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NOTE

Approximate Calculations of Laminar Jets

(Supplementary Note)

By Jun-ichi OKABE

1. A few months ago in a private letter to Professor Yamada of our institute, Mr. A. L. Thomas in Princeton University, New Jersey, U.S.A., wished to know about the accuracy of the approximation employed in my paper "Approximate Calculations of Laminar Jets (*continued*)," *Reports of Research Institute for Fluid Engineering*, vol. 5, no. 1 (1948). Because some of the readers of my paper might be interested in the error estimation of the method, it may be worthwhile to publish the result of the calculation.

2. In what follows we shall confine ourselves to the case of the plane jet. x is measured from the nozzle along its axis, y perpendicular to it. λ is a function of x only, and $2a$ is the breadth of the mouth $(x=0)$ from which the jet is discharged with uniform velocity U, spreading laterally within the outer boundary defined by the equation $y=\Delta(x)$. We write $\eta=y/\Delta$ for shortness; by virtue of symmetry with respect to the x-axis we have only to consider the region $0\leqq y\leqq \Delta$, i.e. $0\leqq\eta\leqq 1$.

Neglecting the pressure gradient, which is presumably very small, we can write down the equation of motion as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

where u and v are the x-, and the y- components of velocity, respectively, ν the kinematic viscosity of the fluid. Writing

$$\theta=u/U,\quad \tau=v/U,$$

and

$$\chi=x/a,\quad \gamma=y/a, \tag{2.2}$$

we obtain

$$\theta\frac{\partial\theta}{\partial\chi}+\tau\frac{\partial\theta}{\partial\gamma}=\frac{1}{R}\frac{\partial^2\theta}{\partial\gamma^2}, \tag{2.3}$$

where $R=Ua/\nu$ is the Reynolds number. Next, if we perform the transformation from the (χ,γ) system to the (λ,η) system, taking into consideration the relation

$$\gamma=\frac{y}{a}=\frac{y}{\Delta}\cdot\frac{\Delta}{a}\equiv\eta\cdot\delta, \tag{2.4}$$

we can easily prove the formulas

$$\frac{\partial}{\partial\chi}=\frac{d\lambda}{d\chi}\left(\frac{\partial}{\partial\lambda}-\frac{d\delta}{d\lambda}\frac{\eta}{\delta}\frac{\partial}{\partial\eta}\right),$$

and

$$\frac{\partial}{\partial\gamma}=\frac{1}{\delta}\frac{\partial}{\partial\eta}. \tag{2.5}$$

(2.3) becomes, therefore,

$$\theta \frac{d\lambda}{d\chi}\left(\frac{\partial \theta}{\partial \lambda}-\frac{d\delta}{d\lambda}\frac{\eta}{\delta}\frac{\partial \theta}{\partial \eta}\right)+\frac{\tau}{\delta}\frac{\partial \theta}{\partial \eta}=\frac{1}{R}\frac{1}{\delta^2}\frac{\partial^2 \theta}{\partial \eta^2}. \tag{2. 6}$$

In order to estimate the error involved in our approximate solution proposed in the paper mentioned above, we shall calculate the quantity Z defined by

$$Z \equiv \theta \frac{d\lambda}{d\chi}\left(\frac{\partial \theta}{\partial \lambda}-\frac{d\delta}{d\lambda}\frac{\eta}{\delta}\frac{\partial \theta}{\partial \eta}\right)+\frac{\tau}{\delta}\frac{\partial \theta}{\partial \eta}-\frac{1}{R}\frac{1}{\delta^2}\frac{\partial^2 \theta}{\partial \eta^2} \tag{2. 7}$$

at several points on a section $x=$ constant (i.e. $\lambda=$ constant). When $Z=0$, obviously the equation of motion (2.6) is found to be satisfied *exactly*.

In the original paper, see (2.1) *loc. cit.*, u or θ is assumed in the form

$$\theta \equiv u/U = (1-e^{-(1-\eta)/\lambda})^3+\frac{3}{\lambda}e^{-1/\lambda}(1-e^{-1/\lambda})^2\eta-\frac{9}{\lambda}e^{-1/\lambda}(1-e^{-1/\lambda})^2\eta^2$$

$$+\frac{9}{\lambda}e^{-1/\lambda}(1-e^{-1/\lambda})^2\eta^3-\frac{3}{\lambda}e^{-1/\lambda}(1-e^{-1/\lambda})^2\eta^4. \tag{2. 8}$$

From u, through the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2. 9}$$

or its transformed form, cf. (2.5),

$$\frac{d\lambda}{d\chi}\left(\frac{\partial \theta}{\partial \lambda}-\frac{d\delta}{d\lambda}\frac{\eta}{\delta}\frac{\partial \theta}{\partial \eta}\right)+\frac{1}{\delta}\frac{\partial \tau}{\partial \eta}=0, \tag{2.10}$$

τ is found to be

$$\tau=\frac{1}{R}\frac{d\lambda}{d\xi}\left\{\frac{d\delta}{d\lambda}(\eta\,\theta-\int_0^\eta \theta\, d\eta)-\delta\frac{\partial}{\partial \lambda}\int_0^\eta \theta\, d\eta\right\}, \tag{2.11}$$

where

$$\xi=\chi/R. \tag{2.12}$$

Therefore Z defined by (2.7) can be obtained in the form

$$RZ=\theta\frac{d\lambda}{d\xi}\left(\frac{\partial \theta}{\partial \lambda}-\frac{d\delta}{d\lambda}\frac{\eta}{\delta}\frac{\partial \theta}{\partial \eta}\right)+\frac{1}{\delta}\frac{d\lambda}{d\xi}\left\{\frac{d\delta}{d\lambda}(\eta\,\theta-\int_0^\eta \theta\, d\eta)\right.$$

$$\left.-\delta\frac{\partial}{\partial \lambda}\int_0^\eta \theta\, d\eta\right\}\frac{\partial \theta}{\partial \eta}-\frac{1}{\delta^2}\frac{\partial^2 \theta}{\partial \eta^2}. \tag{2.13}$$

Making use of the expression of θ given by (2.8), quantities appearing in (2.13) can be evaluated explicitly; namely

$$\frac{\partial \theta}{\partial \lambda}=-\frac{3}{\lambda^2}\Big[e^{-(1-\eta)/\lambda}(1-e^{-(1-\eta)/\lambda})^2(1-\eta)$$

$$+e^{-1/\lambda}(1-e^{-1/\lambda})\left\{1-e^{-1/\lambda}+\frac{1}{\lambda}(-1+3e^{-1/\lambda})\right\}\eta(1-\eta)^3\Big], \tag{2.14}$$

$$\frac{\partial \theta}{\partial \eta}=-\frac{3}{\lambda}e^{-(1-\eta)/\lambda}(1-e^{-(1-\eta)/\lambda})^2$$

$$+\frac{3}{\lambda}e^{-1/\lambda}(1-e^{-1/\lambda})^2(1-6\eta+9\eta^2-4\eta^3), \tag{2.15}$$

$$\int_0^\eta \theta\, d\eta=\eta-3\lambda(e^{-(1-\eta)/\lambda}-e^{-1/\lambda})$$

$$+\frac{3}{2}\lambda\,(e^{-2(1-\eta)/\lambda}-e^{-2/\lambda})-\frac{\lambda}{3}(e^{-3(1-\eta)/\lambda}-e^{-3/\lambda})$$

$$+\frac{1}{\lambda}e^{-1/\lambda}(1-e^{-1/\lambda})^2\left(\frac{3}{2}\eta^2-3\eta^3+\frac{9}{4}\eta^4-\frac{3}{5}\eta^3\right), \tag{2.16}$$

$$\frac{\partial}{\partial\lambda}\int_0^\eta\theta\,d\eta=-3\,(e^{-(1-\eta)/\lambda}-e^{-1/\lambda})+\frac{3}{2}(e^{-2(1-\eta)/\lambda}-e^{-2/\lambda})$$

$$-\frac{1}{3}(e^{-3(1-\eta)/\lambda}-e^{-3/\lambda})-\frac{1}{\lambda}[3\{(1-\eta)\,e^{-(1-\eta)/\lambda}-e^{-1/\lambda}\}$$

$$-3\{(1-\eta)\,e^{-2(1-\eta)/\lambda}-e^{-2/\lambda}\}+\{(1-\eta)\,e^{-3(1-\eta)/\lambda}-e^{-3/\lambda}\}]$$

$$-\frac{1}{\lambda^2}e^{-1/\lambda}(1-e^{-1/\lambda})\left\{1-e^{-1/\lambda}+\frac{1}{\lambda}(-1+3\,e^{-1/\lambda})\right\}$$

$$\cdot\left(\frac{3}{2}\eta^2-3\eta^3+\frac{9}{4}\eta^4-\frac{3}{5}\eta^5\right), \tag{2.17}$$

$$\frac{\partial^2\theta}{\partial\eta^2}=-\frac{3}{\lambda^2}e^{-(1-\eta)/\lambda}(1-e^{-(1-\eta)/\lambda})(1-3\,e^{-(1-\eta)/\lambda})$$

$$+\frac{3}{\lambda}e^{-1/\lambda}(1-e^{-1/\lambda})^2(-6+18\eta-12\eta^2). \tag{2.18}$$

3. We shall compute the value of RZ given by (2.13) in practice at eleven points $\eta=0,\ 0.1,\ 0.2,\ \cdots,\ 0.9$ and 1.0 on the section $\lambda=0.5$ ($x/a\,R=\chi/R=\xi=1.4349$, cf. Table I of the original paper). Although by taking various values of λ and repeating the similar calculations, we should of course arrive at the very definite map of the distribution of the error, yet we shall be able to obtain a general idea about the behavior of the error from the result of the calculation for only one moderate value of λ above chosen as an example.

$d\delta/d\lambda$ in (2.13) can be calculated by means of the method of the numerical differentiation due to L. Collatz.[1] Namely, calculating the value of δ from the formula (2.7) in the original paper for five values of λ: $\lambda=0.497$, 0.498, 0.499, 0.500, 0.501, 0.502 and 0.503, the approximate value of $d\delta/d\lambda$ at $\lambda=0.500$ is given by

$$\left(\frac{d\delta}{d\lambda}\right)_{\lambda=0.500}=\frac{\delta(0.503)-9\,\delta(0.502)+45\,\delta(0.501)-45\,\delta(0.499)+9\delta(0.498)-\delta(0.497)}{60\times0.001}$$

$$=30.332557.^{2)} \tag{3.1}$$

$d\lambda/d\xi$, on the other hand, can be obtained from (2.6) of the original paper, i.e.

$$\frac{d\lambda}{d\xi}=\frac{6\,\lambda\,(1-e^{-1/\lambda})+(1-3\,e^{-1/\lambda})}{(1-e^{-1/\lambda})^4\,\delta^2}, \tag{3.2}$$

so that

$$(d\lambda/d\xi)_{\lambda=0.5}=0.120534997. \tag{3.3}$$

In the following table

$$\zeta_1\equiv\theta\frac{d\lambda}{d\xi}\left(\frac{\partial\theta}{\partial\lambda}-\frac{d\delta}{d\lambda}\frac{\eta}{\delta}\frac{\partial\theta}{\partial\eta}\right), \tag{3.4}$$

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[1] Hidaka, K. *Suchi Sekibun Ho* (*Methods of Numerical Integrations*), vol. 2 (1943), p. 218.

[2] The value of δ corresponding to $\lambda=0.5$ of Table I of the original paper (p. 20) is misprinted and should read 6.879 (6.8786989).

$$\zeta_2 \equiv \frac{1}{\delta}\frac{d\lambda}{d\xi}\left\{\frac{d\delta}{d\lambda}\left(\eta\,\theta - \int_0^\eta \theta\, d\eta\right) - \delta\frac{\partial}{\partial\lambda}\int_0^\eta \theta\, d\eta\right\}\frac{\partial\theta}{\partial\eta}, \tag{3.5}$$

$$\zeta_3 \equiv -\frac{1}{\delta^2}\frac{\partial^2\theta}{\partial\eta^2}, \tag{3.6}$$

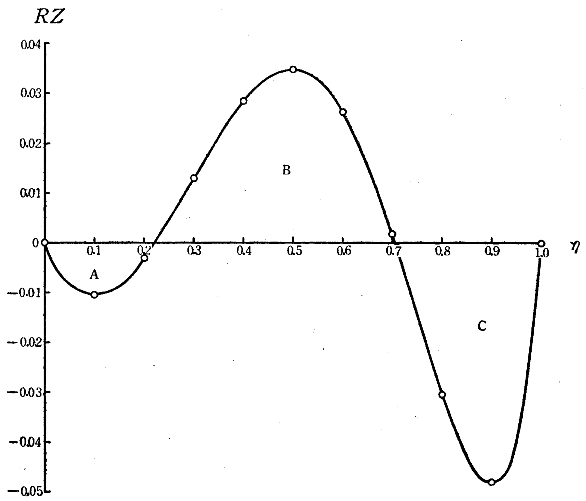
and
$$RZ \equiv \zeta_1 + \zeta_2 + \zeta_3 \tag{3.7}$$

are shown for various values of η.

η	ζ_1	ζ_2	ζ_3	RZ
0	− 0.094612	0	0.094612	0
0.1	− 0.078157	− 0.005504	0.073068	− 0.010593
0.2	− 0.039619	− 0.016547	0.053066	− 0.003100
0.3	0 003584	− 0.024352	0.033816	0.013047
0.4	0.036451	− 0.022399	0.014385	0.028437
0.5	0.049923	− 0.009054	− 0.006112	0.034757
0.6	0.043265	0.010949	− 0.027995	0.026218
0.7	0.024648	0.027083	− 0.049833	0.001898
0.8	0.007462	0.028015	− 0.065898	− 0.030421
0.9	0.000526	0.012555	− 0.060968	− 0.047886
1.0	0	0	0	0

The form of θ given by (2.8) was constructed in such a way that the differential equation (2.3) would be satisfied ***exactly*** at both of the extremities of η, namely on the x-axis ($\eta = 0$) and the outer edge of the jet ($\eta = 1$). This has been verified by the preceding table. The equation of conservation of momentum employed in the original paper to determine δ as a function of λ is nothing but the integrated form of the equation of motion with respect to η, so in the figure of RZ vs. η the following relation should be satisfied:

$$\int_0^1 RZ\, d\eta = 0. \tag{3.8}$$



To prove this equality, A, B, C, three pieces of the foregoing figure drawn on a sheet of paper, were cut out and their weights were measured with a chemical balance. The result obtained is this:

$$B = 2.266\,\text{gr},$$
$$B - (A + C) = 0.036\,\text{gr}.$$

This error (about 1.6 %) should be presumably ascribed to the scarcity of the points through which a smooth curve of RZ had to be drawn and to the cut-off technique of the figure.

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Page 177, line 9, for "Shigematsu" read "Shigekawa."

(*J. O.*)