respectively, signs of which are determined by their positions referring to the axis of the column :— namely, $\bar{a}$, $\bar{b}$ are positive quantities when these eccentric quantities ($\bar{a}$, $\bar{b}$) and (a, b) are on the same side of the axis of the column respectively and negative quantities when these eccentric quantities are in the opposite sides of the axis respectely.

Next, as to the inclinations (θ_{zx}, θ_{yx}) and the bending moments (M_{zx}, M_{yx}) referring to y- and z-axes, we find directly from the results stated in the previous case (A) as follows :—

Namely θ_{zx} and M_{zx} referring to the z-axis :—

$$\left.\begin{aligned} \theta_{xz} &= (\theta_{0z} - c)\cos\alpha s - \frac{M_{0z}}{\alpha E I_z}\sin\alpha s + c \\ M_{\underset{xz}{\rightarrow}} &= \alpha E I_z (c - \theta_{0z})\sin\alpha s - M_{0z}\cos\alpha s \end{aligned}\right\} \quad (2.18_1),$$

in which $\alpha = \sqrt{P_1 P_2}$, $P_1 = \left(1 - \frac{P}{EA}\right)$, $P_2 = P/EI_z$.

And θ'_{yx} & M'_{xy} referring to the y-axis

$$\left.\begin{aligned} \theta'_{xy} &= (\theta'_{0y} - c')\cos\alpha' s - \frac{M'_{0y}}{\alpha' E I_y}\sin\alpha' s + c' \\ M'_{xy} &= \alpha' E I_y (c' - \theta'_{0y})\sin\alpha' s - M'_{0y}\cos\alpha' s \end{aligned}\right\} \quad (2.18_2),$$

in which $\alpha' = \sqrt{P_1 P_2'}$, $P_2' = P/EI_y$.

As well known in the previous case (A), the above equations are general solutions of the inclinations and the bending moments required in the present case and the undetermined constants contained in these equations are found from the binding conditions at the both ends of the column.

(2.5) The Greatest Bending Moment within the Elastic Limit of a Column.

As stated in the previous parts, in the present report, we treat such a case as the both ends of a column are hinged.

First, similarly as the previous methods stated in the section (2.2), we find the inclinations (θ_{0z}, θ_{0y}) and the bending moments ($M_{\underset{xz}{\rightarrow}}$, $M'_{\underset{xy}{\rightarrow}}$).

Namely, applying the binding conditions at the both ends to the 1st equations of Eqs. (2.18_1) and (2.18_2), we have

$$\left.\begin{aligned} \theta_{0z} &= \left\{\left(P_0 c + a\tan\frac{p}{2}\right)\sin p - c\,l\right\}\Big/\left(P_0 - a\,c\tan\frac{p}{2}\right)\sin p \\ \theta'_{0y} &= \left\{\left(P_0' c' + b\tan\frac{p'}{2}\right)\sin p' - c' l\right\}\Big/\left(P_0' - b\,c'\tan\frac{p'}{2}\right)\sin p' \end{aligned}\right\} \quad (2.19),$$

in which

$$P_0 = \sqrt{P_1/P_2} \quad P_0' = \sqrt{P_1/P_2'}, \quad p = \alpha l = l\sqrt{P_1 P_2},$$
$$p' = \alpha' l = l\sqrt{P_1 P_2'}.$$

And, substituting the above equations for θ_{0z} and θ'_{0y} in Eqs. (2.18), we have

$$\left.\begin{aligned} M_{\underset{xz}{\rightarrow}} &= \frac{P}{\sin p}\left[c\,l\sin p\,\xi - a_0\{\sin p\,\xi + \sin p\,(1-\xi)\}\right] \\ M'_{\underset{xy}{\rightarrow}} &= \frac{P}{\sin p'}\left[c' l\sin p'\xi - b_0\{\sin p'\xi + \sin p'(1-\xi)\}\right] \end{aligned}\right\} \quad (2.20),$$

in which $\xi = s/l$.

Further, finding the bending moment (M_{1z}, M'_{1y}) at the upper end from the above equations and the external force respectively, we have the following relations

$$M_{1z} = P(c\,l - a_0) = -P\bar{a}_0$$
$$M'_{1y} = P(c'l - b_0) = -P\bar{b}_0$$
$$\therefore\quad c\,l = a_0 - \bar{a}_0\,, \qquad c'l = b_0 - \bar{b}_0 \qquad (2.21)\,.$$

Here, substituting the above equations for $c\,l$ and $c'l$ in Eqs. (2.20), we have

$$\left.\begin{aligned} M_{\underset{xz}{\rightarrow}} &= -M_{0z}\sin p\,\xi\,(\cot p\,\xi + \eta_0) && a_0 \geqq |\bar{a}_0| \\ &= \quad M_{1z}\sin p\,\xi'(\cot p\,\xi' + \eta_1) && a_0 \leqq |\bar{a}_0| \\ M'_{\underset{xy}{\rightarrow}} &= -M'_{0y}\sin p'\xi\,(\cot p'\xi + \eta_0') && b_0 \geqq |\bar{b}_0| \end{aligned}\right\} (2.22)\,,$$

in which $\xi' = (1-\xi)\,,\quad \eta_0 = (\sin\nu_0 - \cos p)/\sin p,$

$\eta_1 = (\sin\nu_1 - \cos p)/\sin p\,,\quad \eta_0' = (\sin\nu_0' - \cos p')/\sin p'\,,$

$\sin\nu_0 = \bar{a}_0/a_0\,,\quad \sin\nu_1 = a_0/\bar{a}_0\,,\quad \sin\nu'_0 = \bar{b}_0/b_0$

and even if we put the assumptions $a_0 \geqq |\bar{a}_0|$ or $a_0 \leqq |\bar{a}_0|$ and $b_0 \geqq |\bar{b}_0|$ as described in the above equations, we have no restriction in the present problem.

The above equations are the bending formulae required in the present case. Here, for convenience of the latter treatments, let us take a column with circular-section.

Then, from Eqs. (2.22), the resultant bending moment M_x at the cross-section situated at its position x is given by the equation

$$\begin{aligned} M_x^2 &= M^2_{\underset{yx}{\rightarrow}} + M^2_{\underset{zx}{\rightarrow}} \\ &= M_0^2\left\{1 - \kappa\,\frac{\sin^2 p\,\xi}{\sin^2 p} + 2\,\eta_{01}\sin p\,\xi\,\frac{\sin p\,(1-\xi)}{\sin p}\right\} \end{aligned} \qquad (2.23)\,,$$

in which $\eta_{01} = (\sin\nu_{01} - \cos p)/\sin p\,,\quad \sin\nu_{01} = \dfrac{1}{r_0^2}(a_0\,\bar{a}_0 + b_0\,\bar{b}_0)\,,$

$M_0 = P\,r_0\,,\quad \kappa = (1 - \bar{r}_0^2/r_0^2)\,,\quad r \geqq |\bar{r}_0|$

and it is clear that $\eta_{01} \gtreqqless 0$ for $p \gtreqqless p_0\,(=\pi/2 - \nu_{01})$.

Moreover, for finding the maximum bending moment M_m in the column, differentiating the above equation as to ξ, we have

$$\left.\begin{aligned} \frac{\partial(M^2)}{\partial\xi} &= \quad -p\,M_0^2\left\{(\kappa + \eta_{01}\sin 2\,p)\,\frac{\sin 2\,p\,\xi}{\sin^2 p} - 2\,\eta_{01}\cos 2\,p\,\xi\right\} \\ \frac{\partial^2(M^2)}{\partial\xi^2} &= -2\,p^2\,M_0^2\left\{(\kappa + \eta_{01}\sin 2\,p)\,\frac{\cos 2\,p\,\xi}{\sin^2 p} + 2\,\eta_{01}\sin 2\,p\,\xi\right\} \end{aligned}\right\} (2.24)\,.$$

From the 1st equation of Eqs. (2.24), the positions ξ_m of the cross-section, at which the bending moment is extreme value, is given by the equation

$$\tan 2\,\varphi = 2\,\eta_{01}\sin^2 p/(\kappa + \eta_{01}\sin 2\,p) \qquad (2.25)$$

and the extreme value M_m is found from Eqs. (2.23) and (2.25) as follows:—

$$M_m^2 = M_0^2(1 + \eta_{01} \tan \varphi) \qquad (2.26).$$

Here, utilizing the position ξ_m and Eqs. (2.25), the 2nd equation of Eqs. (2.24) becomes

$$\frac{\partial^2(M^2)}{\partial \xi^2} = -\frac{p^2 M_0^2}{\eta_{01}}\left\{\left(\frac{\kappa + \eta_{01} \sin 2p}{\sin^2 p}\right)^2 + (2\eta_{01})^2\right\} \sin 2\varphi$$

and, accordingly, becomes

$$\frac{\partial^2(M^2)}{\partial \xi^2} < 0,$$

because it is clear that $\eta_{01} > 0$ and, considering that ξ_m is always smaller than 1/2 under the assumption $r_0 \geqq |\bar{r}_0|$ as stated in the previous section A, $\sin 2\varphi > 0$ in the range of p as follows:—$\pi > p > p_0$. Therefore, we know that the extreme value is maximum at the real part of the column.

Next, putting $p = p_0(=\pi/2 - \nu_{01})$ and substituting p_0 for p in Eqs. (2.23) and (2.25), we have

$$\tan 2\varphi = 0, \quad (\eta_{01} = 0) \qquad \therefore \quad \xi_m = 0.$$

Namely, the moment M_0 at the lower end is maximum in such a case when $p = p_0$.

Moreover, in the range of $p < p_0$, considering Eqs. (2.23) with the facts ($p < p_0 < \pi$ and $\eta_{01} < 0$), we know easily that the moment M_0 is the greatest value at the real part of the column.

In conclusion, the above descriptions tell us that the greatest bending moment M_g is given by the equation

$$M_g = -M_0\sqrt{1 + \eta_{01} \tan \varphi} \qquad (2.27),$$

in which
in the range of $\quad p \leqq p_0 (= \pi/2 - \nu_{01}), \quad \xi_m = 0 \quad$ and
in the range of $\quad p > p_0, \quad \tan 2\varphi = 2\eta_{01} \sin^2 p/(\kappa + \eta_{01} \sin 2p)$.

(2.6) The Elastic Failure and the Buckling of a Column.

As stated in the introduction, we think that a column receives the elastic failure, that is to say, the buckling when the greatest skin stress in the column reaches the yield point of the material.

First, dividing the external forces acting on the column into the twisting moment about x-axis and force-components X, Y and Z as stated in the sections (2.4) and (2.5), we see that the plane of bending induced by these components X, Y and Z contains the axis of the column itself and has the inclination ϕ to the Y-axis.

Then, similarly as the descriptions in the section (2.3), the greatest skin stress (f_g) in the column can be easily found from the 8th equation of Eqs. (1.1) and Eqs. (2.27) as follows:—

$$f_g = \sigma\{(\pm 1) + k_{01} k_1^2 \Theta_0 \sqrt{1+\eta_{01} \tan\varphi}\} \qquad (2.28),$$

in which $k_{01} = r'/h'$ = eccentricity,

$k_1 = h'\lambda$ = the number determined by the form of the cross-section of the column as stated in the section (2.3),

$\tan\varphi$ = the term already known in Eqs. (2.27).

And the shear stress occurring on the column by the twisting moment $T_{\vec{x}}$ is found from Eqs. (2.16) and the relative formula between shear stress and twisting moment by the equation

$$\tau = \frac{2}{h}\sigma(a_0 c' - b_0 c) \qquad (2.29).$$

Here, let us find the principal stresses at the point which is in the stress states expressed by Eqs. (2.28) and (2.29).

First, as the boundary surface of the column is the principal plane referring to stress states as it is and its principal value (σ_2) is zero, we know easily that the other two principal stresses (σ_1 and σ_3) are found by using the theory of the stress quardric expressing the stress states at any point in the elastic body as follows:

$$\sigma_{1,3} = \frac{1}{2}\{f_g \pm \sqrt{f_g^2 + 4\tau^2}\} \qquad (2.30_1),$$

in which f_g and τ are the stresses expressed by Eqs. (2.28) and (2.29) respectively. Moreover, the maximum shear stress τ_m at its point is found directly from the considerations of Mohr's diagram expressing the stress states as follows:

$$\tau_m = \frac{1}{2}\sqrt{f_g^2 + 4\tau^2} \qquad (2.30_2).$$

Here, as stated in the previous parts, we think also that the column receives the buckling when the greatest stress expressed by Eqs. (2.30_1) or (2.30_2) becomes equal to the yield point of the material.

Then, in the present cases the formulae of the buckling are given as follows:

$$\left.\begin{aligned}(\sigma, e) &= 2(\sigma_l, e_l)/\{F_1 + \sqrt{F_1^2 + 4F_2^2}\}\\ \sigma &= 2\tau_l/\sqrt{F_1^2+4F_2^2}\end{aligned}\right\} \qquad (2.31),$$

in which $F_1 = (\pm 1) + k_{01} k_1^2 \Theta_0 F_3, \quad F_3 = \sqrt{1+\eta_{01}\tan\varphi}, \quad \varphi = p\,\xi_m,$

$$F_2 = \frac{2}{h}(a_0 c' - b_0 c) \doteqdot 2k_0 k'_0 \frac{k_l}{\lambda}(\kappa'_0 - \kappa_0), \quad \kappa_0 = 1 - \sin\nu_0,$$

$\kappa'_0 = 1 - \sin\nu'_0,$

$$\left.\begin{aligned}&p \leqq p_0 (= \pi/2 - \nu_{0l}): \quad \varphi = 0, \quad (\xi_m = 0)\\ &p > p_0: \quad \tan 2\varphi = 2\eta_{01}\sin^2 p/(\kappa + \eta_{01}\sin 2p),\end{aligned}\right\}$$

$$\Theta_0 = \frac{1}{r'}\sqrt{(k_0\,\Theta_{z0})^2 + (k'_0\,\Theta'_{y0})^2}\,, \quad k_0 = a/h\,, \quad k_0' = b/h\,,$$

$$\Theta_{0z} = 1 + a'^2\,\kappa_0\,(\kappa_0 + \eta_0\,p)\,, \quad \Theta'_{y0} = 1 + b'^2\,\kappa'_0\,(\kappa'_0 + \eta'_0 p)\,,$$

$$\eta_0 = (\sin\nu_0 - \cos p)/\sin p\,, \quad \eta_0' = (\sin\nu_0' - \cos p)/\sin p\,,$$

$$p = \lambda\sqrt{e\,(1-e)}$$

and we take the positive one of double signs in the definition of the function F_1 in such a case as the greatest skin stress in the compression side of the column becomes equal to the yield point of the material and the nagative sign in such a case as the greatest skin stress in the tension side of the column becomes equal to the yield point of the material.

In the above results, we have only to take one of Eqs. (2.31) corresponding to such cases as the column receives the buckling owing to compressive stress, tensile stress or shear stress expressed by Eqs. (2.30). Further, from the above equation, the buckling loads $P\,(e = P/EA)$ and their slenderness ratioes (λ) are given approximately by the following equations

$$\left.\begin{array}{ll} \left.\begin{array}{ll} \lambda \doteqdot \dfrac{p}{\sqrt{2\,e_l}}\sqrt{F_1 + (F_1^2 + 4\,F_2^2)^{1/2}} & \text{for normal stress} \\ \lambda \doteqdot \dfrac{p}{\sqrt{2\,\tau_l/E}}(F_1^2 + 4\,F_2^2)^{1/4} & \text{for shear stress} \end{array}\right\}, \\ e \doteqdot (p/\lambda) \end{array}\right\} \quad (2.31')\,.$$

In conclusion, from the above results, we know that from Eqs. (2.31) the buckling load for any slenderness ratio can be found by using the trial method and from Eqs. (2.31′) the relations are found directly with some approximations, if only the quantities of group $(a', b', \bar{a}_0/a_0\ \&\ \bar{b}_0/b_0)$, $(k_0, k'_0, \nu_0\ \&\ \nu'_0)$ etc. are given, and, accordingly, the present problems are solved.

Conculsion. In this report, as stated in the introduction, we think so that the column receives buckling when one of these stresses:— that is, the greatest skin stress, the greatest principal stress and the greatest shear stress, reaches the yield point of the material and treat the buckling of a column which is prismatic bar of uniform cross-section and is hinged at the both ends.

First, we find the buckling formulae of columns loaded as follows:— namely, columns, the cross-section of which are arbitrary forms, are loaded in such loading states as one of the principal axes of its cross-section exists in the plane of bending caused in the column by eccentric loads and columns, the cross-sections of which are circular forms, are loaded in such more general loading states as the principal axes of its cross-section are outside of the plane of bending stated above. In these formulae, needless to say, the eccentricities at the both ends are quite arbitrary with each other.

Next, we find the relations between strains $\left(e=\frac{P}{EA}\right)$ and slenderness ratios λ :— that is $e \sim \lambda$ curves, from the former buckling formulae stated above for the twenty cases :— namely, the ratios of eccentric quantities at the both ends $\bar{a}_0/a_0 = 1,\ 0.707,\ 0,\ -0.707$ and -0.955 with the eccentric values at the lower end surface $k_0 = 0.5,\ 0.1,\ 0.01$ and 0.001 respectively, in which we assume that the material of the column is mild steel with $\sigma_l = 2.23 \times 10^3\ \mathrm{kg/cm^2}$ and $E = 2.08 \times 10^6\ \mathrm{kg/cm^2}$ and compare these curves with the Euler's one.

From the results we know that these curves approach to Euler's one gradually as the eccentricities are smaller and slenderness ratios are larger and, further, see clearly that in the range of slenderness ratioes of usual struts the effect of the eccentricities which generally take place in loading on the buckling of the column are still considerable. After all, from the above descriptions we intend to notice especially that it seems to be no connection indeed between the buckling stated in the report and usual one defined by stability theory well known as Euler's buckling, but there are never discrepancy between them, provided that we compare with each other in the same loading conditions without eccentricities at both ends, because the curves obtained above coincide with Euler's one as the eccentricities at the both ends approach to zero infinitely as shown in Fig. 3 and, moreover, that in the range of slenderness ratios of usual struts the effect of the eccentricities which generally take place in loading on the buckling of column are still considerable and, consequently, in actual designs of struts we need to consider these effects sufficiently.

Last, we are still in the courses of calculating the relative curves for such cases as the column reveives eccentric loads at quite arbitrary positions on the both end surfaces by using the latter buckling formulae stated above and of finding the buckling formulae of the columns under the other binding conditions except hinged ones at the both ends of the column described in the report and, moreover, intend to research the plastic buckling of the columns.

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